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A Family of Graded Epistemic Logics

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**Abstract**

Multi-Agent Epistemic Logic has been investigated in Computer Science [[6](#_bookmark20)] to represent and reason about agents or groups of agents knowledge and beliefs. Some extensions aimed to reasoning about knowledge and probabilities [[5](#_bookmark19)] and also with a fuzzy semantics have been proposed [[7,](#_bookmark21)[14](#_bookmark28)].

This paper introduces a parametric method to build graded epistemic logics inspired in the systematic method to build Multi-valued Dynamic Logics introduced in [[12](#_bookmark22),[13](#_bookmark27)]. The parameter in both methods is the

same: an action lattice [[10](#_bookmark23)]. This algebraic structure supports a generic space of agent knowledge operators, as choice, composition and closure (as a Kleene algebra), but also a proper truth space for possible non bivalent interpretation of the assertions (as a residuated lattice).

*Keywords:* Epistemic Logic, Action Lattice, Modal Logics

# Introduction

The analysis and the applications of concepts such as agent’s knowledge, every- body’s knowledge and common knowledge became a stimulating research field, par- ticularly in the last decades, when *epistemic logics* emerged. Although, the work of

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Hintikka [[9](#_bookmark24)] can be considered the founder of modern Modal Epistemic Logic, most of these logics are heavily influenced by the work of Halpern et al. [[6](#_bookmark20)] on modal logics of knowledge in a multi-agent systems framework. Modal logics of knowledge describe how an agent reasons about his own knowledge and about the knowledge of other agents. We say that an agent knows a fact *ϕ* if *ϕ* is true in every state that the agent considers possible. “The intuition is that if an agent does not have complete knowledge about the world, he will consider a number of possible worlds. These are his candidates for the way the world actually is” [[6](#_bookmark20)].

Much of the agreement and cooperation in a group of agents is reached con- sidering the interaction among the agents and the increasing group knowledge ac- quisition. A fact *ϕ* is mutual knowledge in a group of agents, if each agent knows *ϕ*. This group knowledge is also known as everybody’s knowledge. Suppose, for instance, that each participant in a conference knows that the lecturer will arrive late. The fact that the lecturer will arrive late is mutual knowledge among the participants, but each participant may think that he is the only one who knows about that. However, suppose that one of the participants makes an announcement for the audience: “The lecturer told me that he will arrive late”. From this mo- ment onwards, each participant knows that each participant knows that the lecturer will arrive late, and each participant knows that each participant knows that each participant knows that the lecturer will arrive late, and so on. The participant’s statement turned the fact that was mutually known into a common knowledge fact. There are many situations where we have uncertainty in our knowledge and beliefs. It is not unusual to believe in some fact with some grade of possibility. For instance, *Anna believes that her father has a strong preference for Bob, which means*

*that she believes that he will give a sweet to Bob rather than to Clara. In a scale from 0 to 5, her belief is 4*. This kind of belief is not true or false. In this work we deal with graded knowledge, but atomic propositions are true or false.

In [[6](#_bookmark20)] Multi-Agent Epistemic Logic has been investigated, to represent and rea- son about agents or groups of agents knowledge and beliefs. There are many pro- posals to extend these logics with uncertainty. Some extensions aimed to reasoning about knowledge and probabilities [[5](#_bookmark19)]. In general, this is accomplished extending the language with weighted formulas and adding probabilities to the semantics. There are other attempts that provide a fuzzy or many-valued semantics [[7,](#_bookmark21)[14](#_bookmark28)]. This work goes in the later direction.

The work of Fitting [[7](#_bookmark21)] proposes a many-valued modal logic where the truth values are taken from a lattice. It is presented two semantics, one where the atomic propositions are many valued and a second one where the accessibility relation also is many valued. Also, in [[4](#_bookmark18)], it is presented a many-valued modal logic over a finite residuated lattice. In [[14](#_bookmark28)], it is introduced an epistemic logic based on the work of Fitting. It differs from ours because they work with a particular lattice. Another related work that uses a complete, distributive lattices as semantics for epistemic and doxastic logics is presented in [[8](#_bookmark25)]. More recently, some interesting works have appeared to deal with many valued dynamic epistemic logic [[17,](#_bookmark31)[1,](#_bookmark15)[11](#_bookmark26)].

In [[12,](#_bookmark22)[13](#_bookmark27)], it is proposed a method to build many-valued dynamic logics. Inspired

on this method, we introduce a method to build graded multi-agent epistemic logics. Both methods are based on action lattices [[10](#_bookmark23)]. Using action lattices, we are able to support a generic space of agent knowledge operators, as choice, composition and closure (as a Kleene algebra), but also a proper truth space for possible non bivalent interpretation of the assertions (as a residuated lattice). We use matricial algebra to be able to introduce knowledge representations as weighted graphs, which enables us to capture a wide class of weighted scenarios, from the classic bivalent perspective of knowledge, to other structured, discrete and continuous, domains. It should be notice that, in this work, we only deal with the epistemic notions of knowledge and their duals.

