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A General Method for Forbidden Induced Subgraph Sandwich Problem

NP-completeness [1](#_bookmark0)

## Simone Dantas[2](#_bookmark0)

*IME, Universidade Federal Fluminense, Brazil.*

## Celina M. H. de Figueiredo[2](#_bookmark0)

*COPPE, Universidade Federal do Rio de Janeiro, Brazil.*

## Priscila Petito[2](#_bookmark0)

*FFP, Universidade do Estado do Rio de Janeiro, Brazil.*

## Rafael B. Teixeira[2](#_bookmark0)

*ICE, Universidade Federal Rural do Rio de Janeiro, Brazil.*

**Abstract**

We consider the sandwich problem, a generalization of the recognition problem introduced by Golumbic and Shamir (1993), with respect to classes of graphs defined by excluding induced subgraphs. The Π graph sandwich problem asks, for a pair of graphs *G*1 = (*V, E*1) and *G*2 = (*V, E*2) with *E*1 *⊆ E*2, whether there exists a graph *G* = (*V, E*) with *E*1 *⊆ E ⊆ E*2 that satisfies property Π. We consider the property of being *H*-free, where *H* is a fixed graph. Using a new variant of the SAT problem, we present a general

framework to establish the *NP* -completeness of the sandwich problem for several *H*-free graph classes which

generalizes the previous strategy for the class of Hereditary clique-Helly graphs. We also provide infinite families of 3-connected special forbidden induced subgraphs for which each forbidden induced subgraph sandwich problem is NP-complete.

*Keywords:* algorithms and computational complexity, graph sandwich problems, satisfiability, Linear CNF-formula, 3-sat, forbidden induced subgraph

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2 Email: [sdantas@id.uff.br](mailto:sdantas@id.uff.br), [celina@cos.ufrj.br](mailto:celina@cos.ufrj.br), [ppetito@uerj.br](mailto:ppetito@uerj.br), [rafaelbt@ufrrj.br](mailto:rafaelbt@ufrrj.br)

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# Introduction

All graphs considered here are finite and undirected. Given a graph property Π, the Π graph sandwich problem is defined as follows:

*Input:* A pair (*G*1*, G*2) of graphs with *G*1 = (*V, E*1), *G*2 = (*V, E*2) and *E*1 *⊆ E*2; *Question:* Is there a graph *G* = (*V, E*) with *E*1 *⊆ E ⊆ E*2 that satisfies property Π?

The graph sandwich problem was introduced by Golumbic and Shamir [[6](#_bookmark12)]. Remark that when *G*1 = *G*2 the problem is to decide whether *G*1 satisfies property Π. So the graph sandwich problem generalizes the problem of deciding whether a graph satis- fies a given property. In particular, if the decision problem is NP-complete, then the sandwich problem is also NP-complete. When the property Π is to belong to a class *C* of graphs, we call this problem the *C graph sandwich problem.* Golumbic, Kaplan and Shamir [[7,](#_bookmark13)[8](#_bookmark14)] proved that the interval graph, unit interval graph, permutation graph and comparability graph sandwich problems are all NP-complete; whereas the split graph, threshold graph and cograph sandwich problems are in P.

For an instance (*G*1*, G*2) of the graph sandwich problem with *G*1 = (*V, E*1), *G*2 = (*V, E*2) and *E*1 *⊆ E*2, we say that any element of *E*1 is a *forced edge*, any element of *E*2 *\ E*1 is an *optional edge*, and any other pair of *V × V* is a *forbidden edge*. Every graph *G* = (*V, E*) with *E*1 *⊆ E ⊆ E*2 is called a *sandwich graph* for the pair (*G*1*, G*2). In this case, *E* consists of all forced edges plus some (possibly zero) optional edges and no forbidden edge.

We say that a graph *G contains* a graph *H* if some induced subgraph of *G* is iso- morphic to *H*. A graph *G* is *H-free* if it does not contain *H*. Dantas, de Figueiredo, da Silva and Teixeira [[2](#_bookmark7)], and Dantas, de Figueiredo, Maffray and Teixeira [[3](#_bookmark9)] have studied the *H*-free graph sandwich problems, determining the complexity status of the problem for several graphs *H* (paw and (*Kp \ e*), for every fixed *p ≥* 4, are in P; whereas *Cp*, for every fixed *p ≥* 4, claw and bull are NP-complete). More recently, Couto, Faria, Gravier and Klein [[1](#_bookmark8)], and de Figueiredo and Spirkl [[5](#_bookmark11)] further inves- tigated *H*-free graph sandwich problems and compared the complexities to probe problems, a variation of sandwich problems where the optional edges occur between vertices of a special subset of *V* .

