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A Duality Between Ω-categories and Algebraic Ω-categories [1](#_bookmark0)*,*[2](#_bookmark1)

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**Abstract**

In this paper, we propose a definition of algebraic Ω-categories. Let Ω-**POID** denote the category of Ω-categories with Ω-functors between them such that inverse image of ideals are also ideals, and let Ω- **AlgDom***G* denote the category of algebraic Ω-categories with Scott continuous functors between them having left Ω-adjoints. We show that Ω-**AlgDom***G* and Ω-**POID** are dual equivalent to each other.

*Keywords:* Duality, (Algebraic) Ω-category, Ω-adjunction, Ideals.

# Introduction

The Stone duality and Stone representation come from the classical Stone represen- tation of Boolean algebras [[19](#_bookmark19)], and lead to locale theory as ‘pointless topology’ [2]. Abramsky related the important application of Stone duality in Theoretical Com- puter Science, particularly in Domain Theory of denotational semantics of computer programming languages [[1](#_bookmark2)]. It provides the right framework for understanding the relationship between denotational semantics and program logic. Study of dualities between categories of certain domains were originated by Hofmann, Mislove and

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Stralka [[7](#_bookmark8)] and Lawson [[13](#_bookmark14)]. Therein, there are two basic dualities in domain the- ory: the first is the duality between the category of posets and the category of algebraic domains, many other dualities can be induced by this one; the second one is the duality between the category of domains (i.e., continuous dcpos) and the category of completely distributive lattices.

Quantitative Domain Theory, which models concurrent systems, forms a new branch of Domain Theory, and has undergone active research in the past three decades. Rutten’s generalized (ultra)metric spaces [[17](#_bookmark18)], Flagg’s continuity spaces [[6](#_bookmark7)] and Wagner’s Ω-categories [[24](#_bookmark21)] are examples of quantitative domain theory frame- works. Therein, the Ω-category approach has been payed more and more attentions, including Waszkiewicz [[25](#_bookmark26)], Hofmann and Waszkiewicz [[8](#_bookmark9)] and Lai and Zhang [[12](#_bookmark13)]. And a kind of Lawson duality in framework of Ω-categories has been studied by Hofmann and Waszkiewicz [[9](#_bookmark10)].

Ω-categories are interesting objects for mathematicians and theoretical computer scientists. Firstly, Ω-categories are a special kind of enriched categories, so they can be studied as categories. In 1973, Lawvere [[14](#_bookmark15)] observed that the theory of Ω- categories unifies preordered sets (Ω = *{*0*,* 1*}*, the two point lattice), generalized metric spaces (Ω = [0*, ∞*)*op*), and many other mathematical structures into one framework. Secondly, due to the adjunction *a ∗ b ≤ c ⇔ b ≤ a → c* in the quantale Ω, if we interpret the complete lattice as a set of truth values, the operators *∗* and *→* can be interpreted as the logic connectives conjunction and implication respectively. Therefore, the theory of Ω-categories has a many-valued logic flavor [17]. This feature also leads to the point that Ω-categories can be regarded as generalized preordered sets, or Ω-valued preordered sets. For instance, we can interpret the *A*(*a, b*) as the degree to which *a* is smaller than or equal to *b*, that is, the connection between two points is measured by an element in Ω. Thirdly, Ω-categories are closely related to topology. This can be roughly explained as follows. Generalized metric spaces and many-valued preordered sets are special kinds of Ω-categories, and conversely general Ω-categories can also be studied as Ω-valued quasi-metric spaces or many-valued preordered sets.

The aim of this paper is to study the first duality mentioned in the first para- graph in framework of Ω-categories, that is the duality between the category of Ω-categories and of algebraic Ω-categories. This paper is organized as follows: in Section 2, we recall some basic materials related to Ω-category theory and some preparations are made; in Section 3, we firstly give a definition of an algebraic Ω- categories and then establish a duality between the category of Ω-categories and the category of algebraic Ω-categories.

# Preliminaries and preparations

We refer to [[15](#_bookmark16)] for general category theory, to [[10](#_bookmark11)] for enriched category theory, to

[[16](#_bookmark17)] for quantales, and to [[12](#_bookmark13)] for Ω-categories.

A commutative quantale is a pair (Ω*, ∗*), where Ω is a complete lattice and *∗*

is a commutative, associative, and monotone operation *∗* :Ω *×* Ω *−→* Ω such that

*p ∗* (*−*) has a right adjoint for every *p ∈* Ω. The right adjoint of *p ∗* (*−*) is denoted *p →* (*−*). A commutative quantale is called unital if *∗* has a unit *I*, i.e. *p ∗ I* = *p* for every *p ∈* Ω. It should be noted that the unit *I* need not be the greatest element of Ω. Throughout this paper, (Ω*, ∗,I*), or just Ω, will always denote a commutative, unital quantale if not otherwise specified.

**Proposition 2.1** *Suppose that* (Ω*, ∗,I*) *is a commutative unital quantale, then (I1) p ∗* W*i qi* = W*i*(*p ∗ qi*)*.*

*(I2) I ≤ p → q ⇔ p ≤ q;*

*(I3) I → p* = *p;*

*(I4)* (*p → q*) *∗* (*q → r*) *≤ p → q;*

*(I5)* (W*i pi*) *→ q* = V*i*(*pi → q*)*;*

*(I6) p →* (V*i qi*)= V*i*(*p → qi*)*;*

*(I7)* (*r → p*) *→* (*r → q*) *≥ p → q;*

*(I8)* (*p → r*) *→* (*q → r*) *≥ q → p;*

*(I9) p →* (*q → r*)= (*p ∗ q*) *→ r.*

Categorically speaking, a commutative unital quantale (Ω*, ∗,I*) is just a sym- metric, monoidal closed category with the underlying category being a complete lattice. Therefore, we can develop a theory of categories enriched over Ω [[10](#_bookmark11),[14](#_bookmark15)].

A category enriched over Ω [[14](#_bookmark15)], or an Ω-category, is a set *A* together with an assignment of an element *A*(*a, b*) *∈* Ω to every ordered pair of (*a, b*) *∈ A × A*, such that

1. *I ≤ A*(*a, a*) for every *a ∈ A*;
2. *A*(*a, b*) *∗ A*(*b, c*) *≤ A*(*a, c*) for all *a, b, c ∈ A*.