This paper is organized as follows. Section [2](#_bookmark1) presents all the background needed about Multi-Agent Epistemic Logic. Section [3](#_bookmark4), introduces our method for building graded Multi-Agent Epistemic Logics. It also provides some concepts on Kleene algebras and action lattices. Section [4](#_bookmark9) illustrates the use of our method with two examples. Section [5](#_bookmark12) discusses some conditions where classical axioms of Multi-Agent Epistemic Logic are valid and points out some future work.

# Multi-Agent Epistemic Logic

Multi-Agent Epistemic Logic has been investigated in Computer Science [[6](#_bookmark20)] to rep- resent and reason about agents or groups of agents knowledge and beliefs.

* 1. *Language and Semantics*

**Definition 2.1** *The epistemic language consists of a set* Φ *of countably many proposition symbols, a ﬁnite set A of agents, the boolean connectives ¬ and ∧, a modality Ka for each agent a. The formulas are deﬁned as follows:*

*ϕ* ::= *p |T| ¬ϕ | ϕ*1 *∧ ϕ*2 *| Kaϕ | CGϕ*

*where p ∈* Φ*, a ∈A and G ⊆ A.*

The standard connectives can be presented as abbreviations, namely *⊥ ≡ ¬T*,

V

*ϕ ∨ φ ≡ ¬*(*¬ϕ ∧ ¬φ*), *ϕ → φ ≡ ¬*(*ϕ ∧ ¬φ*) and *EGϕ ≡ a∈G Kaϕ*.

The intuitive meaning of the modal formulas are:

* *Kaϕ* - agent *a* knows *ϕ*;
* *EGϕ* - every agent *a ∈ G* knows *ϕ*;
* *CGϕ* - it is common knowledge among all members of group *G* that it is the case that *ϕ*.

We also introduce, by definition, the dual operators *Bϕ ≡ ¬K¬ϕ* and *MGϕ ≡*

*¬EG¬ϕ*.

**Definition 2.2** *A multi-agent epistemic* frame *is a tuple F* = (*W,* (*Ra*)*a∈A*) *where*

* *W is a non-empty set of states;*
* *Ra is a binary relation over W, for each agent a ∈ A;*

*We also deﬁne the following relations*

* *RG* = *a∈G Ra*
* *∗*

*G*

= (*RG*)*∗, where* (*RG*)*∗ is the reflexive, transitive closure of RG.*

*R*

**Definition 2.3** *A* multi-agent model *is a pair M* = (*F,* **V**)*, where F is a frame and* **V** *is a valuation function* **V** :Φ *→* 2*W .*

In most applications of Multi-Agent Epistemic Logic the relations *Ra* are equiv- alence relations. In this case, models are called *epistemic models* and, in these

structures, if *G* is not the empty group of agents, *R∗* coincides with *R*+, for *R*+

being the transitive closure of *RG*.

*G G G*

**Definition 2.4** *Given a multi-agent model M* = *⟨S,* (*Ra*)*a∈A,V ⟩ and a state s ∈ S. The notion of satisfaction M,s |*= *ϕ is deﬁned as follows*

* *M,s |*= *p iff s ∈ V* (*p*)
* *M,s |*= *¬φ iff M,s |*= *φ*
* *M,s |*= *φ ∧ ψ iff M,s |*= *φ and M,s |*= *ψ*
* *M,s |*= *Kaφ iff for all sj ∈ S* : *sRasj ⇒ M, sj |*= *φ*
* *M,s |*= *CGφ iff for all sj ∈ S* : *sR∗ sj ⇒ M, sj |*= *φ*

*G*

*It is easy to see that M,s |*= *EGφ iff for all sj ∈ S* : *sRGsj ⇒ M, sj |*= *φ.*

**Example 1 (An adaptation from [**[**18**](#_bookmark32)**])** *Suppose a father has three envelopes, each containing:* **0***,* **1** *and* **2** *dollars inside respectively.The father has three children:* **a***nna,* **b***ob and* **c***lara. Each child receives one envelope and does not know content of the envelopes of the other children.*

*We use proposition symbols* 0*x,* 1*x,* 2*x for x ∈ {a, b, c} meaning that “child x has envelope* **0***,* **1** *or* **2***. We name each state by the envelope that each child has in that state, for instance* 012 *is the state where child* **a** *has* **0***, child* **b** *has* **1** *and child* **c** *has* **2***. A state name underlined means actual state. The following epistemic model represents the epistemic state of each agent* [6](#_bookmark3) *.*

*Hexa* = *⟨S, Ra, Rb, Rc,V ⟩:*

* + *S* = *{*012*,* 021*,* 102*,* 120*,* 201*,* 210*}*
  + *Ra* =

*{*(012*,* 012)*,* (012*,* 021)*,* (021*,* 021)*,... },*

*...*

012 *a*

*b*

*c*

102 *a*

*b*

021

*c*

*b*

*c*

120

* + *V* (0*a*)= *{*012*,* 021*}, V* (1*a*)=

*{*102*,* 120*}, ...*

201 *a* 210

*It is not difficult to see that* 012 *|*= *Bb*0*a and* 012 *|*= *BaKc*2*c hold, but*

021 *|*= *Eac*2*b does not hold.*

6 We omit the reflexive loops in the picture

# Parametric construction of Graded Epistemic Logics

We introduce, in this paper, a parametric method to build graded epistemic logics inspired in the systematic method to build multi-valued dynamic logics introduced in [[12,](#_bookmark22)[13](#_bookmark27)]. The methods are based in the same parameter: an action lattice [[10](#_bookmark23)].