Dourado, Petito, Teixeira and de Figueiredo [[4](#_bookmark10)] proved the NP-completeness of the Hereditary clique-Helly graph sandwich problem, where the class of Hereditary clique-Helly graphs is defined by a set of four forbidden induced subgraphs, the so-called ocular graphs. Here, we develop this study by providing a general method to prove NP-completeness, which generalizes their strategy, for infinite families of forbidden subgraphs.

In order to complete this task, we have introduced a suitable variant of the NP- complete Linear conjunctive normal form 3-SAT problem (lcnf 3-sat) [[9](#_bookmark15)], which we call *k*-girth lcnf 2-3-sat.

# *k*-GIRTH LCNF 2-3-SAT

We begin this section by proposing a suitable variant of the NP-complete lcnf 3- sat [[9](#_bookmark15)], which we call *k*-girth linear conjunctive normal form 2-3-SAT problem (*k*- girth lcnf 2-3-sat). According to [[9](#_bookmark15)], in the linear conjunctive normal form 3-SAT problem (lcnf 3-sat) the clauses have size 3 and each pair of distinct clauses have 0 or 1 variable in common.

Let *I* = (*X, C*) be an instance of 3-sat, where *X* denotes the set of variables, and *C* denotes the set of clauses. The bipartite graph of clauses and variables *B*(*I*) = (*V I, EI* ) is a graph constructed from a general instance *I* = (*X, C*) of 3- sat as follows. For each clause *c* of *C*, there exists a vertex *c* that belongs to *V I* . For each variable *x* of *X*, there exists a vertex *x* that belongs to *V I* . If a clause *c* contains the literal *x* or *x*, then the edge *cx* belongs to *EI* . We remark that the *linear property* of two distinct clauses having 0 or 1 variable in common implies that the girth of *B*(*I*) is greater than 4. The proposed variant is stated as follows:

*k*-girth linear cnf 2-3-sat (*k*-girth lcnf 2-3-sat)

Instance: set *X* = *{v*1*,..., vn}* of variables, collection *C* = *{c*1*,..., cm}* of clauses over *X* such that each clause *c ∈ C* has size 2 *≤ |c|≤* 3 and, for all *c, cj ∈ C*, *c /*= *cj*,

*|c ∩ cj| ≤* 1, and the bipartite graph of clauses and variables has girth greater than

*k*.

Question: Is there a truth assignment for *X* such that each clause in *C* has at least one true literal?

**Theorem 2.1** *k*-girth lcnf 2-3-sat *problem is NP-Complete.*

**Proof:** Given an instance (*X, C*) of lcnf 3-sat, we construct an instance (*Xj, Cj*) of *k*-girth lcnf 2-3-sat as follows.

Let *X* = *{v*1*,..., vn}* be the set of variables, *C* = *{c*1*,... cm}* be a collection of clauses over *X* such that each clause *c ∈ C* has size *|c|* = 3 and, for all *c, cj ∈ C*, *c /*= *cj*, *|c ∩ cj| ≤* 1. This last constraint ensures that the girth *g* of the bipartite graph of clauses and variables *B*(*I*) = (*V I, EI* ) is greater than 4. In order to increase the value of *g* we proceed as follows. Set *Xj* := *X* and *Cj* := *C*. For each clause *cj* = *{l*1*, l*2*, l*3*} ∈ C*, 1 *≤ j ≤ m*, we introduce two new auxiliary variables, *Xj* := *Xj ∪ {xc , yc }*, and we replace *cj* by three new clauses *cj* , *cjj* and *cjjj*, that is,

*j j j j j*

*Cj* := (*Cj \ cj*) *∪ {{l*1*, xc }*, *{l*2*, xc , yc }*, *{l*3*, yc }}*.

*j j j j*

It is easy to see that this transformation is done in polynomial time. The linear

property implies that the set of clauses *C* has a size linear in *n*, which implies that the resulting set of clauses *Cj* has a size linear in *n*, since the preceding set of clauses is linear in *n*, and each clause *cj ∈ C* is replaced by three clauses with pairwise intersection of at most one variable with all clauses of the new *Cj* (because these new clauses do not duplicate variables of *cj*, and distinct additional auxiliary variables are added for each clause *cj ∈ C*, 1 *≤ j ≤ m*).