For all *a, b ∈* Ω, let Ω(*a, b*)= *a → b*. Then (Ω*, →*) becomes an Ω-category [[14](#_bookmark15)]. The **L**-preordered sets [[3](#_bookmark3)] for **L** a complete residuated lattice, the generalized metric space [[14](#_bookmark15),[23](#_bookmark22)] and the *V*-continuity space in [[5](#_bookmark6),[6](#_bookmark7)] are special cases of Ω-categories.

Suppose that *A* is an Ω-category. Let *Aop*(*a, b*)= *A*(*b, a*) for all *a, b ∈ A*. Then *Aop* is also an Ω-category, called the opposite of *A*. If *B* is a subset of *A*, let *B*(*x, y*) = *A*(*x, y*) for all *x, y ∈ B*. Then *B* becomes an Ω-category, called a (full) subcategory of *A*. An Ω-functor between Ω-categories *A* and *B* is a map *f* : *A −→ B* such that *A*(*a, b*) *≤ B*(*f* (*a*)*,f* (*b*)) for all *a, b ∈ A*. An Ω-functor *f ∈* [*Aop,* Ω] (resp., *f ∈* [*A,* Ω]) is always called a lower set (resp., an upper set) in *A*.

Given two Ω-categories *A* and *B*, denote the set of all the Ω-functors from *A*

to *B* by [*A, B*]. For all *f, g ∈* [*A, B*], let [*A, B*](*f, g*) = V

*x∈A*

*B*(*f* (*x*)*, g*(*x*)). Then

[*A, B*] becomes an Ω-category, called the functor category from *A* to *B* [[10](#_bookmark11)]. All

Ω-categories and Ω-functors form an ordinary category, denoted by Ω-**Cat**.

For an ordinary set *X*, Ω*X* the set of all maps from *X* to Ω, the members are called Ω-sets of *X*. The family Ω*X* is also an Ω-category, which is the same to [*X,* Ω]

by regarding *X* as a discrete Ω-category. That is to say, Ω*X* (*f, g*) = V

*x∈X*

*f* (*x*) *→*

*g*(*x*) (*∀f, g ∈* Ω*X* )

**Definition 2.2** A pair of Ω-functors *f ∈* [*A, B*]*,g ∈* [*B, A*] is said to be an Ω-

adjunction, in symbols *f E g* : *A ~ B*, if *B*(*f* (*a*)*, b*)= *A*(*a, g*(*b*)) for all *a ∈ A, b ∈*

*B*. In this case, we say *f* is a left Ω-adjoint of *g* and *g* is a right Ω-adjoint of *f* . Sometimes we also say that (*f, g*) is an Ω-adjunction between *A* and *B*.

**Theorem 2.3** *[*[*12*](#_bookmark13)*] Suppose f* : *A −→ B and g* : *B −→ A are two maps (need not be* Ω*-functors) between* Ω*-categories. Then the following conditions are equivalent:*

1. (*f, g*) *is an* Ω*-adjunction.*
2. *For all a ∈ A, b ∈ B, A*(*a, g*(*b*)) = *B*(*f* (*a*)*, b*)*.*
3. *f and g are functors and I ≤ A*(*a, gf* (*a*))*,I ≤ B*(*fg*(*b*)*, b*) (*∀a ∈ A, b ∈ B*)*.*

**Example 2.4** A fundamental example of Ω-adjunctions are that induced by Kan extension. Let *f* : *A −→ B* be an Ω-functor. For each *ψ ∈* [*B,* Ω], define *f→*(*ψ*)= *ψ ◦ f* . Then we obtain a functor *f→* : [*B,* Ω] *−→* [*A,* Ω], which has a left Ω-adjoint

*f→* : [*A,* Ω] *−→* [*B,* Ω] given by *f→*(*φ*)(*y*)= W

*x∈A*

*φ*(*x*) *∗ B*(*f* (*x*)*, y*) (*∀y ∈ B*) for each

*φ ∈* [*A,* Ω]. That is *f→ E f→* : [*A,* Ω] *~* [*B,* Ω] is an Ω-adjunction. Since if *f* : *A −→*

*B* is an Ω-functor then so is *f* : *Aop −→ Bop*, we have *f→ E f→* : [*Aop,* Ω] *~* [*Bop,* Ω] is an Ω-adjunction.

Let *A* be an Ω-category. For *φ ∈* Ω*A*, define **y**(*φ*)(*x*)= W

*a∈A*

*A*(*x, a*) *∗ φ*(*a*) (*∀x ∈*

1. For *x ∈ A*, by **y**(*x*) we mean the Ω-set **y**(*Ix*), where *Ix* is the Ω-set sending *x*

to the unit *I* and others to 0. In fact, **y**(*x*)(*y*)= *A*(*y, x*) for any *x, y ∈ A*.

**Definition 2.5** ([[12](#_bookmark13),[26](#_bookmark27)]) In an Ω-category *A*, an Ω-set *φ* of *A* is called a directed set in *A* if

## W

*x∈A*

*φ*(*x*) *≥ I*;

* 1. *∀x, y ∈ A, φ*(*x*) *∗ φ*(*y*) *≤* W

*z∈A*

*φ*(*z*) *∗ A*(*x, z*) *∗ A*(*y, z*).

A directed set is called an ideal if it is a lower set additionally. The set of all ideals

in *A* is denoted by *I*(*A*), then *I*(*A*) is a subcategory of [*Aop,* Ω]. Clearly, for each

*x ∈ A*, **y**(*x*) *∈ I*(*A*).

**Proposition 2.6** *(1) For any x ∈ A, J ∈ I*(*A*)*, I*(*A*)(**y**(*x*)*,J* )= *J* (*x*)*.*

*(2) Let f* : *A −→ B be an* Ω*-functor, then f→*(*J* ) *∈ I*(*A*) *for any J ∈ I*(*B*)*.*

**Proof.** Straightforward. *2*

**Lemma 2.7** *For φ ∈* Ω*A, we have*

1. *for any ψ ∈* Ω*A,* Ω*A*(*φ, ψ*) *∗ φ ≤ ψ.*
2. *for any x, y ∈ A, A*(*x, y*) *∗* Ω*A*(*φ,* **y**(*x*)) *≤* Ω*A*(*φ,* **y**(*y*))*.*
3. **y**(*φ*) *is the smallest lower set which is larger than or equal to φ under point- wise order in* Ω*A;*
4. *if φ is directed then* **y**(*φ*) *is an ideal;*
5. *for an* Ω*-functor f ∈* [*A, B*]*, if φ is directed set in A then f→*(*φ*) *∈ I*(*B*)*.*

**Proof.** (1), (2) and (3) are straightforward.

1. Suppose that *φ* is directed. Then

## W

*x∈A*

**y**(*φ*)(*x*) *≥* W

*x∈A*

*φ*(*x*) *≥ I*.

