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A Hybridization of Irreflexive Modal Logics

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Abstract

This paper discusses a bimodal hybrid language with a sub-modality (called the irreflexive modality) as- sociated with the intersection of the accessibility relation R and the inequality /=. First, we provide the Hilbert-style axiomatizations (with and without the COV-rule) for logics of our language, and prove the Kripke completeness and the finite frame property for them. Second, with respect to the frame expressive power, we compare our language containing the irreflexive modality with the hybrid languages √ and √(E). Finally, we establish the Goldblatt-Thomason-style characterization for our language.

*Keywords:* Modal and Hybrid Logics, Kripke Completeness, Finite Frame Property, (Modality for) Irreflexivity, Gabbay-style rule, Goldblatt-Thomason-style Characterization.

# Introduction and Motivation

As is well known, the standard modal propositional language ł with ♦ cannot define all the natural assumptions about the accessibility relation *R* on the set *W* of states, e.g., irreflexivity and antisymmetry. It has known that we can prove the completeness with respect to the frames with undefinable properties by adding

Gabbay-style non-orthodox rules without changing the language [[10](#_bookmark33),[11](#_bookmark34)]. In order to overcome the lack of expressive power of ł, on the other hand, extensions with various tools have been proposed, e.g., the difference operator D [[8](#_bookmark31)], nominals *i* [[4](#_bookmark27)], the global modality E [[12](#_bookmark35)], and the satisfaction operator @i [[6](#_bookmark29)].

For the same reason, the author of the present paper [[16](#_bookmark39),[18](#_bookmark41),[19](#_bookmark42)] proposed a new modest extension ł(◆) of ł, which consisted in adding an operator ◆ associated with the intersection of the accessibility relation *R* and the inequality /= (written (*R*∩ /=)). E.g., in tense logic◆,*ϕ* means that *ϕ* holds in some future instant *different from now*. He [[16](#_bookmark39),[18](#_bookmark41)] has already proved that various normal modal logics (some

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of which are kinds of Lemmon-Scott’s Axioms ♦m n*p* → j♦k*p* (*m, n, j, k* ∈ *ω*)) in his extended language enjoy Kripke completeness (for some logics containing K, even in the predicate extension [[17](#_bookmark40)]) and the finite model property. He [[19](#_bookmark42)] has also shown that ♦*p* → ◆*p* defines irreflexivity and ◆◆*p* → ◆*p* defines the conjunction of antisymmetry and transitivity, though antisymmetry is not independently definable. The definability of these properties of frames witnesses the strength of his extension. In addition, the author and Sato recently have provided the Van Benthem-style and

Goldblatt-Thomason-style Characterizations for his extension [[20](#_bookmark43)].

We can point out that ◆ has a close connection with topological semantics for modal logics. ♦ has been topologically interpreted in at least two ways: as the closure operator or as the derivative operator. ‘♦ as derivative’ has been studied by e.g., [[9](#_bookmark32)]. By the recent work [[2](#_bookmark24)] (especially, Lemmas 1.62 and 1.64), we can understand that ‘♦ as derivative’ has the same meaning as our ◆ in the relational semantics. Thus, we can claim that the author’s studies about ◆ contain, as a special case, a relational-semantical study of ‘♦ as derivative’. In fact, when ♦

satisfies the axioms for S4, ◆, associated with (*R*∩ /=), satisfies those fowr K4 [[2](#_bookmark24), sec.3.1.1], which is the basic axiom system for ‘♦ as derivative’ (Note that, if ♦ satisfies the reflexivity axiom T: *p* → ♦*p* at least, ♦*p* is equivalent to ◆*p* V *p*).

Topological semantics for hybrid logic have been studied by, e.g., [[21](#_bookmark44),[23](#_bookmark45)]. Most of

these works deal with ♦ as the closure operator. There has been no study that tried to interpret ♦ as the derivative operator in the framework of hybrid logics. This paper is a preliminary step toward this. In this paper, we hybridize our ł(◆) and intend to give a relational basis for topological hybrid logics with ‘♦ as derivative’.

We add a new sort of variables, *nominals*, which will be formulas used to denote points in the domains in the Kripke frames (7(◆) denotes this hybridization of ł(◆)). Nominals also allow us to define irreflexivity. Therefore, the expressive powers of nominals and the irreflexive modality ◆ overlap. We reveal the difference between the irreflexive modality and nominals with respect to the expressive powers and meta-logical properties.

Hybridization of our extension is also inspired by [[13](#_bookmark36)] (cf. [[1](#_bookmark25), sec.2.2]). They showed that intersection of accessibility relations can be defined using nominals. For ◆, this means that the axiom scheme ◆*i* — ♦*i* Λ D*i* holds, where *i* is a nominal and D is the difference operator associated with the inequality /=. In addition, they also showed that complement of the accessibility relation can be defined by nominals. Since the axiom scheme *i* — ч D*i* holds, we obtain the following scheme:

* *i* — ♦*i* Λ ч *i*.

We use such a scheme to provide our Hilbert-style axiomatization by using the traditional notion of possibility forms [[14](#_bookmark37)] (and the COV-rule [[13](#_bookmark36)]). We also show that hybridization of ł(◆) preserves Kripke completeness and the finite frame property (Section 3). Second, with respect to the frame expressive power, we com-

pare our languages containing the irreflexive modality with the hybrid languages 7 and 7(E). Finally, we give the Goldblatt-Thomason-style characterization for 7(◆) (Section 4).

# Basic Notions

* 1. *Syntax and Semantics*

The *irreflexive hybrid language* 7(◆) is obtained by the basic modal language ł containing 0 with an infinite set of nominals Nom = { *i, j, k,...* } and the irreflexive modality ◆. The set of irreflexive hybrid formulas is defined as

*ϕ* ::= T| *p* | *i* | ч *ϕ* | *ϕ* Λ *ψ* | 0*ϕ* | ◆*ϕ,*

where *p* ∈ Prop, the set of all proposition letters, and *i* is a nominal. It is usually assumed that the set of nominals Nom as well as Prop is countable.

We define the following languages and their formulas similarly: The *hybrid lan- guage* 7 without ◆, the *irreflexive modal language* ł(◆) without Nom, and their extension with the *global modality* E (written, e.g., 7(◆*,* E)). In addition to the usual abbreviations for material implication →, disjunction V, logical equivalence

—, the falsum ⊥, we use the following: *ϕ* := ч 0 ч *ϕ*, □*ϕ* := ч ◆ ч *ϕ*. We use

*ϕ*, *ψ*, *θ,...* to denote formulas and Γ, Δ, *...* to sets of formulas. A formula *ϕ* is

called *pure* if it contains no proposition letters (nominals are allowed).