We claim that instance (*X, C*) is satisfiable if, and only if, (*Xj, Cj*) is satisfi- able because clause *{l*1*, l*2*, l*3*}* is logically equivalent to the set of clauses *{l*1*, xcj }*,

*{l*2*, xcj , ycj }*, *{l*3*, ycj }*.

*Hc*



*Hx*



Fig. 1. Clause subgraph *Hc*, variable subgraph *Hx*, and their connection.

This procedure is reflected in the updated bipartite graph *B*(*I*) as follows. Let *C* be a cycle in *B*(*I*) that contains the clause vertex *cj*. After applying the procedure above, the clause vertex *cj* is replaced by three new clause vertices, say *cj* , *cjj* and

*j j*

*cjjj*; and we also add to *V I* the two new auxiliary variable vertices *x**c*

*j*

*j*

and *ycj* . Let

*l*1 = *x* or *l*1 = *x* be a literal of the clause *cj* = *{l*1*, l*2*, l*3*}*. Hence, the edge *xcj* is

replaced by the path *x, cj , xc , cjj*. So, the size of any cycle in *B*(*I*) that contains *cj*

*j j j*

is increased by at least 2 vertices. Now, it is clear that by repeating the procedure

at most *k/*2 times, we are guaranteed to build in polynomial time an instance *I*

such that the girth *g* of *B*(*I*) is greater than *k*, since *k* is a fixed number. *2*

# *H*-free graph sandwich problem

We are interested in a special structure of the forbidden graph. The forbidden graph *H* is required to have a matching of size 2, say *A* = *{a*1*aj , a*2*aj }*, and to

1 2

have an anti-matching (i.e. a matching in the complement graph *H*) of size 3, say

*B* = *{b*1*bj , b*2*bj , b*3*bj }*.

1 2 3

Given an instance (*X, C*) of *k*-girth lcnf 2-3-sat, we construct an instance

(*G*1*, G*2) of *H*-free graph sandwich problem as follows (see Figure [1](#_bookmark1)). In what follows, each induced variable subgraph *Hx* and each induced clause subgraph *Hc* is a copy of *H*.

For each variable *x* of *X*, there exists an induced variable subgraph *Hx* in *G*2, such that the edges *axajx* and *axajx* are the unique optional edges of *Hx* in the set

*E*2 *\ E*1.

1 1 2 2

For each three-sized clause *c* = *{l*1*, l*2*, l*3*}* of *C*, there exists an induced clause subgraph *Hc* in *G*1, such that for each literal *li*, *i ∈ {*1*,* 2*,* 3*}*, we include the addi-

*c ′c*

tional optional edge *bibi* in the set *E*2 *\ E*1.

For each two-sized clause *c* = *{l*1*, l*2*}* of *C*, there exists an induced clause sub- graph *Hc* in *G*1, such that for each literal *li*, *i ∈ {*1*,* 2*}*, we include the additional

optional edge *bcb′c* in the set *E \ E* . Note that, in this case, the edge *bcb′c* is

*i i* 2 1 3 3

*c ′c*

forbidden, i.e., *b*3*b*3 */∈ E*2.

Whenever a variable *x* occurs as positive (resp. negative) literal *li* in clause

*c*, then the edge *axajx* (resp. *axajx*) is equivalent to the edge *bcbjc*, by identifying

1 1 2 2 *i i*

*ax* = *bc* and *ajx* = *bjc* (resp. *ax* = *bc* and *ajx* = *bjc*), *i ∈ {*1*,* 2*,* 3*}* (resp. *i ∈ {*1*,* 2*}* in

1 *i* 1 *i* 2 *i* 2 *i*

case of a two-sized clause).

This concludes the construction of the particular instance (*G*1*, G*2) of *H*-free graph sandwich problem.

We claim that this construction gives sufficient conditions to analyze the NP- completeness of *H*-free graph sandwich problems by studying some properties of graph *H* and of the structure of problem *k*-girth lcnf 2-3-sat.