* 1. *Kleene algebras, action lattices and graded knowledge representation*

Action lattices support a generic space of agent knowledge operators, as choice, com- position and closure (as a Kleene algebra), but also a proper truth space for possible non bivalent interpretation of the assertions (as a residuated lattice). Observe that the original motivations of Kozen to introduce action lattices were very different for these ones. Originally, the residues were introduced within *action algebra* [[16]](#_bookmark30) as a necessary technicality to obtain a finitely-based equational variety to reason about imperative programs. Then, Kozen adjusted this notion into the *action lat- tice* in [[10](#_bookmark23)] by introducing and axiomatizing a meet operation, in order to recover the closeness by matricial formation of the Kleene Algebras [[3](#_bookmark16)]. We overview, in the following, the action algebra with some relevant examples in the context of our purpose. A lot of other examples and properties can be found in [[12](#_bookmark22)]. The structure of Kleene algebra will be used to model the set of agent knowledge operators over a set of agents *A*. In our setting, the valuations of propositions are crisp, i.e., true or false. This forces the integrability on the action lattices adopted.

|  |  |  |  |
| --- | --- | --- | --- |
| *a* + (*b* + *c*)= (*a* + *b*)+ *c* | (1) | *a*; *x ≤ x ⇒ a∗*; *x ≤ x* | (11) |
| *a* + *b* = *b* + *a* | (2) | *x*; *a ≤ x ⇒ x*; *a∗ ≤ x* | (12) |
| *a* + *a* = *a* | (3) | *a*; *x ≤ b ⇔ x ≤ a → b* | (13) |
| *a* +0 = 0+ *a* = *a* | (4) | *a → b ≤ a →* (*b* + *c*) | (14) |
| *a*; (*b*; *c*)= (*a*; *b*); *c* | (5) | (*x → x*)*∗* = *x → x* | (15) |
| *a*;1 = 1; *a* = *a* | (6) | *a ·* (*b · c*)= (*a · b*) *· c* | (16) |
| *a*; (*b* + *c*)= (*a*; *b*)+ (*a*; | *c*)(7) | *a · b* = *b · a* | (17) |
| (*a* + *b*); *c* = (*a*; *c*)+ (*b*; | *c*)(8) | *a · a* = *a* | (18) |
| *a*;0 = 0; *a* =0 | (9) | *a* + (*a · b*)= *a* | (19) |
| 1+ *a* + (*a∗*; *a∗*) *≤ a∗* | (10) | *a ·* (*a* + *b*)= *a* | (20) |

Fig. 1. Axiomatisation of action lattices (from [[10](#_bookmark23)])

**Definition 3.1 (Kleene Algebra)** *A* Kleene algebra *is an idempotent (and thus partially ordered) semiring endowed with a closure operator ∗, i.e. it consists of a tuple* (*A,* +*,* ; *,* 0*,* 1*,∗* ) *where A is a non empty set,* + *and* ; *are binary operations,*

*∗ is an unary operation and* 0*,* 1 *are constants satisfying the axioms (*[*1*](#_bookmark5)*)–(*[*12*](#_bookmark5)*) (the relation ≤ is the natural order induced by the operation* +*: a ≤ b iff a* + *b* = *b).*

Note that ([4](#_bookmark5)) implies that 0 is the minimum element in any Kleene algebra. Conway shown in [[3](#_bookmark16)] that we can endow the class of all matrices over a Kleene algebra with a Kleene structure. We recall this procedure here: given a Kleene algebra

**A** = (*A,* +*,* ; *,* 0*,* 1*,∗* ) we define a Kleene algebra M*n*(**A**) = (*Mn*(**A**)*,* **+***,* **;***,* **0***,* **1***, ∗*) as follows:

1. *Mn*(**A**) is the space of (*n × n*)-matrices over **A**
2. for any *A, B ∈ Mn*(**A**), define *M* = *A*+*B* by *Mij* = *Aij* + *Bij*, *i, j ≤ n*.
3. for any *A, B ∈ Mn*(**A**), define *M* = *A* **;** *B* by *Mij* = Σ*n* (*Aik*; *Bkj*) for any

*k*=1

*i, j ≤ n*.

1. **1** and **0** are the (*n × n*)-matrices defined by **1***ij*

for any *i, j ≤ n*.