A *bimodal frame* is a triple *F* = ⟨ *W, R, S* ⟩ of a non-empty set *W* , called a

*domain*, and two binary relations *R*, *S* on *W* . A *bimodal model* is a pair *M* =

⟨ *F,V* ⟩ of a bimodal frame *F* = ⟨ *W, R, S* ⟩ and a mapping *V* : Prop ∪ Nom → У(*W* ) *with* |*V* (*i*)| = 1 for any *i* ∈ Nom. A *unimodal frame* and a *unimodal model* are defined similarly. | *M* | (or | *F* |) means the domain of a model *M* (or, a frame *F*, respectively). For any binary relation *R* on *W* , *R*[*w*] denotes { *w*' | *wRw*' }.

For a bimodal model *M* = ⟨ *W, R, S, V* ⟩ (or unimodal model *M* = ⟨ *W, R, V* ⟩), *w* ∈ *W* and a formula *ϕ* of 7(◆) (or 7, respectively), the satisfaction relation *M,w* H *ϕ* is defined as usual. We assume that 0 is associated with *R*, i.e., *M,w* H 0*ϕ* ⇐⇒ [*wRw*' and *M, w*' H *ϕ*] for some *w*', and that ◆ is associated with *S*. For E, the satisfaction is defined as follows: *M,w* H E*ϕ* ⇐⇒ *M, w*' H *ϕ* for some *w*'.

A formula *ϕ* is *valid in a model M* (written *M* H *ϕ*) if *M,w* H *ϕ* for any *w*

in *M*. *ϕ* is *valid in a frame F* (written *F* H *ϕ*) if ⟨ *F,V* ⟩ H *ϕ* for any valuation

*V* : Prop ∪ Nom → У(| *F* |). *ϕ* is *satisﬁable in a model M (or a frame F)* if *M* /Hч *ϕ* (or *F* /Hч *ϕ*, respectively). *ϕ* is *valid in a class* F *of frames* (written F H *ϕ*) if it is valid in every *F* ∈ F. For a set of formulas, these notions are defined similarly. A set Γ of formulas *deﬁnes a class* F *of frames* if, for all frames *F*, *F* H Γ ⇐⇒ *F* ∈ F. A class F of frames is *modally deﬁnable* if there is some set of formulas that defines F.

A bimodal frame ⟨ *W, R, S* ⟩ satisfying *S* = (*R*∩ /=) is called a7n (◆)*-frame*, where *w*(*R*∩ /=*w*)' means that *wRw*' and *w* /=*w*'. The notion of 7(◆)*-model* is defined similarly. Thus, for any 7(◆)-model *M*, *M,w* H ◆*ϕ* ⇐⇒ [*w*(*R*∩ /=*w*)' implies *M, w*' H *ϕ*] for some *w*'. Observe that an 7(◆)-frame ⟨ *W, R,* (*R*∩ /=⟩) (or -model) is determined by the unimodal frame ⟨ *W, R* ⟩ (or model, respectively). Therefore, we often regard ⟨ *W, R* ⟩ as 7(◆)-frame ⟨ *W, R,* (*R*∩ /=⟩.)

* 1. *Unimodal and Bimodal p-morphisms*

Definition 2.1 Let *F* = ⟨ *W, R, S* ⟩ and *F*' = ⟨ *W* '*, R*'*, S*' ⟩ be bimodal frames. A mapping *f* : *W* → *W* ' is a *bimodal p-morphism from F to F*' (written *f* : *F* → *F*') if it satisfies the following:

* (*R*-forth) If *wRv*, then *f* (*w*)*R*'*f* (*v*),
* (*R*-back) If *f* (*w*)*R*'*v*', then *wRv* and *f* (*v*)= *v*' for some *v* ∈ *W* , and (*S*-forth) and (*S*-back) defined similarly.

For bimodal models *M* and *M*', *f* : |*M*| → |*M*'| is *a bimodal p-morphism from M to M*' (written *f* : *M* → *M*') if *f* : *F* → *F*' with *w* ∈ *V* (*a*) ⇐⇒ *f* (*w*) ∈ *V* '(*a*) for each *a* ∈ Prop ∪ Nom and each *w* ∈ |*M*|.

If there is a bimodal *p*-morphism *f* from *M* to *M*' such that *f* is surjective as a mapping between domains, *M*' is called *a p-morphic image of M* (written *M* → *M*').

Given two unimodal frames (or models) unimodal *p*-morphism between frames (or models, respectively), is defined by using the clauses (*R*-forth) and (*R*-back).

It is known that the following holds (see, e.g., Blackburn’s thesis [[3](#_bookmark26), Lemma 3.2.2]. He stated the tense logical analogue, though the generalization to 7(◆) (or 7) is obvious).

Fact 2.2 *Let M and M*' *be bimodal (or unimodal) models with f* : *M* → *M*'*. Then, for each formula ϕ of* 7(◆) *(or* 7*, respectively), and each w in M, M,w* H *ϕ* ⇐⇒ *M*'*,f* (*w*) H *ϕ.*

Note that a surjective *p*-morphism between frames does not preserve validity for the modal language with nominals [[3](#_bookmark26), sec 3.2.3].

The following is an application of Fact [2.2](#_bookmark1), i.e., a *unimodal p*-morphism *between models* preserves the satisfaction:

Proposition 2.3 *There exists no formula ϕ*(*q*) *of* 7 *containing q such that M,w* H

* *ψ* ⇐⇒ *M,w* H *ϕ*(*ψ*)*, for any ψ, any model M and any w* ∈ |*M*|*. In other words,*
* *is not deﬁnable in* 7 *in the level of models.*

Proof. Suppose for contradiction that ◆ is definable in 7 in the level of models. Then we define ◆ ч *i* by 7-formulas *ϕ*( ч *i*). Consider two models and a unimodal *p*-morphism: *M*1 = ⟨{ *a, b, c* }*,* {⟨ *a, b* ⟩*,* ⟨ *a, c* ⟩} ∪ { *b, c* }×{ *b, c* }*, V*1 ⟩ where *V*1(*i*)=

{ *a* } for any *i* ∈ Nom, *M*2 = ⟨{ 0*,* 1 }*,* {⟨ 0*,* 1 ⟩*,* ⟨ 1*,* 1 ⟩ }*, V*2 ⟩ where *V*2(*i*) = { 0 } for any *i* ∈ Nom, and *f* (*a*) = 0 and *f* (*b*) = *f* (*c*) = 1. Then, *M*1*,b* H *ϕ*( ч *i*), but *M*2*,f* (*b*) /H*ϕ*( ч *i*) hence *M*1*,b* /H*ϕ*( ч *i*) by Fact [2.2](#_bookmark1), a contradiction.