* 1. For any *x, y ∈ A*,

**y**(*φ*)(*x*) *∗* **y**(*φ*)(*y*) = W

*a,b∈A*

*≤* W

*A*(*x, a*) *∗ φ*(*a*) *∗ A*(*y, b*) *∗ φ*(*b*)

*φ*(*c*) *∗ A*(*a, c*) *∗ A*(*b, c*) *∗ A*(*x, a*) *∗ A*(*y, b*)

*a,b,c∈A*

*≤* W *φ*(*c*) *∗ A*(*x, c*) *∗ A*(*y, c*)

*c∈A*

*≤* W W

*φ*(*c*) *∗ A*(*z, c*) *∗ A*(*x, z*) *∗ A*(*y, z*)

*c∈A z∈A*

= W ( W *φ*(*c*) *∗ A*(*z, c*)) *∗ A*(*x, z*) *∗ A*(*y, z*)

*z∈A c∈A*

## = W

*z∈A*

**y**(*φ*)(*z*) *∗ A*(*x, z*) *∗ A*(*y, z*)*.*

Then **y**(*φ*) is directed.

1. By Proposition 5.3 in [[26](#_bookmark27)], we know that *f→*(*φ*) is directed, and by (4),

Ω

*f→*(*φ*)= **y**(*f→*(*φ*)) is an ideal. *2*

Ω

Let *A* be an Ω-category. An element *b ∈ A* is called a colimit [[10](#_bookmark11)] of a functor *f ∈*

[*K, A*] weighted by *φ ∈* [*Kop,* Ω] if for each *y ∈ A*, *A*(*b, y*)= V

*k∈K*

*φ*(*k*) *→ A*(*f* (*k*)*, y*).

Weighted colimits, when they exist, are unique up to isomorphism. It is written by

*b* = colim*φf* if *b* is a colimit of *f* weighted by *φ*.

Consider an Ω-category *A* as an Ω-preordered set, an element *b ∈ A* is called

a join of *φ* : *A —→* Ω, in symbols *b* = *Hφ*, if *A*(*b, x*) = V

*y∈A*

*φ*(*y*) *→ A*(*y, x*) for any

*x ∈ A*. In fact, if *φ* is a lower set in *A*, then *Hφ* = colim*φ*id, where id : *A —→ A* is

the identical functor (cf. Example 3.2(4) and Proposition 3.3(2) in [[12](#_bookmark13)]).

**Proposition 2.8** *In an* Ω*-category A, for φ ∈* Ω*A, if Hφ exists then so does H***y**(*φ*)

*and Hφ* = *H***y**(*φ*)*.*

**Proof.** Suppose that *a* = *Hφ*, we only need to show that for any *x ∈ A*,

## V

*y∈A*

**y**(*φ*)(*y*) *→ A*(*y, x*) = V

*y∈A*

*φ*(*y*) *→ A*(*y, x*)*.* In fact, V

*y∈A*

**y**(*φ*)(*y*) *→ A*(*y, x*) =

V V (*φ*(*z*) *∗ A*(*y, z*)) *→ A*(*y, x*) = V *φ*(*z*) *→* V (*A*(*y, z*) *→ A*(*y, x*)) =

*y∈A z∈A*

*z∈A*

*y∈A*

## V

*y∈A*

*φ*(*y*) *→ A*(*y, x*)*.* *2*

**Proposition 2.9** *[*[*4*](#_bookmark5)*,*[*10*](#_bookmark11)*,*[*12*](#_bookmark13)*,*[*20*](#_bookmark23)*] If f* : *A —→ B has a right* Ω*-adjoint, that is f is a left* Ω*-adjunction. Then f preserves the existing joins, that is f* (*Hφ*)= *Hf→*(*φ*)*.*

**Proof.** Easily following from Theorem 3.11 in [[12](#_bookmark13)] and Proposition 2.9 above. See also Theorem 4.5 in [[26](#_bookmark27)]. *2*

By *a class of weights* [[2](#_bookmark4),[10](#_bookmark11),[11](#_bookmark12)] is meant a functor Φ : Ω-**Cat** *—→* Ω-**Cat** such that (1) for every Ω-category *A*, Φ(*A*) *⊆* [*Aop,* Ω]; (2) Φ(*A*) contains the image

of the Yoneda embedding **y** : *A —→* [*Aop,* Ω]; (3) Φ(*f* ) = *f→* for every Ω-functor *f* : *A —→ B*. The class of weights *P* given by *P*(*A*) = [*Aop,* Ω] is the largest class of weights. The class of weights *Y* given by *Y*(*A*)= *{***y**(*a*)*| a ∈ A}* is the smallest class of weights. The correspondence *I* : *A —→ I*(*A*) is a class of weights (Lemma 5.3 in [[12](#_bookmark13)]).

Let Φ be a class of weights. An Ω-category is call Φ-cocomplete if for any *φ ∈* Φ(*K*) and any functor *f ∈* [*K, A*], colim*φf* always exists. Let Φ be a class of wights. A functor *f ∈* [*A, B*] between Φ-cocomplete Ω-categories is called Φ- cocontinuous if it preserves colimits weights in Φ, that is colim*φg* = colim*φ*(*fg*) for all *φ ∈* Φ(*K*) and *g ∈* [*K, A*].