= 1 if *i* = *j*

0 otherwise

and **0***ij*

= 0,

1. for any *M* = [*a*] *∈* M (**A**), *M* **\*** = [*a∗*]; for any *M* = ⎡ *A B* ⎤ *∈ M*

*n*

(**A**), *n >* 1,

1

where *A* and *D* are square matrices, define

⎡ *F* **\***

⎣ *D∗*; *C*; *F∗ D***\*+**(*D***\* ;** *C* **;** *F* **\* ;** *B* **;** *D***\***) ⎦

*F* **\* ;** *B* **;** *D***\*** ⎤

*M* **\*** =

⎣ *C D* ⎦

where *F* = *A* + *B* **;** *D***\* ;** *C*. Note that this construction is recursively defined from the base case (*n* = 2) where the operations of the base action lattice **A** are used.

In the present work we take advantage of this matricial algebra to be able to operate knowledge representations as weighted graphs or, more precisely, weighted labelled transition systems. As we will see, this abstract structure capture a wide class of weighted scenarios, from the classic bivalent perspective of knowledge, to other structured, discrete and continuous, domains.

Moreover, as stated, we are interested in the definition of Graded Epistemic logics with non necessarily boolean degrees of truth. In this view, in order to be able to interpret other logical connectives, we extend our Kleene Algebra of knowledge with some additional structure - namely, with a residue for the interpretation of the logical implication and an infimum to interpret the logical conjunction. This can be found in the following notion of action lattice introduced by D. Kozen in [[10](#_bookmark23)]. Note, however, that the seminal motivation for this definition was quite distinct of the stated one. In particular, it aimed to adjust the finitely-based equational variety “action algebra” of Pratt [[16](#_bookmark30)], to an algebra closed under the matricial constructions. Let us recall this notion:

**Definition 3.2** *An* action lattice *is a tuple* **A** = (*A,* +*,* ; *,* 0*,* 1*,∗ , →, ·*)*, where A is a non empty set,* 0 *and* 1 *are constants, ∗ is an unary operation in A and* +*,* ; *, → and*

*· are binary operations in A satisfying the axioms enumerated in Figure* [*1*](#_bookmark6)*, where the relation ≤ is induced by* + *in the sense that a ≤ b iff a* + *b* = *b. An* integral action lattice *is an action lattice satisfying a ≤* 1*.*

Beyond the bivalent *{*0*,* 1*}*-action lattice we consider the following two action

lattice that will be used to illustrate our method in Section [4](#_bookmark9). More examples and properties of action lattices can be found in [[12](#_bookmark22)].

**Definition 3.3 (L~ ukasiewicz arithmetic lattice)** *The* L- ukasiewicz

arithmetic lattice *is the structure* ***L-***

= ([0*,* 1]*,* max*, ⊙,* 0*,* 1*,∗ , →,* min)*, where*

*x → y* = min(1*,* 1 *− x* + *y*)*,*

*x ⊙ y* = max(0*,y* + *x −* 1) *and*

*x∗* = 1*.*

**Definition 3.4 (Finite Wajsberg hoops)** *We consider now an action lattice en- dowing the ﬁnite* Wajsberg hoops *[*[*2*](#_bookmark17)*] with a suitable star operation. Hence, for a ﬁx natural k >* 0 *and a generator a, we deﬁne the structure* **W***k* = (*Wk,* + *,* ; *,* 0*,* 1*,∗ , →*

*, ·*)*, where Wk* = *{a*0*, a*1*, ··· , ak},* 1= *a*0 *and* 0= *ak, and for any m, n ≤ k,*

*am* + *an* = *amin{m,n} am*; *an* = *amin{m*+*n,k}* (*am*)*∗* = *a*0

*am → an* = *amax{n−m,*0*} am · an* = *amax{m,n}*

* 1. *A method to build Graded Epistemic Logics*

In this section we introduce a method to build multi-agent epistemic logics parame- terized by an action lattice. The “on-demand grading” of the logic is only reflected in its semantics; the syntax is the same as in the standard case. The proposition assignment is crisp and only the agent’s relations are graded on the underlying ac- tion lattice. This non orthodox feature is naturally expressed on the definition of satisfaction.

Let us fix a complete action lattice **A** = (*A,* +*,* ; *,* 0*,* 1*,∗ , →, ·*). We introduce, in the following, a method to generate an **A***-graded epistemic logic GE* (**A**):

* + - Signatures (At*,* Ag) where At is a set of atomic propositions and Ag is a finite set of agents.
    - Sentences are the standard sentences of Multi-Agent Epistemic Logic:

*ϕ* ::= *p |⊥| ϕ ∧ ϕ | ϕ ∨ ϕ | ϕ → ϕ | Kaϕ | Baϕ | EGϕ | MGϕ | CGϕ*

where *p ∈* At, *a ∈* Ag, *G ⊆* Ag. Note that, we are explicitly considering the disjunction connective and the dual operators of the ones introduced in Defini- tion [2.1](#_bookmark2) , since here these operators are not definable because we do not have, in general, a negation.

* + - Models are structures (*W, R, V* ) where *W* is a finite non empty set of states, with cardinality *n*; *R* is an Ag-family of (*n × n*)-matrices of M(**A**) and *V* : At *× W →*

*{*0*,* 1*}* is a valuation function. We use the notation *Ra*(*w, wj*) to denote the cell (*w, wj*) of the matrix *Ra*.