**Theorem 3.1** *Let H be a graph, containing a matching of size 2 and an anti- matching of size 3. If the particular instance* (*G*1*, G*2) *constructed above admits an H-free sandwich graph G, then there exists a truth assignment that satisﬁes instance* (*X, C*) *for the k*-girth lcnf 2-3-sat*.*

**Proof:** Suppose *G* is an *H*-free sandwich graph. So, every *Hc* clause subgraph of *G*1 is destroyed by using at least one optional edge of set *B*. However, no *Hx* variable subgraph is created by adding both edges of *A*.

We now define the truth assignment for (*X, C*): if an edge of *B* belongs to *G\G*1 then set the truth value of the corresponding literal to true. Suppose that two edges are corresponding to the positive and negative literals of the same variable *x*. This generates an *H* induced subgraph in *G*, corresponding to the *Hx* variable subgraph for *x*, a contradiction. *2*

The converse theorem is not so straightforward and it requires a deeper study of the structure of the graph *H* and of the bipartite graph of clauses and variables *B*(*I*).

Every rule we use in order to construct a sandwich graph from a truth assign- ment of *k*-girth lcnf 2-3-sat needs to ensure that no side-effect *H* subgraph is generated.

It is quite easy to give a rule that destroys all *Hc* clause subgraphs of *G*1 without creating an *Hx* variable subgraph of *G*2.

Our attempt consists in giving a simple rule (every optional edge corresponding to a true literal is added to *G*1 to form a sandwich graph *G*), and then search for a side-effect *H* sandwich subgraph. We remark that every vertex of the constructed instance (*G*1*, G*2) belongs to an *Hc* clause subgraph or to an *Hx* variable subgraph, and possibly to both an *Hc* clause subgraph and an *Hx* variable subgraph. Although a vertex may belong to at most one *Hx* variable subgraph, possibly a vertex may belong to several *Hc* clause subgraphs. A *side-effect H* is an induced subgraph isomorphic to *H* of some sandwich graph *G*, such that *H* is neither associated to an *Hc* clause subgraph nor to an *Hx* variable subgraph.

In the special structure of the forbidden graph *H*, in order to have the converse theorem, we require further the forbidden graph *H* to be 3-connected.

**Theorem 3.2** *Let H be a 3-connected graph, containing a matching of size 2 and an anti-matching of size 3. If there exists a truth assignment that satisﬁes instance*

(*X, C*) *of the k*-girth lcnf 2-3-sat*, then the particular instance* (*G*1*, G*2) *con-*

*structed above admits an H-free sandwich graph G.*

**Proof:** Let *H* be a 3-connected graph, containing a matching of size 2 and an anti-matching of size 3, and let *l* be the size of the largest chordless cycle of *H*. Let



*Hx*

*H*

*ax*

*i*

*ai*

*′x*

Fig. 2. The removal of vertices *ax* and *a′x* disconnects *H*, a contradiction.

*i* *i*

parameter *k* = 2*l*.

Suppose there exists a truth assignment that satisfies instance *I* = (*X, C*) of

the *k*-girth lcnf 2-3-sat, and consider the simple rule that adds to *G*1 every

optional edge corresponding to a true literal in order to define a sandwich graph *G*. Since the simple rule destroys all *Hc* clause subgraphs of *G*1 without creating an *Hx* variable subgraph of *G*2, it remains to prove that the sandwich graph *G* contains no side-effect induced subgraph *H*, associated to neither an *Hc* clause subgraph nor to an *Hx* variable subgraph. Assume to get a contradiction that *G* contains such a side-effect induced subgraph *H*. By construction, there exists a variable *x* such that the side-effect subgraph *H* must contain at least one among the two vertices

*ax* and *ajx*, endvertices of the optional edge *axajx*. The two vertices *ax* and *ajx* are

*i i i i i* *i*

the only connection between an *Hc* clause subgraph and an *Hx* variable subgraph,

or between two clause subgraphs *Hcj* and *Hcj′* , by the linear property. Note that the optional edge *axajx* belongs to a unique variable subgraph *Hx*. Note further

*i i*

that the optional edge *axajx*

must be associated to at least one clause subgraph

*i i*

*Hc*. Suppose first that the removal of the two vertices *ax* and *ajx* disconnects the

*i* *i*

side-effect induced subgraph *H*, this gives a contradiction since *H* is a 3-connected

graph (see Figure [2](#_bookmark4)).