**Proposition 2.10** *[*[*2*](#_bookmark4)*,*[*12*](#_bookmark13)*] An* Ω*-category A is* Φ*-cocomplete iff Hφ exists for any φ ∈* Φ(*A*)*. A functor f ∈* [*A, B*] *is* Φ*-cocontinuous iff f* (*Hφ*) = *Hf→*(*φ*) *for any φ ∈* Φ(*A*)*.*

**Proof.** This proposition can be implied by using Proposition 3.5, Corollary 3.5 and Corollary 4.6 in [[12](#_bookmark13)]. *2*

**Corollary 2.11** *An* Ω*-category A is I-cocomplete iff HI exists for any I ∈ I*(*A*)*. A functor f ∈* [*A, B*] *is I-cocontinuous iff f* (*HI*)= *Hf→*(*I*) *for any I ∈ I*(*A*)*.*

An *I*-cocontinuous functor is called Scott continuous in some papers, e.g. [[26](#_bookmark27)], it is a counterpart of a Scott continuous map in domain theory.

# Algebraic Ω-category and its dual to Ω-category

Let *L* be an *I*-cocomplete Ω-category. Define **w** : *L × L —→* Ω by

**w**(*a, b*)=

*J∈£*(*L*)

*L*(*b, HJ* ) *→ J* (*a*) (*∀a, b ∈ L*)*.*

We call **w** the way below relation on *L* (which is denoted by *⇓* in [[26](#_bookmark27)]). For *x ∈ L*, if **w**(*x, x*) *≥ I*, then we call *x* a compact element in *L* and denote by *K*(*L*) the set of all compact elements in *L*.

Let *L* be an *I*-cocomplete Ω-category and *x ∈ L*. Define a map *kx* : *L —→* Ω by *kx* = **y**(*x*)*|K*(*L*), **y**(*x*) restricted on *K*(*L*), that is *kx*(*y*) = *e*(*y, x*) if *y ∈ K*(*L*) and otherwise 0. If *kx* is directed in *L* (or equivalently, *kx ∈ I*(*K*(*L*))) and *x* = *Hkx* for any *x ∈ L*, then we call *L* an algebraic Ω-category. The algebraic Ω-category of a generalization of the algebraic fuzzy dcpos in [[26](#_bookmark27)] for Ω a complete residuated lattice and that in [[22](#_bookmark24)] for Ω a complete Heyting algebra.

The aim of this section is to establish a duality between the following two cate- gories:

The one is Ω-**POID**: objects are Ω-categories, morphisms are maps between them such that inverse image of ideals are still ideals (maps like that the one *f* : *A —→ B* between Ω-categories such that *f→*(*I*) *∈ I*(*A*) for all *I ∈ I*(*B*), it is routine to show that such a map is automatically an Ω-functor).

The other is Ω-**AlgDom***G*: objects are algebraic Ω-categories, morphisms are Scott continuous maps between them which having left Ω-adjoints.

The duality between Ω-**POID** and Ω-**AlgDom***G* will show the reasonableness of the definition of algebraicness of Ω-categories.

* 1. *A functor* Ω*-***POID** *from to* Ω*-***AlgDom***op*

*G*

**Proposition 3.1** *For any* Ω*-category A, I*(*A*) *is I-cocomplete as a full subcategory of* [*Aop,* Ω]*.*

**Proof.** Suppose that Φ *∈ I*(*I*(*A*)). We will show that *H*Φ= W

*J∈£*(*A*)

Φ(*J* ) *∗ J* . Put

## W

*J∈£*(*A*)

Φ(*J* ) *∗ J* = *φ*.

**Step 1.** *φ ∈ I*(*A*). In fact,

* + 1. *φ* is a lower set. For any *x, y ∈ A*,

*φ*(*x*) *→ φ*(*y*) *≥*

*J∈£*(*A*)

(Φ(*J* )*∗J* (*x*)) *→* (Φ(*J* )*∗J* (*y*)) *≥*

*J∈£*(*A*)

*J* (*x*) *→ J* (*y*) *≥ A*(*y, x*)*.*

## W

*φ*(*x*)= W W Φ(*J* )*∗J* (*x*)= W

Φ(*J* )*∗*( W

*J* (*x*)) *≥* W

Φ(*J* ) *≥*

*x∈A*

*I*.

*x∈A J∈£*(*A*)

*J∈£*(*A*)

*x∈A*

*J∈£*(*A*)

* + 1. For any *x, y ∈ A*,

*φ*(*x*) *∗ φ*(*y*)

## = W

*J*1*,J*2*∈£*(*A*)

*≤* W

Φ(*J*1) *∗ J*1(*x*) *∗* Φ(*J*2) *∗ J*2(*y*)

Φ(*J* ) *∗ I*(*A*)(*J*1*,J* ) *∗ I*(*A*)(*J*2*,J* ) *∗ J*1(*x*) *∗* Φ(*J*2) *∗ J*2(*y*)

*J*1*,J*2*,J∈£*(*A*)

*≤* W

*J∈£*(*A*)

Φ(*J* ) *∗ J* (*x*) *∗ J* (*y*)

*≤* W W

Φ(*J* ) *∗ J* (*z*) *∗ A*(*x, z*) *∗ A*(*y, z*)

*J∈£*(*A*) *z∈A*

## = W ( W

*z∈A J∈£*(*A*)

Φ(*J* ) *∗ J* (*z*)) *∗ A*(*x, z*) *∗ A*(*y, z*)

## = W

*z∈A*

*φ*(*z*) *∗ A*(*x, z*) *∗ A*(*y, z*)*.*

**Step 2.** *H*Φ= *φ*. In fact, for any *φ*1 *∈ I*(*A*),

*I*(*A*)(*φ, φ*1) = [*Aop,* Ω](*φ, φ*1)

## = V

*x∈A*

*φ*(*x*) *→ φ*1(*x*)

## = V V

(Φ(*J* ) *∗ J* (*x*)) *→ φ*1(*x*)

*x∈A J∈£*(*A*)

## = V V

Φ(*J* ) *→* (*J* (*x*) *→ φ*1(*x*))

*J∈£*(*A*) *x∈A*

## = V

*J∈£*(*A*)

Φ(*J* ) *→* ( V

*x∈A*

*J* (*x*) *→ φ*1(*x*))

## = V

*J∈£*(*A*)

Φ(*J* ) *→ I*(*A*)(*J, φ*1)*.*

*2*

**Corollary 3.2** *Suppose that* Φ *∈ I*(*I*(*A*))*. Then for any x ∈ A,* (*H*Φ)(*x*) = Φ(**y**(*x*))*.*

**Proof.** By Proposition 3.1,

(*H*Φ)(*x*)=

*J∈£*(*A*)

Φ(*J* ) *∗ J* (*x*)=

*J∈£*(*A*)