* + - Satisfaction:

*·* (*w |*= *⊥*)=0

* (*w |*= *p*)= *V* (*p, w*), for any *p ∈* At
* (*w |*= *ϕ ∧ ϕj*)= (*w |*= *ϕ*) *·* (*w |*= *ϕj*)
* (*w |*= *ρ ∨ ρj*)= (*w |*= *ρ*)+ (*w |*= *ρj*)
* (*w |*= *ϕ → ϕj*)= (*w |*= *ϕ*) *→* (*w |*= *ϕj*)

*·* (*w |*= *Ka ϕ*)= V*w′∈W* *Ra*(*w, wj*) *→* (*wj |*= *ϕ*)

* (*w |*= *Ba ϕ*)=  V*w′∈W*

*·* (*w |*= *EG ϕ*)=

*w′∈W*

*·* (*w |*= *MG ϕ*)= *w′∈W* *RG*(*w, wj*); (*wj |*= *ϕ*)

*Ra*(*w, wj*); (*wj |*= *ϕ*)

*RG*(*w, wj*) *→* (*wj |*= *ϕ*)

* (*w |*= *C ϕ*)= where *RG* = Σ*a∈G Ra*

*G* V*w′∈W*

*G*

# Examples

*R∗* (*w, wj*) *→* (*wj |*= *ϕ*)

We have already discussed an example of epistemic logic in the background section. Such example can be seen as an instantiation of our method over the *{*0*,* 1*}* standard action lattice (see [[12](#_bookmark22)]). We present two more examples, namely one that deals with discrete degrees of knowledge and, on the same context, another one that admits knowledge ranging over a continuous scale.

**Example 2** *Consider here the Graded Epistemic Logic generated by the Wajsberg hoop* **W**5 *over {a*0*, a*1*, a*2*, a*3*, a*4*, a*5*} (Deﬁnition* [*3.4*](#_bookmark7)*). Recall that the order in* **W**5 *is a*5 *< a*4 *< a*3 *< a*2 *< a*1 *< a*0*. In order to simplify the example, we denote ak by* 5 *− k, for k* = 0*,...,* 5*. This logic is useful to reasoning about the following variant of Example* [*2*](#_bookmark10)*.*

*Suppose now that the children are jealous and they have the following beliefs:*

1. **a***nna believes that the father has a strong preference for* **b***ob, which means that she believes that he will give the envelop with higher value to* **b***ob than to* **c***lara. In a scale from 0 to 5, her belief is 4; Conversely, her belief that the envelop bob received has a smaller value is 1.*
2. **c***lara also believes that the father has a preference for* **b***ob . In a scale from 0 to 5, her belief is 3; and conversely, her belief that the envelop bob received has a smaller value is 1. But if she has the envelop* **2** *then she believes that the father has no preference between* **a***nna and* **b***ob; in that case her belief is 4.*
3. **b***ob does not believe that the father has any preference between* **a***nna and* **c***lara. So his belief is 3 indifferently about any situation.*

*The following draws represent the beliefs of* **a***nna,* **b***ob and* **c***lara. We draw it separately for clarity sake. Moreover, we omit the reflexive loops in the picture with value 5.*

*We evaluate some formulas in this model. In order to simplify the calculations we use the fact that a*5 *→ an* = *a*0(*i.e.,* 0 *→ x* = 5) *and a*5; *an* = *a*5 *(i.e.,* 0; *x* = 0)*.*

012 *|*= *Bb*0*a* = W ,*Rb*(012*,* 012); 012 *|*= 0*a, Rb*(012*,* 210); 210 *|*= 0*a*, = W*{*5; 5*,* 3; 0*}* =5

102 ¸¸

4

1

012,¸ 021

zz

1

4

˛

120

z

102

012

3

021

3

120

3

201 4 zz

201 210

,¸ 210

1

102

4

012

02 1

3

1

4

12¸0

201

.

1

210

¸

,

Fig. 2. anna’s, **b**ob’s and **c**lara’s beliefs

012 *|*= *BaKc*2*a* = W ,*Ra*(012*,* 012); 012 *|*= *Kc*2*a, Ra*(012*,* 021); 021 *|*= *Kc*2*a*,

= W ,5; ,*Rc*(012*,* 012) *→* 012 *|*= 2*a, Rc*(012*,* 102) *→* 102 *|*= 2*a*,*,* 4; ,*Rc*(021*,* 021) *→* 021 *|*= 2*a, Rc*(021*,* 201) *→* 201 *|*= 2*a*,,

= W ,5; ,5 *→* 0*,* 4 *→* 0,*,* 4; ,5 *→* 0*,* 1 *→* 5,,

= W ,*a*0; ,*a*0 *→ a*5*, a*1 *→ a*5,*, a*1; ,*a*0 *→ a*5*, a*4 *→ a*0,,

= W ,*a*0; *a*5*, a*1; *a*5, = *a*5(= 0)

*To calculate Mac*2*b at* 021 *we ﬁrst calculate the matrix of Rac* = *Ra* + *Rc.*