Remark that *f* : |*M*1| → |*M*2| in the proof is not a *bimodal p*-morphism between

7(◆)-models. Consider the following model and mapping *g* : |*M*3| → |*M*1|:

*M*3 = ⟨ *ω* ∪{ *x* }*,* {⟨ *m, m* ⟩*,* ⟨ *m, m* +1 ⟩*,* ⟨ *x, m* ⟩| *m* ∈ *ω* }*, V*3 ⟩*,*

where *V*3(*i*) = { *x* } for any *i* ∈ Nom, and *g*(*x*) = *a*, *g*(2*m*) = *b* and *g*(2*m* + 1) = *c*

for any *m* ∈ *ω*. Then, *g* is a bimodal *p*-morphism between 7(◆)-models.

Next, we introduce the notion of realizations. This is a generalization of bull- dozing in hybrid logics (see, e.g., [[3](#_bookmark26), ch.5]) and the tricks as Koymans [[15](#_bookmark38), Theorem 4.3.3] and de Rijke [[8](#_bookmark31), Theorem 3.2].

Definition 2.4 Let *M* be a bimodal model, *M*' an 7(◆)-model. If a bimodal *p*- morphism *f* : *M*' → *M* is surjective as a mapping between domains, we call *f* a

* + *-realizer* and *M*' a ◆*-realization* of *M*.

Proposition 2.5 *Suppose that a bimodal model M =* ⟨ *W, R, S, V* ⟩ *satisﬁes* (*R*∩ /=

) ⊂ *S* ⊂ *R and that, for any i* ∈ Nom*, wSw fails where* { *w* } *= V* (*i*)*. Then, M has a* ◆*-realization.*

Proof. Let *C* = { *w* ∈ | *M* || *wSw* }, *W* − = *W* \ *C*, 2 = { 0*,* 1 }, and *W* ' = *W* − ∪ (*C* ×2), where we may assume *W* − ∩ (*C* ×2) = ∅. *f* : *W* ' → *W* is defined as follows: *f* (*x*) := *x* if *x* ∈ *W* −; *w* if *x* = ⟨ *w, i* ⟩ for some *i* ∈ 2 and some *w* ∈ *C*. Write *M*'

= ⟨ *W* '*, R*'*, S*'*,V* ' ⟩, where *R*' = {⟨ *x, y* ⟩| *f* (*x*)*Rf* (*y*) }, *S*' = (*R*'∩ /=), an*V*d '(*a*) = *f* −1[*V* (*a*)] = { *x* | *f* (*x*) ∈ *V* (*a*) } for any *a* ∈ Prop ∪ Nom. Note that *V* ' is a valuation since |*V* (*i*)| ≥ 1 by the surjectiveness of *f* and |*V* (*i*)| ≤ 1 by the injectiveness of *f* T *W* −. It suffices to prove that *f* is a bimodal *p*-morphism from ⟨ *W* '*, R*'*, S*' ⟩ to

⟨ *W, R, S* ⟩. Since *f* is surjective by definition and *xR*'*y* is equivalent to *f* (*x*)*Rf* (*y*) for any *x, y* ∈ *W* ', we have (*R*-forth) and (*R*-back). It is easy to see that (*S*-forth) and (*S*-back) hold.

# Kripke Completeness and Finite Frame Property

We axiomatize the irreflexive hybrid logic of 7(◆) using Goldblatt’s notion of a

*possibility form* [[14](#_bookmark37)].

Definition 3.1 Fix an arbitrary symbol $ not occurring in 7(◆). A *possibility form* (PF) of $ (written: *m*($)) are defined inductively as follows: (1) $ is a PF of

$. (2) If *m* is a PF of $ and *ϕ* is a formulas of 7(◆), then *ϕ* Λ *m*, 0*m* and ◆*m* are PFs of $.

Given *m*($) and *ϕ* of 7(◆), we use *m*(*ϕ*) to denote the formula obtained by replacing the unique occurrence of $ in *m* by *ϕ*.

Definition 3.2 Axiomatization of K√(◆) is given as follows:

(Taut) *ϕ*, for all classical tautologies *ϕ*;

( 1) (*p* → *q*) → ( *p* → *q*);

(□1) □(*p* → *q*) → (□*p* → □*q*);

(□2) *i* → ч ◆*i*; (M1) *p* Λ □*p* → *p*; (M2) *p* → □*p*;

(N1) *m*1(*i* Λ *p*) → ч *m*2(*i* Λ ч *p*) for any nominal *i* and any PFs *m*1 and *m*2;

(MP) from *ϕ* → *ψ* and *ϕ*, we may infer *ψ*;

( -rule) from *ϕ*, we may infer *ϕ*;

(Sub) from *ϕ*, we may infer *ϕσ*, where *σ* is a substitution that uniformly replaces proposition letters by formulas and nominals by nominals.

Axiomatization of K+ is defined as axiomatization of K√(◆) plus the following

√(◆)

additional rule [[13](#_bookmark36)]:

(COV) From ч *m*(*i*), we may infer ч *m*(T), for any PF *m* and any nominal *i*

which does not appear in *m*.

A K√(◆)-logic is any set of formulas containing all the axioms of K√(◆) and closed under all its rules. For every Λ of 7(◆), K√(◆)Λ denotes the smallest logic containing Λ. For a K√(◆)-logic Λ, a set Γ and a formula *ϕ* of 7(◆), the deducibility

relation Γ ▶Λ *ϕ* is defined as usual [[5](#_bookmark28), Definition 4.4]. The notion of K+ -logic

√(◆)

is defined similarly to K√(◆)-logic, except for closing in addition under the (COV)-

rule. The notion of K+

√(◆)

Λ and the deducibility relation are defined similarly. For

▶Λ, we usually drop the subscript if it is clear from the context.

From (□2), (M1) and (M2), we deduce that ◆*i* — 0*i* Λ ч *i*. Remark that

*p* → ч ◆*p* is not valid on any 7(◆)-frames, though (□2) is valid.