Φ(*J* ) *∗ I*(*A*)(**y**(*x*)*,J* ) *≤* Φ(**y**(*x*))

since Φ is a lower set. For the other direction,

(*H*Φ)(*x*)=

*J∈£*(*A*)

Φ(*J* ) *∗ J* (*x*) *≥* Φ(**y**(*x*)) *∗* **y**(*x*)(*x*) *≥* Φ(**y**(*x*))*.*

*2*

**Proposition 3.3** *In the I-cocomplete* Ω*-category I*(*A*)*, for any J ∈ I*(*A*)*, we have*

**w**(*J, J* )= *J* (*x*) *∗ I*(*A*)(*J,* **y**(*x*))*.*

*x∈A*

*It follows that for each x ∈ X,* **y**(*x*) *is a compact element in I*(*A*)*.*

**Proof.** By the definition of way below relation **w**, for any *J ∈ I*(*A*),

**w**(*J, J* )=

Φ*∈£*(*£*(*A*))

*I*(*A*)(*J, H*Φ) *→* Φ(*J* )*.*

On the one hand, for any *x ∈ A,* Φ *∈ I*(*I*(*A*)), we have

*J* (*x*) *∗ I*(*A*)(*J, H*Φ) *∗ I*(*A*)(*J,* **y**(*x*))

*≤ I*(*A*)(*J,* **y**(*x*)) *∗ J* (*x*) *∗* (*J* (*x*) *→* (*H*Φ)(*x*))

*≤ I*(*A*)(*J,* **y**(*x*)) *∗* (*H*Φ)(*x*)

= *I*(*A*)(*J,* **y**(*x*)) *∗* Φ(**y**(*x*))

*≤* Φ(*J* )*.*

This shows that *J* (*x*) *∗ I*(*A*)(*J,* **y**(*x*)) *≤ I*(*A*)(*J, H*Φ) *→* Φ(*J* ). By the arbitrariness

of *x ∈ A* and Φ, we have **w**(*J, J* ) *≥* W

*x∈A*

*J* (*x*) *∗ I*(*A*)(*J,* **y**(*x*)).

On the other hand, define

Φ*J* (*φ*)= *J* (*x*) *∗ I*(*A*)(*φ,* **y**(*x*)) (*∀φ ∈ I*(*A*))*.*

*x∈A*

If Φ*J ∈ I*(*I*(*A*)) and *J* = *H*Φ*J* , then **w**(*J, J* ) *≤* Φ*J* (*J* )= W

*x∈A*

*J* (*x*) *∗ I*(*A*)(*J,* **y**(*x*)).

In fact, (i) for any *φ*1*, φ*2 *∈ I*(*A*),

Φ*J* (*φ*1) *→* Φ*J* (*φ*2) *≥* (*J* (*x*) *∗ I*(*A*)(*φ*1*,* **y**(*x*))) *→* (*J* (*x*) *∗ I*(*A*)(*φ*2*,* **y**(*x*)))

*x∈A*

*≥* *I*(*A*)(*φ*1*,* **y**(*x*)) *→ I*(*A*)(*φ*2*,* **y**(*x*)) *≥ I*(*A*)(*φ*2*, φ*1)*.*

*x∈A*

Then Φ*J* is a lower set.

1. W Φ*J* (*φ*) = W W *φ*(*x*) *∗ I*(*A*)(*φ,* **y**(*x*)) *≥* **y**(*x*)(*x*) *∗*

*φ∈£*(*A*)

*I*(*A*)(**y**(*x*)*,* **y**(*x*)) *≥ I*.

1. *∀φ*1*, φ*2 *∈ I*(*A*),

Φ*J* (*φ*1) *∗* Φ*J* (*φ*2)

*φ∈£*(*A*) *x∈A*

## = W

*x,y∈A*

*≤* W

*J* (*x*) *∗ I*(*A*)(*φ*1*,* **y**(*x*)) *∗ J* (*y*) *∗ I*(*A*)(*φ*2*,* **y**(*x*))

*J* (*z*) *∗ A*(*x, z*) *∗ A*(*y, z*) *∗ I*(*A*)(*φ*1*,* **y**(*x*)) *∗ I*(*A*)(*φ*2*,* **y**(*x*))

*x,y,z∈A*

*≤* W

*z∈A*

*J* (*z*) *∗ I*(*A*)(*φ*1*,* **y**(*z*)) *∗ I*(*A*)(*φ*2*,* **y**(*z*))

*≤* W W

*J* (*z*) *∗ I*(*A*)(*φ,* **y**(*z*)) *∗ I*(*A*)(*φ*1*, φ*) *∗ I*(*A*)(*φ*2*, φ*)

*z∈A φ∈£*(*A*)

## = W

*φ∈£*(*A*)

Φ*J* (*φ*) *∗ I*(*A*)(*φ*1*, φ*) *∗ I*(*A*)(*φ*2*, φ*)*.*

In (iii), the fact that *I*(*A*)(*—,* **y**(*z*)) = **y**(**y**(*z*)) is an ideal in *I*(*A*) for any *z ∈ A* is used.

By (i)-(iii), Φ*J* is an ideal in *I*(*A*).

1. It is easy to show that W

*φ∈£*(*A*)

*I*(*A*)(*φ,* **y**(*x*)) *∗ φ* = **y**(*x*) for all *x ∈ A*. By

Proposition 3.1,

*H*Φ*J* = W

*φ∈£*(*A*)

Φ*J* (*φ*) *∗ φ*

## = W W

*J* (*x*) *∗ I*(*A*)(*φ,* **y**(*x*)) *∗ φ*

*φ∈£*(*A*) *x∈A*

## = W

*x∈A*

## = W

*x∈A*

*J* (*x*) *∗* ( W

*φ∈£*(*A*)

*J* (*x*) *∗* **y**(*x*)

*I*(*A*)(*φ,* **y**(*x*)) *∗ φ*)

= **y**(*J* )= *J.*

Note that **y**(*x*)= *H***y**(**y**(*x*)) = W

*φ∈£*(*A*)

*I*(*A*)(*φ,* **y**(*x*)) *∗ φ*. *2*

**Proposition 3.4** *For any J ∈ I*(*A*)*, kJ is directed and HkJ* = *J. Thus I*(*A*) *is an algebraic* Ω*-category.*

**Proof.** For any *φ ∈ K*(*I*(*A*)), *kJ* (*φ*) = *I*(*A*)(*φ, J* ) and especially for *x ∈ A*, *kJ* (**y**(*x*)) = *I*(*A*)(**y**(*x*)*,J* )= *J* (*x*).