012 021 102 120 201 210

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | 012 | 5 | 4 | 4 | 0 | 0 | 0 |
| 021 | 1 | 5 | 0 | 0 | 1 | 0 |
| *Rac* = | 102 | 4 | 0 | 5 | 4 | 0 | 0 |
|  | 120 | 0 | 0 | 1 | 5 | 0 | 1 |
|  | 201 | 0 | 0 | 0 | 4 | 5 | 4 |
|  | 210 | 0 | 0 | 0 | 4 | 1 | 5 |

*Then we have,*

021 *|*= *Mac*2*b* = W ,*Rac*(021*,* 012); 012 *|*= 2*b, Rac*(021*,* 021); 021 *|*= 2*b, Rac*(021*,* 201); 201 *|*= 2*b*, = W*{*1; 5*,* 5; 5*,* 1; 0*}* =5

*If we consider the group knowledge we have*

021 *|*= *Eac*2*b* = ,*Rac*(021*,* 012) *→* 012 *|*= 2*b, Rac*(021*,* 021) *→* 021 *|*= 2*b, Rac*(021*,* 201) *→* 201 *|*= 2*b*,

= *{*1 *→* 0*,* 5 *→* 5*,* 1 *→* 0*}* = *{*4*,* 5*,* 4*}* =4

**Example 3** *Consider now the Graded Epistemic Logic generated by the L- ukasie-*

*wicz arithmetic lattice* ***L-***

= ([0*,* 1]*,* max*, ⊙,* 0*,* 1*,∗ , →,* min) *(Deﬁnition* [*3.3*](#_bookmark8)*). This*

*logic is adequate to reasoning about knowledge expressed in the continuous scale*

[0*,* 1]*. Let us look at the following variant of Example* [*2*](#_bookmark10)*. Suppose now that the children have the following beliefs:*

1. **a***nna believes that the father has a strong preference for* **b***ob, which means that she believes that he will give the envelop with higher value to bob than to* **c***lara. Her belief is* 4 *; moreover her belief that the value is less is* 1

5 5

1. **c***ath also believes that the father has a preference for* **b***ob. Her belief is* 3 *. But if she has the envelop* **2** *then she believes that the father has no preference between* **a***nna and* **b***ob. In such case her belief is* 1*.*

5

1. **b***ob does not believe that the father has any preference between* **a***nna and* **c***lara. So, his beliefs are all* 1*.*

*The draws in Fig.* [*3*](#_bookmark11) *represent the beliefs of* **a***nna,* **b***ob and* **c***lara. We draw it separately for clarity sake.*

1 4

5 4 5

5



012

1

5

,

¸

021

4

5

zz

1

5

4

5

v˛

zz z z

012

1

021

1

102 ¸¸

201,¸ 210

1

1 5 4

5 4 5

120

102

120

zz 5 z z

201 210

1

1

1

vz

102

3

5

2 3

5 5

120,¸¸

3

5

2

012

5

210

,

¸

zz

,

z

2

¸ 5

021 ,¸

3

5

2

5 ,¸

201

5 vz

Fig. 3. **a**nna’s, **b**ob’s and clara’s beliefs

*We will evaluate the same formulas as in previous example:*

012 *|*= *Bb*0*a* = W ,*Rb*(012*,* 012) *⊙* 012 *|*= 0*a, Rb*(012*,* 210) *⊙* 210 *|*= 0*a*,

= W*{*1 *⊙* 1*,* 1 *⊙* 0*}* =1

012 *|*= *BaKc*2*a* = W ,*Ra*(012*,* 012) *⊙* 012 *|*= *Kc*2*a, Ra*(012*,* 021) *⊙* 021 *|*= *Kc*2*a*,

= W ,1 *⊙* ,*Rc*(012*,* 012) *→* 012 *|*= 2*a, Rc*(012*,* 102) *→* 102 *|*= 2*a*,*,*

4 *⊙* ,*Rc*(021*,* 021) *→* 021 *|*= 2*a, Rc*(021*,* 201) *→* 201 *|*= 2*a*,,

W5 , ,

= 1 *⊙*

5

1 *→* 0*,* 1 *→* 0,*,* 4 *⊙* ,

1 *→* 0*,* 2 *→* 1,,

= W ,1 *⊙* ,0*,* 0,*,* 4 *⊙* ,0*,* 1,,

5

W , 4 , 5 W ,

= 1 *⊙* 0*,* 5 *⊙* 0 =

0*,* 0*}* =0

*To calculate Mac*2*b at* 021 *we ﬁrst calculate the matrix of Rac* = *Ra* + *Rc.*

012

021

102

120

201

210

012 021 102 120 201 210

4

1

1

5

1

0

0

0

5

4

5

0

0

3

5

0

1

0

1

1

5

0

0

0

0

4

5

4

5

0

3

5

0

2

5

0

0

2

5

1

0

0

0

3

5

4

5

4

5

*Rac* =

*Then we have,*

021 *|*= *Mac*2*b* = W ,*Rac*(021*,* 012) *⊙* 012 *|*= 2*b, Rac*(021*,* 021) *⊙* 021 *|*= 2*b, Rac*(021*,* 201) *⊙* 201 *|*= 2*b*,