Fact 3.3 *The following are theorems and a derived rule of* K√(◆)*, therefore, of*

+

K

√(◆)

*:* (i) *m*1(*i* Λ *ϕ*) Λ ч *m*1( ч (*i* Λ *ψ*)) → *m*1(*i* Λ *ϕ* Λ *ψ*)*;* (ii) *m*1(*i* Λ *ϕ*) Λ *m*2(*i* Λ *ψ*) →

*m*1(*i* Λ *ϕ* Λ *ψ*)*;* (iii) ▶ *ϕ* =⇒ ▶ ч *m*( ч *ϕ*)*, where m*1*, m*2 *and m are PFs and*

*i* ∈ Nom*.*

Let K be the class of all 7(◆)-frames.

Proposition 3.4 *For any ϕ of* 7(◆)*,* ▶KH(◆) *ϕ* =⇒ K H *ϕ and* ▶K+

H(◆)

*ϕ* =⇒

K H *ϕ.*

Proof. By induction on ▶KH(◆) *ϕ*. Note that (M1) defines (*R*∩ /=⊂)

*S*, (M2)

defines *S* ⊂ *R* and (□2) defines the irreflexivity of *S*. With respect to (N1) and

(COV), adding ◆ does not change the usual argument for soundness of hybrid logic

(see, e.g., [[12](#_bookmark35), Proposition 5.1] for (COV)).

* 1. *Kripke Completeness without (COV)*

Let Γ be a set of 7(◆). We say that Γ is *maximal* if, for any *ϕ* of 7(◆), *ϕ* ∈ Γ

or ч *ϕ* ∈ Γ. For any K√(◆)-logic (or, K+

√(◆)

-logic, respectively) Λ, we say that Γ is

Λ*-consistent* if Γ /Λ▶⊥ and Γ is Λ*-maximal consistent* if Γ is maximal and consistent

with respect to Λ.

Definition 3.5 The canonical *bimodal* frame *F*KH(◆) is defined as follows:

* *W* = { *w* | *w* is a K√(◆)-maximal consistent set }
* *wRw*' ⇐⇒ [ *ϕ* ∈ *w* =⇒ *ϕ* ∈ *w*'] for any *ϕ* of 7(◆).
* *wSw*' ⇐⇒ [□*ϕ* ∈ *w* =⇒ *ϕ* ∈ *w*'] for any *ϕ* of 7(◆).

*N om*(*w*) denotes { *i* ∈ Nom | *i* ∈ *w* } for any *w* in *F*KH(◆) .

Definition 3.6 Let *F* = ⟨ *W, R, S* ⟩ and *F*' = ⟨ *W* '*, R*'*, S*' ⟩ be two bimodal frames.

*F*' is *(bimodally) generated subframe* of *F* (written *F*' > *F*) if

* + (i) *W* ' ⊂ *W* ,
  + (ii) *R*' = *R* ∩ (*W* ')2,
  + (iii) *w* ∈ *W* ' implies *R*[*w*] ⊂ *W* ' and
  + (ii') and (iii') about *S*, *S*' defined similar to (ii) and (iii).

For a subset *X* ⊂ *W* , the *subframe generated by X* (notation: *F*X) is the smallest generated subframe of *F* whose domain contains *X*. The *point-generated frame by w* (notation: *F*w) is *F*{ w } where *w* is called the *root* of the frame.

(N1) (especially, Fact [3.3](#_bookmark3) (ii)) forces that in every point-generated subframe, every nominal is satisfied in at most one point as follows:

Proposition 3.7 *Let w be in F*KH(◆) *and v, v*' *in F*KH(◆) *. Then, v* /=*v*' *implies*

w

*N om*(*v*) ∩ *N om*(*v*') *=* ∅*.*

Lemma 3.8 *Let F*KH(◆)

w

*=* ⟨ *W* '*, R*'*, S*' ⟩ *be a point-generated subframe of F*KH(◆) *.*

*Then, (i)* (*R*'∩ /=⊂) *S*'*, (ii) S*' ⊂ *R*' *and (iii) for any v in W* ' *with N om*(*v*) /=∅*,*

*vS*'*v fails.*

Proof. Clearly, (ii) holds. It is easy to prove that (iii) by (□2) (see, e.g., [[3](#_bookmark26), Lemma 5.1.1]). We prove (i) as follows: Assume that *xRx*' and *x* /=*x*'. *B* ∈ *x* and ч *B* ∈ *x*' for some *B* of 7(◆) since *x* and *x*' are maximal consistent. Suppose for contradiction that not *xSx*'. Then, □*C* ∈ *x* and ч *C* ∈ *x*' for some *C* of 7(◆).

We have ч *B* л ч *C* ∈ *x*' whence *B* V *C*

∈*/ x*'. Since *xRx*', (*B* V *C*) ∈*/*

*x*. By

*B,* □*C* ∈ *x*, we have *B* V *C,* □(*B* V *C*) ∈ *x* and so (*B* V *C*) л □(*B* V *C*) ∈ *x*, which

implies (*B* V *C*), a contradiction.

Lemma 3.9 *Let F*KH(◆) *=* ⟨ *W* '*, R*'*, S*' ⟩ *be a point-generated subframe of F*KH(◆) *.*

w

*Fix* ∈*/ W, the domain of F*KH(◆) *. Deﬁne M*٨ *=* ⟨ *W* ٨*, R*٨*, S*٨*,V* ٨ ⟩ *as follows: W* ٨

*= W* ∪{}*; R*٨ *= R*'*; S*٨ *= S*'*;*

{ *v* ∈ *W* ' | *a* ∈ *v* } *if a* ∈ Prop ∪ x∈W ' *N om*(*x*)

(*a*) =

*V* ٨

{} *if a* ∈ Nom \ x∈W ' *N om*(*x*)

*Then,* (i) (*R*٨∩ /=⊂) *S*٨ *and S*٨ ⊂ *R*٨*;* (ii) *for any v in W* ٨ *with* { *v* } = *V* (*i*) *for some i* ∈ Nom*, vS*٨*v fails;* (i) *for any w* ∈ *W* ' *and for any ϕ of* 7(◆)*, M*٨*,w* H *ϕ*

⇐⇒ *ϕ* ∈ *w.*

Proof. (i) and (ii) are obvious by Lemma [3.8](#_bookmark5) and the construction. We can prove

(iii) by induction on *ϕ*. Note that *V* ٨ is a valuation, i.e., |*V* ٨(*i*)| =1 for any *i* ∈ Nom, by Lemma [3.7](#_bookmark4) and the construction.