## W

*φ∈£*(*A*)

*kJ* (*φ*) *≥* W

*x∈X*

*I*(*A*)(**y**(*x*)*,J* )= W

*x∈A*

*J* (*x*) *≥ I*.

1. For any *φ*1*, φ*2 *∈ K*(*I*(*A*)), we have **w**(*φi, φi*) *≥ I* (*i* = 1*,* 2), by Proposition

## 3.3, W

*x∈A*

*I*(*A*)(**y**(*x*)*, φi*) *∗ I*(*A*)(*φi,* **y**(*x*)) *≥ I* (*i* = 1*,* 2) (note that *φi*(*i* = 1*,* 2) need

not be equal to **y**(*x*) for some *x ∈ X*),

*kJ* (*φ*1) *∗ kI* (*φ*2)

*≤* W

*x,y∈A*

*I*(*A*)(*φ*1*,J* ) *∗ I*(*A*)(*φ*2*,J* ) *∗ I*(*A*)(**y**(*x*)*, φ*1) *∗ I*(*A*)(*φ*1*,* **y**(*x*)) *∗ I*(*A*)(**y**(*y*)*, φ*2)

*∗ I*(*A*)(*φ*2*,* **y**(*y*))

*≤* W

*x,y∈A*

## = W

*x,y∈A*

*≤* W

*I*(*A*)(**y**(*x*)*,J* ) *∗ I*(*A*)(**y**(*y*)*,J* ) *∗ I*(*A*)(*φ*1*,* **y**(*x*)) *∗ I*(*A*)(*φ*2*,* **y**(*y*))

*J* (*x*) *∗ J* (*y*) *∗ I*(*A*)(*φ*1*,* **y**(*x*)) *∗ I*(*A*)(*φ*2*,* **y**(*y*))

*J* (*z*) *∗ A*(*x, z*) *∗ A*(*y, z*) *∗ I*(*A*)(*φ*1*,* **y**(*x*)) *∗ I*(*A*)(*φ*2*,* **y**(*y*))

*x,y,z∈A*

*≤* W

*z∈A*

## = W

*z∈A*

*J* (*z*) *∗ I*(*A*)(*φ*1*,* **y**(*z*)) *∗ I*(*A*)(*φ*2*,* **y**(*z*))

*kJ* (**y**(*z*)) *∗ I*(*A*)(*φ*1*,* **y**(*z*)) *∗ I*(*A*)(*φ*2*,* **y**(*z*))

*≤* W

*φ∈£*(*A*)

*kJ* (*φ*) *∗ I*(*A*)(*φ*1*, φ*) *∗ I*(*A*)(*φ*2*, φ*)*.*

1. By Proposition 3.1, *HkJ* = W

*φ∈£*(*A*)

*kJ* (*φ*) *∗ φ ≥* W

*x∈A*

*J* (*x*) *∗* **y**(*x*) = *J* and

*HkJ* = W

*φ∈£*(*A*)

*kJ* (*φ*) *∗ φ* = W

*φ∈£*(*A*)

*I*(*A*)(*φ, J* ) *∗ φ ≤ J* . Hence *HkJ* = *J* . *2*

**Theorem 3.5** *Suppose that f* : *A —→ B is a morphism in* Ω*-***FPOID***. Deﬁne* **Id**(*f* )= *f→|£*(*B*) : *I*(*B*) *—→ I*(*A*) *by* **Id**(*f* )(*J* )= *f→*(*J* ) (*∀J ∈ I*(*B*))*. Then* **Id**(*f* ) *is a morphism in* Ω*-***AlgDom***G.*

**Proof. Id**(*f* ) : *I*(*B*) *—→ I*(*A*) is a map since *f→*(*J* ) *∈ I*(*A*) for any *J ∈ I*(*B*). Since *f→ ~ f→* : [*Aop,* Ω] *—→* [*Bop,* Ω] is an Ω-adjunction, by Proposition 2.10, *f→* : [*Bop,* Ω] *—→* [*Aop,* Ω] preserves arbitrary joins and then **Id**(*f* ) = *f→|£*(*B*) : *I*(*B*) *—→ I*(*A*) preserves joins of ideals and so is Scott continuous.

Define *g* : *I*(*A*) *—→ I*(*B*) by *g*(*Jj*)= *f→*(*Jj*) (*∀Jj ∈ I*(*A*)), that is *g* = *f→|£*(*A*).

By Lemma 2.5(5), *g* is a map and for any *Jj ∈ I*(*A*)*, J ∈ I*(*B*),

*I*(*B*)(*g*(*Jj*)*,J* )= [*Bop,* Ω](*f→*(*Jj*)*,J* ) = *I*(*A*)(*Jj,f→*(*J* )) = *I*(*A*)(*Jj,* **Id**(*f* )(*J* ))*.*

Thus (*g,* **Id**(*f* )) is an Ω-adjunction by Theorem 2.3. *2*

Proposition 3.4 and Theorem 3.5 show that **Id**:Ω-**POID***—→* Ω-**AlgDom***op* is a functor which transfers **Id**(*A*)= *I*(*A*) for any Ω-category *A* and **Id**(*f* )= *f→|£*(*B*) : *I*(*B*) *—→ I*(*A*) for any Ω-functor *f ∈* [*A, B*].

*G*

* 1. *A functor from* Ω*-***AlgDom***G to* Ω*-***POID***op*

For any algebraic Ω-category *L*, *K*(*L*) is an Ω-category as a full subcategory of *L*.