Hence, the side-effect subgraph *H* must contain a chordless cycle *S* that contains at least one of *ax* or *ajx* (see Figure [3](#_bookmark5)). Note that the size of *S* is less than or equal

*i* *i*

to the size of the largest chordless cycle of *H*, that is, *|S|≤ l*. Since the removal of

the two vertices *ax* and *ajx* does not disconnect the side-effect induced subgraph *H*,

*i* *i*

cycle *S* has vertices of at least two different variable subgraphs. We claim that the

induced subgraph in *B*(*I*) constructed from *S* by taking the corresponding vertices of variable and clause subgraphs of *S*, has a cycle *RI* . Otherwise, this subgraph is a tree in *B*(*I*) and there exists a variable vertex *xi* with two adjacent clause vertices *cj* and *cj′* , such that vertices of the corresponding *Hcj* and *Hcj′* in *S* would be disconnected by the removal of the two vertices *ax* and *ajx*, again a contradiction.

*i* *i*

We observe that the cycle *RI* has size at most 2*l* because, in the worst case, every

edge of *S* is incident to two different variable subgraphs.

In this case, *RI* has *l* vertices corresponding to *Hx* variable subgraphs and *l* vertices corresponding to *Hc* clause subgraphs. This contradicts the fact that, by definition, the girth of *B*(*I*) is greater than 2*l*, which concludes the proof. *2*



Multishared vertex

Cycle *S* of *H*

Fig. 3. An example of a cycle *S* of *H*. Note the possible shared vertex by two clause subgraphs.

We present next two applications by considering two particular graphs *H*. De- note by *Kp* the complete graph on *p* vertices, denote by 3*K*2 the graph consisting of an induced matching with three edges, and call *p*-*wheel* a graph consisting of a chordless cycle on *p* vertices and an additional vertex *u* adjacent to all *p* vertices on the cycle. Denote by *E*[3*K*2] the edge-set of the 3*K*2 graph.

**Corollary 3.3** *If H is Kp \ E*[3*K*2]*, for p ≥* 6*, then the H-free graph sandwich problem is NP-complete.*

**Proof:** Any two non incident edges of *Kp \ E*[3*K*2] is a matching of size 2. The missing 3*K*2 is an anti-matching of size 3. Finally, *Kp \ E*[3*K*2] is 3-connected. Thus, Theorems [3.1](#_bookmark2) and [3.2](#_bookmark3) can be applied. *2*

**Corollary 3.4** *If H is p-wheel, for p ≥* 6*, then H-free graph sandwich problem is* *NP-complete.*

**Proof:** Any two non incident edges of the *p*-cycle of a *p*-wheel is a matching of size 2. Any induced *p*-cycle, *p ≥* 6, contains an anti-matching of size 3. Finally, a *p*-wheel, *p ≥* 6, is 3-connected. Thus, Theorems [3.1](#_bookmark2) and [3.2](#_bookmark3) can be applied. *2*

# 4 Concluding remarks

In the present work, we provide a new variant of SAT called *k*-girth lcnf 2-3- sat, which yields the classification of the graph sandwich problem for *H*-free graph classes such that the forbidden graph *H* is 3-connected, *H* contains an anti-matching (i.e. *H* contains a matching) of size 3 and *H* contains a matching of size 2.

In particular, we prove that, when *H* is *Kp \ E*[3*K*2] ora *p*-wheel, for *p ≥* 6, the

*H*-free graph sandwich problem is NP-complete.

Furthermore, our results establish an interesting dichotomy: for every fixed *p ≥* 6, the (*Kp \ e*)-free graph sandwich problem is in P [[2](#_bookmark7)], whereas both the (*Kp \ E*[3*K*2])-free graph sandwich problem (from Corollary [3.3](#_bookmark6)) and the (*Kp \ E*[2*K*2])-free graph sandwich problem are NP-complete. The NP-completeness of the (*Kp \ E*[2*K*2])-free graph sandwich problem is implied by the NP-completeness of the *C*4-free graph sandwich problem [[2](#_bookmark7)].

The strategy used to prove these results provides a tool to classify as NP- complete the graph sandwich problem for several families of graphs defined by forbidden induced subgraphs. For instance, we remark that *K*6 *\ E*[3*K*2] is iso- morphic to the power of cycle *C*2, and that the *H*-free graph sandwich problem is NP-complete when *H* is the power of cycle *Cp*, for *n ≥* 6 and *p < [n/*2*♩*.

6

*n*

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