Theorem 3.10 *Let ϕ be a formula of* 7(◆)*.* K H *ϕ* =⇒ ▶KH(◆) *ϕ.*

Proof. Suppose that ч *ϕ* is K√(◆)-consistent. ч *ϕ* ∈ *w* for some K√(◆)-maximal consistent set *w*. Take the point-generated subframe *F*KH(◆) of *F*KH(◆) . Note that

w

*M*٨ is a bimodal model. We need to duplicate the *S*٨-reflexive points in *M*٨. By Lemma [3.9](#_bookmark6) (i), (ii) and Proposition [2.5](#_bookmark2), there exists a ◆-realization *N* such that *f* : *N* → *M*٨ for some *f* . However, by Lemma [3.9](#_bookmark6) (iii), *M*٨*,w* H ч *ϕ*. Thus,

*N,x* H ч *ϕ* for some *x* in *N* with *f* (*x*) = *w*. Thus, we conclude that K /H*ϕ*.

* 1. *Kripke Completeness with (COV)*

Let *ϕ* be a pure formula of 7(◆). It is well known that *ϕ* defines the first-order property [[3](#_bookmark26), p.92]. *M* = ⟨ *W, R, V* ⟩ is called *named* if, for all *w* ∈ *W* , there is some *i* ∈ Nom such that *V* (*i*) = { *w* }. The following fact is quite useful for the general completeness theorem for pure axioms [[5](#_bookmark28), Lemma 7.22].

Fact 3.11 *Let M =* ⟨ *F,V* ⟩ *be a named model and ϕ a pure formula. Suppose that for all substitution σ, M* H *ϕσ. Then, F* H *ϕ.*

We expand the language 7(◆) with the set *J* of new nominals as there are formulas in the language. 7(◆)[*J*] denotes the expanded language. Let Λ be a

+

K

√(◆)

-logic.

Definition 3.12 Γ of 7(◆)[*J*] is *a named* Λ*-maximal consistent set* if Γ is a Λ- maximal consistent set and Γ satisfies the following: for any PF *m* in 7(◆)[*J*], *m*(T) ∈ Γ =⇒ [*m*(*j*) ∈ Γ for some *j* ∈ *J* which does not appear in *m*].

Lemma 3.13 *Every* Λ*-consistent set* Γ *of* 7(◆) *can be extended to a named* Λ*- maximal consistent set* Γ+ *of* 7(◆)[*J*]*.*

Note that adding the new nominals *J* preserves consistency. For the proof, see, e.g., [[22](#_bookmark46), Lemma 5.3.9].

Definition 3.14 Let Γ be a named Λ-maximal consistent set. Define Nom' as

{ *j* ∈ Nom ∪ *J* | *m*(*j*) ∈ Γ for some PF *m* }. We define the equivalence relation ∼ on Nom' as follows: *m*(*j* л *k*) ∈ Γ for some PF *m*. The quotient Nom'*/* ∼ is defined as usual and [*j*] ∈ Nom'*/* ∼ denotes the equivalence class of *j*. *F*Λ = ⟨ *W, R, S* ⟩ is defined as follows: *W* = Nom'*/* ∼; [*j*]*R*[*k*] ⇐⇒ *m*(*j* л 0*k*) ∈ Γ for some PF *m*; [*j*]*S*[*k*] ⇐⇒ *m*(*j* л ◆*k*) ∈ Γ for some PF *m*.

Γ

Since ▶KH(◆) ◆*i* — 0*i* л ч *i*, we can easily prove the following:

Lemma 3.15 *In F*Λ*,* (*R*∩ /=) =*S holds.*

Γ

Lemma 3.16 *Fix* ∈*/ W. Deﬁne M*٨ *=* ⟨ *W* ٨*, R*٨*, S*٨*,V* ٨ ⟩ *as follows: W* ٨ *= W* ∪

{}*; R*٨ *= R; S*٨ *= S;*

٨ { [*j*] ∈ *W* | *m*(*a* л *j*) ∈ Γ for some PF *m* } *if a* ∈ Prop ∪ Nom'

*V*

(*a*) =

{} *if a* ∈ (Nom ∪ *J*) \ Nom'

*Then,* (i) *V* ٨ *is a valuation from* Nom ∪ *J to* У(*W* ٨)*.* (ii) (*R*٨∩ /=) =*S*٨*;* (iii) *for any* [*j*] ∈ *W and any ϕ of* 7(◆)[*J*]*, M*٨*,* [*j*] H *ϕ* ⇐⇒ [*m*(*j* л *ϕ*) ∈ Γ *for some PF m*]*.*

(i) is obvious by the construction and Fact [3.3](#_bookmark3) (ii). Clearly (ii) holds by Lemma

[3.15](#_bookmark9). We can prove (iii) by induction on *ϕ*. Observe that *M*٨ is a named model.

Theorem 3.17 *Let λ be a pure formula and ϕ a formula of* 7(◆)*. Then,* Kλ H *ϕ*

=⇒ ▶K+

H(◆)

{ λ } *ϕ, where* Kλ *is the class of frame deﬁned by λ* [3](#_bookmark13) *.*

Proof. Suppose that ч *ϕ* is K+

√(◆)

{ *λ* }-consistent. ч *ϕ* ∈ Γ holds for some named

+

K

√(◆)

{ *λ* }-maximal consistent set Γ. Since T ∈ Γ and Γ is named, *j* ∈ Γ for

some *j* ∈ *J*. Thus, *j* л ч *ϕ* ∈ Γ. By Lemma [3.16](#_bookmark10) (iii), *M*٨*,* [*j*] H ч *ϕ* hence *F*٨

= ⟨ *W* ٨*, R*٨*, S*٨ ⟩ /H*ϕ* (Note that *M*٨ is a 7(◆)-model). For any substitution *σ*,

▶K+

H(◆)

{ λ } *λσ*, which implies *M*٨ H *λ* and so *F*٨ H *λ* by Fact [3.11](#_bookmark8). Since *λ* defines

Kλ, *F*٨ ∈ Kλ, which implies Kλ /H*ϕ*.

* 1. *Finite Frame Property and Decidability*

Let Kfinite be the class of all finite 7(◆)-frames. In this subsection, we prove that

K√(◆) and K+

√(◆)

are complete with respect to Kfinite, i.e., we prove the finite frame

property of K√(◆) and K+ .