**Lemma 3.6** *Suppose that g* : *L —→ M is a Scott continuous functor between two algebraic* Ω*-categories which has a left* Ω*-adjoint gE* : *M —→ L. Then gE*(*K*(*M* )) *⊆ K*(*L*)*.*

**Proof.** For any *a ∈ K*(*M* ), we need to show *gE*(*a*) *∈ K*(*L*), that is *L*(*gE*(*a*)*, HJ* ) *≤*

*J* (*gE*(*a*)) for all *J ∈ I*(*L*). In fact,

*L*(*gE*(*a*)*, HJ* )= *M* (*a, g*(*HJ* )) = *M* (*a, Hg→*(*J* )) *≤ g→*(*J* )(*a*)

= *J* (*b*) *∗ L*(*a, g*(*b*)) = *J* (*b*) *∗ M* (*gE*(*a*)*, b*) *≤ J* (*gE*(*a*))*.*

*b∈B*

*b∈B*

*2*

**Lemma 3.7** *For J ∈ I*(*K*(*L*))*, consider J as an* Ω*-set of L, we have* **y**(*J* ) *∈ I*(*L*)*, where*

**y**(*J* )(*x*)= *J* (*a*) *∗ L*(*x, a*) (*∀x ∈ L*)

*a∈K*(*L*)

*is that deﬁned in the paragraph above Deﬁnition 2.5.*

**Proof.** By Lemma 2.5(3), **y**(*J* ) is a lower set and for any *x ∈ A*, and W

*x∈L*

**y**(*J* )(*x*) *≥*

## W

*x∈L*

*J* (*x*) *≥ I*. For any *x*1*, x*2 *∈ L*,

**y**(*J* )(*x*1) *∗* **y**(*J* )(*x*2) = W

*a*1*,a*2*∈K*(*L*)

*≤* W

*J* (*a*1) *∗ L*(*x*1*, a*1) *∗ J* (*a*2) *∗ L*(*x*2*, a*2)

*J* (*a*) *∗ L*(*a*1*, a*) *∗ L*(*x*1*, a*1) *∗ L*(*a*2*, a*) *∗ L*(*x*2*, a*2)

*a*1*,a*2*,a∈K*(*L*)

*≤* W

*a∈K*(*L*)

*J* (*a*) *∗ L*(*x*1*, a*) *∗ L*(*x*2*, a*)

*≤* W

*a∈L*

**y**(*J* )(*a*) *∗ L*(*x*1*, a*) *∗ L*(*x*2*, a*)*.*

*2*

**Proposition 3.8** *Let L be an algebraic* Ω*-category. Then* **y**(*x*)*|K*(*L*) *∈ I*(*K*(*L*)) *for any x ∈ L.*

**Proof.** Clearly **y**(*x*)*|K*(*L*) = *kx ∈ I*(*K*(*L*)).

## W

*a∈K*(*L*)

**y**(*x*)*|K*(*L*)(*x*)= W

*a∈K*(*L*)

*kx*(*a*) *≥ I*.

1. For any *a*1*, a*2 *∈ K*(*L*),

**y**(*x*)*|K*(*L*)(*a*2) *∗ K*(*L*)(*a*1*, a*2)= *L*(*a*2*, x*) *∗ L*(*a*1*, a*2) *≤ L*(*a*1*, x*)= **y**(*x*)*|K*(*L*)(*a*1)*,*

thus **y**(*x*)*|K*(*L*) is a lower set in *K*(*L*).

1. For any *a*1*, a*2 *∈ K*(*L*),

**y**(*x*)*|K*(*L*)(*a*1) *∗* **y**(*x*)*|K*(*L*)(*a*2)

= *kx*(*a*1) *∗ kx*(*a*2)

*≤* W

*a∈L*

*kx*(*a*) *∗ L*(*a*1*, a*) *∗ L*(*a*2*, a*)

## = W

*a∈K*(*L*)

## = W

*a∈K*(*L*)

*kx*(*a*) *∗ K*(*L*)(*a*1*, a*) *∗ K*(*L*)(*a*2*, a*)

**y**(*x*)*|K*(*L*)(*a*) *∗ ∗K*(*L*)(*a*1*, a*) *∗ K*(*L*)(*a*2*, a*)*.*

*2*

**Theorem 3.9 K** : Ω*-***AlgDom***G —→* Ω*-***POID***op* (*L '→ K*(*L*)*, g '→ gE*) *is a func- tor.*

**Proof.** Suppose that *g* : *L —→ M* is a morphism in Ω-**AlgDom***G*, we need to show that *gE* : *K*(*M* ) *—→ K*(*L*) is a morphism in Ω-**POID**. Suppose that *J ∈ I*(*K*(*L*)), by Lemma 2.7(4), **y**(*J* ) *∈ I*(*L*), by Proposition 2.8, *HJ* = *H***y**(*J* ).

Put *c* = *HJ* , then we have *J* (*a*) = **y**(*c*)(*a*) for all *a ∈ K*(*L*) and then *J* = **y**(*c*)*|K*(*L*). In fact, *J* (*a*) *≤ L*(*a, c*) = **y**(*c*)(*a*) since *c* = *HJ* . Conversely, since *a ∈ K*(*L*), we have

*I ≤* **w**(*a, a*) *≤ L*(*a, H***y**(*J* )) *→* **y**(*J* )(*a*)= *L*(*a, c*) *→* **y**(*J* )(*a*)

and

**y**(*c*)(*a*)= *L*(*a, c*) *≤* **y**(*J* )(*a*)=

*x∈K*(*L*)

*J* (*x*) *∗ L*(*a, x*) *≤ J* (*a*)

since *J* is a lower set in *K*(*L*).

We will show that (*gE*)*→*(*J* )= **y**(*g*(*c*))*|K*(*M* ). For any *b ∈ L*(*M* ), (*gE*)*→*(*J* )(*b*)= *J* (*gE*(*b*)) = **y**(*c*)(*gE*(*b*)) = *L*(*gE*(*b*)*, c*)= *M* (*b, g*(*c*)) = **y**(*g*(*c*))(*b*)*.*

Hence (*gE*)*→*(*J* ) *∈ I*(*K*(*M* )) by Proposition 3.8. *2*

By the proof of Theorem 3.9, we have

**Proposition 3.10** *For any algebraic* Ω*-category L, all ideals in K*(*L*) *has the form*

**y**(*x*)*|K*(*L*) *for some x ∈ L.*

**Proof.** Let *id* : *L —→ L* be the identical functor. Then the left Ω-adjoint of *id* is still *id*, thus for any ideal *J* in *K*(*L*), *J* = *id→*(*J* )= **y**(*x*)*|K*(*L*), where *x* is the join of *J* in *L*. *2*

* 1. *Duality between* Ω*-***AlgDom***G and* Ω*-***POID**

For any Ω-category *A*, define *ηA* : *A —→ K*(*I*(*A*))*, x '→* **y**(*x*) (*∀x ∈ X*).