= W*{* 1 *⊙* 1*,* 4 *⊙* 1*,* 2 *⊙* 0*}* = W*{* 1 *,* 4 *,* 0*}* = 4

5 5 5 5 5 5

1. *All instantiations of propositional tautologies*,
2. *Ka*(*ϕ → ψ*) *→* (*Kaϕ → Kaψ*),
3. *Kaϕ → ϕ*,
4. *Kaϕ → KaKaϕ* (+ *introspection*)
5. *¬Kaϕ → Ka¬Kaϕ* (*− introspection*)
6. *CGϕ ↔ EGCGϕ*
7. *CG*(*ϕ → EGϕ*) *→* (*ϕ → CGϕ*)

Fig. 4. Axioms of Epistemic Logic [[6,](#_bookmark20)[18](#_bookmark32)]

*If we consider the group knowledge we have*

021 *|*= *Eac*2*b* = ,*Rac*(021*,* 012) *→* 012 *|*= 2*b, Rac*(021*,* 021) *→* 021 *|*= 2*b, Rac*(021*,* 201) *→* 201 *|*= 2*b*,

= *{* 1 *→* 1*,* 4 *→* 1*,* 2 *→* 0*}* = *{*1*,* 1*,* 3 *}* = 3

5 5 5 5 5

# How epistemic *GE*(A) logics are?

The abstract study of the properties of the logic generated by the method proposed in previous sections, in terms of properties of the underlying action lattice, is very challenging. Obviously, there are aspects that have to be studied instantiation- by-instantiation. In this section, however, we discuss this issue in a systematic perspective, trying to respond the question *How epistemic GE* (**A**) *logics are?* by studying the validity of the standard axioms of Epistemic Logic (cf. Fig. 4) in the generated logics.

We obtain some results for specific classes of generated logics, with respect to specific classes of action lattices and imposing constrains on the achieved models. The latter also happens in the standard Epistemic Logic, which the completeness is established for a restricted class of models, for instance, the epistemic ones (i.e., models whose accessible relations are equivalence relations) [[18](#_bookmark32)].

We follow the strategy adopted in [[12,](#_bookmark22)[13](#_bookmark27)] (in the context of generated graded dynamic logics). The integrability (*a ≤* 1) on action lattices provides a nice proof strategy to work at this generic level: as it is well known, in any integral action lattice, we have

(*a → b*)=1 *⇔ a ≤ b* (21)

**Theorem 5.1** *Let* **A** *be an integral* ;*-idempotent,* ;*-commutative action lattice. The property*

1. *Ka*(*ϕ → ψ*) *→* (*Kaϕ → Kaψ*)

*is valid in the logic GE* (**A**)*.*

**Proof.** The proof can be extracted from Lemma 9 of [[12](#_bookmark22)]. *2*

In a similar way, but by imposing commutativity on the operation ; we can extract the proof for the axiom (vii):

**Theorem 5.2** *Let* **A** *be an integral action lattice such that* ;= *·. Then the property*

(vii) *CG*(*ϕ → EGϕ*) *→* (*ϕ → CGϕ*)

*is valid in the logic GE* (**A**)*.*

**Proof.** This can be directly adapted from Lemma 10 of [[12](#_bookmark22)]. *2*

So, we have to study the remaining axioms, specifically the ones that distinguish Epistemic Logic from other modal logics - the axioms (iii), (iv), (v) and (vi). In this view, we have to impose further properties on the structure of the models. In particular, we have to generalize the reflexivity and transitivity conditions for our graded setting to guarantee the validity of (iii) and (iv). What the conditions needed for the cases (iii) and (iv) are still in study.

**Definition 5.3** Let **A** be an action lattice and *M* be a model in *GE* (**A**). We say that *M* is *graded-reflexive* if for any *a ∈* Ag, *w ∈ W* ,

*Ra*(*w, w*) = 1 (22)

and that it is *graded-transitive*, whenever any *a ∈* Ag

for any *w, wj, wjj ∈ W, Ra*(*w, wjj*) *≥ Ra*(*w, wj*); *Ra*(*wj, wjj*) (23)

**Theorem 5.4** *Let* **A** *be an integral action lattice. Then, the axiom*

1. *Kaϕ → ϕ,*

*is valid in graded-reflexive models.*

**Proof.** Since **A** is integral, we have by ([21](#_bookmark13)) that it is sufficient to prove that, for any model *M* , and for any state *w ∈ W* , (*w |*= *Kaϕ*) *≤* (*w |*= *ϕ*). In this view, we observe that:

(*w |*= *Kaϕ*)

= *{|*= defn*}*

*w* *′∈W*

*Ra*(*w, wj*) *→* (*wj |*= *ϕ*)

*≤ {*infimum properties*}*

*Ra*(*w, w*) *→* (*w |*= *ϕ*)

= *{*([22](#_bookmark14))*}*

1 *→* (*w |*= *ϕ*)