√(◆)

Definition 3.18 Let *M* = ⟨ *W, R, S, V* ⟩ be a bimodal model and Σ a subformula- closed set of formulas. *w* ~Σ *v* ⇐⇒ [*M,w* H *ϕ* ⇐⇒ *M,v* H *ϕ*] for every *ϕ* ∈ Σ. The quotient *W/* ~Σ is defined as usual and [*w*] ∈ *W/* ~Σ denotes the equivalence

class of *w*. A bimodal model *M*f = ⟨ *W* f *, R*f *, S*f *,V* f ⟩ is called *a bimodal ﬁnest*

Σ

*ﬁltration of M through* Σ if *W* f = *W/* ~Σ, [*w*]*R*f [*v*] ⇐⇒ E *w*' ∈ [*w*] E *v*' ∈ [*v*] *w*'*Rv*', [*w*]*S*f [*v*] ⇐⇒ E *w*' ∈ [*w*] E *v*' ∈ [*v*] *w*'*Sv*' and *V* f : Prop ∪ Nom → У(*W* f ) is a function satisfying,

f { [*w*] ∈ *W* f | *w* ∈ *V* (*a*) } if *a* ∈ Prop ∪ (Nom ∩ Σ)

*V* (*a*)=

{ [*v*] } if *a* ∈ Nom \ Σ

where [*v*] is an arbitrarily fixed element of *W* f since *W* f /=∅.

Note that *V* f is a valuation (see [[3](#_bookmark26), Lemma 2.4.2]). The following fact holds (see, e.g., [[3](#_bookmark26), Theorem 2.4.1]) since a finest filtration is a special case of the usual filtration.

Fact 3.19 *Suppose that* Σ *is subformula closed. For any ϕ* ∈ Σ *and any w in M, M,w* H *ϕ* ⇐⇒ *M*f *,* [*w*] H *ϕ.*

Σ

Lemma 3.20 *(i)* (*R*∩ /=⊂) *S implies* (*R*f ∩ /=⊂) *S*f *, (ii) S* ⊂ *R implies S*f ⊂ *R*f *.*

3 One of the referees of the present paper comments on (the preliminary version of) Theorem [3.17](#_bookmark11) as follows: it seems to follow directly from the existing general completeness results for pure axioms since ◆i ↔ ♦i∧ ¬ i

is pure. If we remove (□2), (M1), (M2) from the axiomatization of K+

H(◆)

and add ◆i ↔ ♦i ∧ ¬ i and

(□-rule) to the axiomatization, then we can immediately get the desired results. Note, however, that in the

definition of K+

H(◆)

-logic, we need not require (□-rule), which is derivable from (M2) and ( -rule).

Proof. Clearly, (ii) holds. To prove (i), suppose [*w*](*R*f ∩ /=*w*)['], i.e., *vRv*' for some *v* ∈ [*w*] and some *v*' ∈ [*w*']. Since [*w*] /=*w*['], *v* /=*v*' whence *v*(*R*∩ /=*v*'). By (*R*∩ /=⊂) *S*, *vSv*' whence [*w*]*S*f [*w*'].

Definition 3.21 A valuation *V* on *F* is *S-irreflexivity respecting* if *w* ∈ *V* (*i*) is *S*- irreflexive for all *i* ∈ Nom. A bimodal model *M* = ⟨ *F,V* ⟩ is *S-irreflexivity respecting* if *V* is *S*-irreflexivity respecting.

Lemma 3.22 *Let M =* ⟨ *W, R, S, V* ⟩ *be an S-irreflexivity respecting bimodal model and* Σ *a subformula closed ﬁnite set. Then, there exists ﬁnite* Σ' *such that* Σ ⊂ Σ' *and a bimodal ﬁnest ﬁltration M*f ' *is S-irreflexivity respecting.*

Σ

Proof. Define Σ1 as follows: Σ1 = Σ (if Σ ∩ Nom /=∅); Σ1 = Σ ∪{ *j* } where *j* is the fixed element of Nom (if Σ ∩ Nom = ∅). Fix *j*0 ∈ Σ1 ∩ Nom. Take the bimodal

finest filtration *M*f

Σ

1

with *V* f (*k*) = *V* f (*j*0) for all nominals *k* ∈*/* Σ1.

Assume that *i* ∈ Nom ∩ Σ1. *M,w* H *i* holds for some *w*. Since *M* is an *S*-

irreflexivity respecting model, *wSw* fails. It follows from *i* ∈ Σ1 that *M*f *,* [*w*] H *i*

Σ1

by Fact [3.19](#_bookmark12). Note that [*w*] = *V* (*i*). Since *M*f ' is the finest filtration and *wSw*

Σ

fails, [*w*]*S*f [*w*] fails. Assume that *i* ∈ Nom \ Σ1. Let *V* f (*i*) = { [*v*] }. Since *V* f (*i*) =

*V* f (*j*0), *S*f -irreflexivity of [*v*] holds. Thus, *M*f is *S*-irreflexivity respecting.

Σ1

Theorem 3.23 *Let ϕ be a formula of* 7(◆)*.* Kfinite H *ϕ* =⇒ ▶K+ *ϕ.*

H(◆)

Proof. Suppose that ч *ϕ* is K√(◆)-consistent. Construct the *M*٨ as in the proof of Theorem [3.17](#_bookmark11). Let Σ be the set of all subformulas of ч *ϕ*. By Lemmas [3.15](#_bookmark9), [3.16](#_bookmark10) and [3.22](#_bookmark14), there exists finite Σ' such that Σ ⊂ Σ' and *M*٨f is finite and *S*-irreflexivity respecting, and satisfies (*R*f ∩ /=⊂) *S*f and *S*f ⊂ *R*f . Apply Proposition [2.5](#_bookmark2) as in the proof of Theorem [3.10](#_bookmark7). Note that the construction in the proof of Proposition

Σ'

[2.5](#_bookmark2) does not change the finite cardinality.

Theorem 3.24 *Let ϕ be a formula of* 7(◆)*.* Kfinite H *ϕ* =⇒ ▶KH(◆) *ϕ.*

Proof. Suppose that ч *ϕ* is K√(◆)-consistent. Construct the *M*٨ as in the proof of Theorem [3.10](#_bookmark7). Apply the similar argument as in the proof of Theorem [3.23](#_bookmark15).

Corollary 3.25 K√(◆) *and* K+

√(◆)

*are decidable.* [4](#_bookmark16)

# Frame Definablity

* 1. *Relations between nominals and the irreflexive modality*

Recall that Γ defines a class F of frames if, for all frames *F*, *F* H Γ ⇐⇒ *F* ∈ F.