**Theorem 3.11** *η* : *id*Ω*-***POID** *—→* **K** *◦* **Id** *is a natural transformation.*

1. *ηA*

) *K*(*I*(*A*))

*f f→*

v ) v

1. *ηB K*(*I*(*B*))

**Figure 1**

**Proof.** For *f* : *A —→ B* a morphism in Ω-**POID**, **Id**(*f* ) = *f→|£*(*B*) : *I*(*B*) *—→*

*I*(*A*). By Theorem 3.5, the left Ω-adjoint of **Id**(*f* ) is **K***◦***Id**(*f* )= *f→|£*(*A*) : *I*(*A*) *—→ I*(*B*).

We need to show that *f→ ◦ ηA* = *ηB ◦ f* . In fact for any *x ∈ A*, for any *y ∈ B*,

*f→*(*ηA*(*x*))(*y*)= *f→*(**y**(*x*))(*y*)= **y**(*x*)(*a*)*∗Bop*(*f* (*a*)*, y*)= *B*(*y, f* (*a*))*∗A*(*a, x*)*.*

On one hand,

*a∈B*

*a∈A*

## W

*a∈A*

*B*(*y, f* (*a*)) *∗ A*(*a, x*) *≤* W

*a∈A*

*B*(*y, f* (*a*)) *∗ B*(*f* (*a*)*,f* (*x*))

*≤ B*(*y, f* (*x*)) = **y**(*f* (*x*))(*y*)= *ηB*(*f* (*x*)(*y*);

on the other hand,

*B*(*y, f* (*a*)) *∗ A*(*a, x*) *≥ B*(*y, f* (*x*)) *∗ A*(*x, x*) *≥ ηB*(*f* (*x*)(*y*)*.*

*a∈A*

Hence *f→*(*ηA*(*x*))(*y*)= *ηB*(*f* (*x*)(*y*). Therefore *f→ ◦ ηA* = *ηB ◦ f* . *2*

**Proposition 3.12** *Deﬁne a transformation ε* : **Id** *◦* **K** *—→ id*Ω*-***AlgDom***G by for any L ∈* Ω*-***AlgDom***G, εL* : *I*(*K*(*L*)) *—→ L, J '→ HJ* (*∀J ∈ I*(*K*(*L*))*. Then ε is a natural isomorphism. The inverse of ε of given by ε−*1(*x*)= **y**(*x*)*|K*(*L*)*.*

*L*

**Proof.** *ε*(*ε−*1(*x*)) = *H*(**y**(*x*)*|K*(*L*)) = *Hkx* = *x* and *ε−*1(*ε*(*J* )) = *ε−*1(*HJ* ) =

**y**(*HJ* )*|K*(*L*) = *J* . *2*

By Propositions 3.11 and 3.12,

**Theorem 3.13 Id** *is the left adjoint of* **K***.*

In order to show the isomorphism between **Id** and **K**, we need two additional conditions for the quantale Ω:

(Q1) *I ≤* W *A* implies *I ≤ x* for some *x ∈ A ⊆* Ω;

(Q2) *I ≤ x ∗ y* implies *I ≤ x* or *I ≤ y* for any *x, y ∈* Ω.

The following example gives such a quantale which is nontrivial, *∗ /*= *∧* and

*I /*= 1.

**Example 3.14** Let Ω = *{*0*, a, b,* 1*}* be the diamond lattice, that is 0 *≤ a, b ≤* 1 and

*a /≤ b, b /≤ a*. Define *∗* :Ω *×* Ω *—→* Ω by

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| \* | 0 | a | b | 1 |
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | b | 1 |
| b | 0 | b | b | b |
| 1 | 0 | 1 | b | 1 |

Clearly, *∗* is monotone and *a* is the unit and the conditions (Q1) and (Q2) are satisfied. We now only need to show that *x∗*(*a∨b*)= (*x∗a*)*∨*(*x∗b*) or *x∗*1= *x∨*(*x∗b*) for any *x ∈* Ω. In fact, if *x* = 0 or *x* = *a*, then it holds; if *x* = 1, it holds since 1 *∗* 1= 1; if *x* = *b*, then *x∗* 1= *b* = *b∨b* = *x∨* (*x∗b*). Then (Ω*, ∗, a*) is a commutative unital quantale (furhtermore, *∗* is idempotent).

**Proposition 3.15** *If (Q1) and (Q2) hold for* Ω*, then the compact elements in I*(*A*)

*have the form* **y**(*x*) (*x ∈ A*)*. In this case,* Ω*-***AlgDom***G is dual to* Ω*-***POID***.*

**Proof.** Let *A* be an Ω-category. Suppose that *φ* is a compact element in *I*(*A*),

by Proposition 3.3, we have W

*x∈A*

*I*(*A*)(*φ,* **y**(*x*)) *∗ I*(*A*)(**y**(*x*)*, φ*) *≥ I*. By (Q1), we

have *I*(*A*)(*φ,* **y**(*x*)) *∗ I*(*A*)(**y**(*x*)*, φ*) *≥ I* for some *x ∈ A*. By (Q2) *I*(*A*)(*φ,* **y**(*x*)) *≥*

*I, I*(*A*)(**y**(*x*)*, φ*) *≥ I*, which implies *φ* = **y**(*x*). *2*

# 4 Conclusions

By introducing a definition of algebraicity of Ω-categories, we show that the category of algebraic Ω-categories (with certain morphisms) and the category of Ω-functors

(with certain morphisms) are dual equivalent to each other. The transformation from an Ω-category to an algebraic Ω-category exactly is the ideal completion (i.e., *I*-completion), and the that from an algebraic *Q*-category to an Ω-category just is the restriction to the compact objects of an algebraic Ω-category.

Such a duality could be generalized to one between Ω-categories and Φ-algebraic Ω-categories for Φ is a (saturated) class of weights. For Φ being *I*, an *I*-algebraic Ω- category just is an algebraic Ω-category in this paper. For Φ is the maximal class *P*, a *P*-algebraic Ω-category just is a totally algebraic cocomplete *Q*-categories in [[21](#_bookmark25)]. There are also many interesting examples of other classes of weights in framework of metric spaces studied in [[18](#_bookmark20)].

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