= *{*in any action lattice 1 *→ a* = *a* (cf. [[12](#_bookmark22)])*}*

(*w |*= *ϕ*) *2*

**Theorem 5.5** *Let* **A** *be an integral* ;*-commutative action lattice. Then, the axiom*

1. *Kaϕ → KaKaϕ* (+ *introspection*)*, is valid in graded-transitive models.*

**Proof.** Since **A** is integral, we have by ([21](#_bookmark13)) that it is sufficient to prove that, for any model *M* , and for any state *w ∈ W* , (*w |*= *Kaϕ*) *≤* (*w |*= *KaKaϕ*). In this view, we observe that:

for any *wj, wjj ∈ W, Ra*(*w, wjj*) *≥ Ra*(*w, wj*); *Ra*(*wj, wjj*)

*⇔ {* ;-commutative*}*

for any *wj, wjj ∈ W, Ra*(*w, wjj*) *≥ Ra*(*wj, wjj*); *Ra*(*w, wj*)

*⇔ {a ≤ b ⇒ b → c ≤ a → c* (cf. [[12](#_bookmark22)])*}*

for any *wj, wjj ∈ W, Ra*(*w, wjj*) *→* (*wjj |*= *ϕ*) *≤*

(*Ra*(*wj, wjj*); *Ra*(*w, wj*)) *→* (*wjj |*= *ϕ*)

*⇔ {*infimum properties*}*

for any *wjj ∈ W, Ra*(*w, wjj*) *→* (*wjj |*= *ϕ*) *≤*

*w* *′∈W*

(*Ra*(*wj, wjj*); *Ra*(*w, wj*)) *→* (*wjj |*= *ϕ*)

*⇔ {*in any action lattice *a →* (*b → c*)= (*b*; *a*) *→ c* (cf. [[12](#_bookmark22)])*}*

for any *wjj, Ra*(*w, wjj*) *→* (*wjj |*= *ϕ*) *≤*

*w* *′∈W*

*Ra*(*w, wj*) *→* (*Ra*(*wj, wjj*) *→* (*wjj |*= *ϕ*))

*⇔ {*inf. monotocity*}*

*w′* *′∈W*

*Ra*(*w, wjj*) *→* (*wjj |*= *ϕ*) *≤*

*Ra*(*w, wj*) *→* (*Ra*(*wj, wjj*) *→* (*wjj |*= *ϕ*))

*w′,w′′∈W*

*⇔ {*in any complete action lattice, *x →* ( *i∈I yi*)= *i∈I* (*x → yi*) (cf. [[12](#_bookmark22)])*}*

*w′* *′∈W*

*Ra*(*w, wjj*) *→* (*wjj |*= *ϕ*) *≤*

*w* *′∈W*

*Ra*(*w, wj*) *→*

*w′′∈W*

(*Ra*(*wj, wjj*) *→* (*wjj |*= *ϕ*))

*⇔ {|*= defn twice*}*

(*w |*= *Kaϕ*) *≤* (*w |*= *KaKaϕ*)

*2*

# Conclusions and future work

This paper starts with a research program on the parametric generation of graded epistemic logics. The approach is based on the application of the method intro- duced in Section [3](#_bookmark4), and should be explored as an effective source of logics to reason on agent knowledge scenarios with distinct degrees of Knowledge/Belief.

The generality of the method was illustrated with three graded epistemic logics (note that the standard Multi-Agent Epistemic Logic corresponds to the instan- tiation of the action lattice **2**), but a lot of other examples can be considered - from a *{false, unknown, true}*-three valued Epistemic Logic, achieved by instanti- ating the action lattice **3** to a more ‘esoteric’ Graded Epistemic Logic to deal with knowledge/belief scenarios involving resource aware constraints (built on the Floyd Warshall algebra - see [[12](#_bookmark22)]). Beyond of their philosophical interest, the study of each one of these instantiations as a logic with ‘its own rights’ is very challenging. Indeed, as discussed in Section [5](#_bookmark12), it is possible to characterize specific classes of graded epistemic logics (parametric on specific subclasses of action lattices and by imposing further condition on the models) that preserves the essence of the bivalent epistemic logic.

There is, however, a lot of work to do in this line of research. To establish suffi- cient conditions for validating the negative introspection axiom (and of (vi)) is still work in progress for us. It seems that, beyond of a generalization of the Euclidean property on models, some new conditions should be imposed in the action lattices, particularly with respect to their negation (note that, in its generic form, there is no negation involution in general). The parametric generation of calculus and the study of complexity of generated epistemic logic w.r.t. to specific classes of action lattices are also in our agenda. Another interesting line of research is to investigate the concepts of simulation and bisimulation for our knowledge representations on the lines proposed in [[19,](#_bookmark33)[15](#_bookmark29)] for generic fuzzy labelled transition systems.

Finally, it would be interesting to investigate whether our approach allows for the representation of epistemic actions. Public announcements or private commu- nications. More interesting is to look for epistemic actions that make sense only in this (or similar) setting. For example, one can think of situations in which the agent has a belief of some grade *n*, and then some new information ’downgrades’ or ’upgrades’ this belief (some form of belief revision, but now in a ’graded’ fashion).

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