Fact 4.1 *In* 7(◆)*, the following deﬁne the corresponding properties of R. (i)* 0*p* → ◆*p: irreflexivity, (ii)* 00*p* → ◆*p: strict partial ordering, (iii)* ◆◆*p* → ◆*p: antisymmetry and transitivity, (iv)* (*p* → 0*p*) Λ (◆◆*p* → ◆*p*)*: partial ordering.*

4 Complexity of our decidability remains open. One of the referees suggests that it may be in PSPACE.

As for (iii), note that *S*-transitivity is equivalent in the 7(◆)-frames to the conjunction of *R*-transitivity and *R*-antisymmetry. When adding ◆ to standard modal language without nominals, we cannot define antisymmetry [[20](#_bookmark43)] [5](#_bookmark17) [6](#_bookmark18) . We can, on the other hand, define it in 7 by (*i* Λ 0(*j* Λ 0*i*)) → *j* [[22](#_bookmark46), p.45]. Thus, nominals increase the frame-expressive power of the irreflexive modality.

In this subsection, we deal with the following question:

(Q) Does the irreflexive modality ◆ increase the frame-expressive power of hybrid language?

Some uses of ◆ are inessential. For example, ◆(*i*Λ *ϕ*) and *ϕ*V◆*ϕ* can equally well be written (even in the level of pointed models) without using ◆, as 0(*i*Λ*ϕ*)Λч *i* and *ϕ* V 0*ϕ*, respectively. In the level of frames, the following proposition demonstrates another inessential use of ◆.

Proposition 4.2 *(1)* ◆T *(of* 7(◆)*) deﬁnes* 6 *x* E *y* [*xRy and x* /=*y*]*. (2) i* → 0 ч *i*

*(of* 7*) also deﬁnes the same property.*

As for (1), observe that ◆T defines the seriality of *S*. For (2), the standard translation for hybrid languages, e.g., [[5](#_bookmark28), Exercises 7.3.1] tells us the correspondence.

We have not obtained the full answer to (Q) yet. We, however, will give two partial answers to this. First answer is that ◆ increases the expressivity of 7(E), where 7(E) means 7 with the global modality E, whose accessibility relation is the total relation on *W* . Second answer is that, with respect to the *pure formulas*, ◆ increases the definability of 7.

* + 1. *A Comparison in the Setting of Adding the Global Modality*

Here, we consider the extended languages 7(◆*,* E) and 7(E). Recall that we define

*M,w* H E*ϕ* ⇐⇒ *M, w*' H *ϕ* for some *w*'. We can easily prove the following:

Proposition 4.3 E◆T *deﬁnes* E *x* E *y* [*xRy and x* /=*y*]*.*

E *x* E *y* [*xRy* and *x* /=*y*], however, is undefinable in 7(E) [7](#_bookmark19) . To show this, we introduce the frame construction, *ultraﬁlter morphic images* [[22](#_bookmark46), Definition 4.2.5].

Definition 4.4 (1) Given a binary relation *R* on a set *W* , we define a unary opera- tion *m*R on У(*W* ): *m*R(*X*)= { *w* ∈ *W* | *R*[*w*] ∩ *X* /=∅ }. (2) The *bimodal ultraﬁlter* *extension ue F* of *F* = ⟨ *W, R, S* ⟩ is the frame ⟨ *W ue, Rue, Sue* ⟩, where *W ue* is the set of (principal and non-principal) ultrafilters over *W* , *uRueu*' ⇐⇒ for any *X* ⊂ *W* , *X* ∈ *u*' implies *m*R(*X*) ∈ *u*, and *Sue* defined similarly by *m*S. *The unimodal ultra-* *ﬁlter extension* of *F* = ⟨ *W, R* ⟩ is ⟨ *W ue, Rue* ⟩.

Definition 4.5 Let *F* and *G* be (unimodal) frames. *G* is a *bimodal ultraﬁlter mor-*

5 Consider *F* = ⟨{ a, b }, { a, b }2 ⟩ and *F*' = ⟨ ω, {⟨ m, m ⟩, ⟨ m, m +1 ⟩| m ∈ ω } ⟩. Note that *F*' is antisym- metric and *F* is not. Define f by f (2m) = a, f (2m + 1) = b for any m ∈ ω. Then, we can prove that f is a surjective bimodal p-morphism (in the sense of standard modal language) f : *F*' → *F*, which violates antisymmetry.

6 Even if we add the global modality E in addition to ◆, we cannot define it. This is because the surjective bimodal p-morphism with respect to R and (R∩ /=) preserve frame-validity of the irreflexive modal language with the global modality.

7 Contrary to E◆T, E(i → ♦ч i) does not define E x E y [xRy and x /=y]. E(i → ♦ч i) defines 6 x E y .x /=y.

*phic image* of *F* if there is a surjective bimodal *p*-morphism *f* : *F* → *ue G* such that

|*f* −1[{ *u* }]| = 1 for all principal ultrafilters *u* in *ue G*. *Unimodal ultraﬁlter morphic images* are defined similarly.

It is known that validity of 7(E)-formulas is preserved under taking unimodal ultrafilter morphic images [[22](#_bookmark46), Proposition 4.2.6].

Proposition 4.6 E *x* E *y* [*xRy and x* /=*y*] *is undeﬁnable in* 7(E)*.*

We prove this by the example used in the proof of [[22](#_bookmark46), Proposition 4.2.7].

Proof. Take any frame *F* = ⟨ *W,* idW ⟩ with |*W* | ≥ *ω*, where idW is the identity relation on *W* . Consider the ultrafilter extension ⟨ *W ue,* (idW )*ue* ⟩ of *F*. Note that *W ue* contains non-principal ultrafilters since |*W* | ≥ *ω*. It is easy to prove that (idW )*ue* = idW *ue* .

Next, we ‘bulldoze’ the non-principal ultrafilters in *W ue* as follows: Let *P* the set of all principal ultrafilters and *NP* the set of all non-principal ultrafil- ters. Define *G* = ⟨ *W* '*, R*' ⟩ where *W* ' =*P* ∪ (*NP* × { 0*,* 1 }) and *R*' = idW ' ∪

{⟨⟨ *u,* 0 ⟩*,* ⟨ *u,* 1 ⟩ ⟩*,* ⟨⟨ *u,* 1 ⟩*,* ⟨ *u,* 0 ⟩⟩ | *u* ∈ *NP* }. Consider *f* : *W* ' → *W ue* as *u* ∈ *P*

'→ *u* and ⟨ *u, i* ⟩ ∈ *NP* × { 0*,* 1 } '→ *u*. Then, we can prove that *F* is an ultrafilter morphic image of *G*. *G* satisfies E *x* E *y* [*xRy* and *x* /=*y*] but *F* does not. Thus, by the validity preservation under ultrafilter morphic images [[22](#_bookmark46), Proposition 4.2.6], we can conclude that E *x* E *y* [*xRy* and *x* /=*y*] is undefinable in 7(E).

Thus, 7(◆*,* E) is more expressive than 7(E) with respect to the frame definabil- ity. We can also conclude that ł(◆*,* E), ł(◆) with E, is the different extension from 7(E) (i.e., by the Gargov and Goranko Translation [[12](#_bookmark35)], ł(D), the unimodal language with the *difference operator*, associated with the inequality /=).

* + 1. *A Comparison with respect to the Pure Formulas*

Recall that a formula *ϕ* of 7(◆) (7 or 7(E)) is called *pure* if it contains no propo- sition letters.

The following proposition holds clearly:

Proposition 4.7 *(1)* □⊥ *deﬁnes* 6 *x* 6 *y* [*xRy* ⇒ *x* = *y*]*. (2)* E□⊥ *(of* 7(◆*,* E)*)deﬁnes*

E *x* 6 *y* [*xRy* ⇒ *x* = *y*]*. (3)* 0□⊥ *deﬁnes* 6 *x* E *y* [*xRy and* 6 *z* [*yRz* ⇒ *y* = *z*]]*.*

In 7(◆) (even in ł(◆)), we can define the atomicity by 0□⊥ without the assumption of transitivity.

We will prove that the properties in (2), (3) of Proposition [4.7](#_bookmark20) are not definable by pure formulas of 7(E). To prove this, we introduce the following new frame construction:

Definition 4.8 Let *F* and *G* be (unimodal) frames. *G* is an *unimodal ultraﬁlter morphic domain* of *F* if there is a surjective unimodal *p*-morphism *f* : *ue G* → *F* such that *f* T *P* is the identity function on *P* , where *P* is the set of all principal ultrafilters *u* in *ue G*.

Ultrafilter morphic domains are different from ultrafilter morphic images in the

direction of surjective unimodal *p*-morphisms.

It is easy to show the following Lemma. Observe, however, that the surjective- ness of *f* is needed for the case where *ϕ* is E*ψ*.

Lemma 4.9 *Let M and N be (unimodal) models. Let f* : *M* → *N be a surjective unimodal p-morphism. Then, for any formula ϕ in* 7(E) *and any w* ∈ |*M*|*, M,w* H *ϕ* ⇐⇒ *N,f* (*v*) H *ϕ.*

Proposition 4.10 *The validity of pure formulas in* 7(E) *is preserved under taking ultraﬁlter morphic domains.*

Proof. Let *f* : *ue G* → *F* be a surjective unimodal *p*-morphism such that *f* is injective with respect to the set of all principal ultrafilters. Let *ϕ* be a pure for- mula in 7(E). We prove the contraposition. Suppose that *G* /H*ϕ*, which im- plies ⟨ *G,V* ⟩*,w* /H*ϕ* for some *V* and *w* ∈ |*G*|. From this, we can prove that

⟨ *ue G,V ue* ⟩*, u*w /*ϕ*H where *u*w = { *X* ∈ У(|*G*|) | *w* ∈ *X* } (see, e.g., the second para- graph in the proof of [[22](#_bookmark46), Proposition 4.2.6]). Observe that this inference holds even

for *any* formulas in 7(E).

Then, we define a valuation *V* ' (for Nom) on *F* as follows: *V* '(*i*) = *f* [*V ue*(*i*)]. This is a valuation since, for any *i* ∈ Nom, the element of *V ue*(*i*) is the principal ultrafilter *u*v where *V* (*i*)= { *v* } and so |*V* '(*i*)| = |*f* [*V ue*(*i*)]|= 1 by the injectiveness of *f* with respect to the set of all principal ultrafilters. Clearly, for any *u* in *ue G* and any *i* ∈ Nom, *u* ∈ *V ue*(*i*) ⇐⇒ *f* (*u*) ∈ *V* '(*i*). Since ⟨ *ue G,V ue* ⟩*, u*w /H*ϕ* and *ϕ* is pure, we deduce, by Lemma [4.9](#_bookmark21), that ⟨ *F,V* ' ⟩*,f* (*u*w) /H*ϕ* hence *F* /H*ϕ*.

Proposition 4.11 *(1)* E *x* 6 *y* [*xRy* ⇒ *x* = *y*] *and (2)* 6 *x* E *y* [*xRy and* 6 *z* [*yRz* ⇒

*y* = *z*]] *are not deﬁnable by pure formulas in* 7(E)*, i.e., in* 7*.*

Proof. Consider the frame *N* = ⟨ N*,<* ⟩, the natural numbers in their usual order- ing. Take the ultrafilter extension *ue N* = ⟨ N*ue, <ue* ⟩ of *N*. We can divided N*ue* into two disjoint parts, *P* , the set of all principal ultrafilters, and *NP* , the set of all non-principal ultrafilters. As pointed out in [[5](#_bookmark28), Example 2.58], this frame consists of an isomorphic copy *P* of natural numbers, followed by an uncountable cluster containing *NP* . Observe that for any pair *u*, *u*' of ultrafilters, if *u*' is non-principal, then *u <ue u*' [[5](#_bookmark28), Example 2.58].

Construct *G* = ⟨ *W* '*, R*' ⟩ as follows: *W* ' = *P* ∪{∗ }, *R*' = {⟨ *u, u*' ⟩ ∈ *P* 2 | *u <ue u*' }

∪ *P* ×{ ∗ } ∪ { ⟨ ∗*,* ∗⟩ }. Define *f* : N*ue* → *W* ' as *u* ∈ *P* '→ *u* and *u* ∈ *NP* '→ ∗. Then, it is easy to see that *f* is a surjective unimodal *p*-morphism and *f* is an identity function with respect to *P* . Thus, *N* is an ultrafilter morphic domain of *G*.

*G* satisfies (1), (2) but *F* does not. Thus, by Proposition [4.10](#_bookmark22), we conclude that (1), (2) are undefinable in 7(E).

Therefore, the set of pure formulas of 7(◆) is more expressive than that of 7

with respect to the frame definability.

* 1. *Goldblatt-Thomason-style Characterization*

In this subsection, we use some notions from first-order model theory, e.g., submodel, elementary embedding, *ω*-saturatedness. The reader unfamiliar with them can refer to, e.g., [[7](#_bookmark30)]. In addition, we drop the usual assumption that Nom and Prop of 7(◆) have countable members.

The notion of *unimodal* generated subframes are defined similar to bimodal generated subframes without using the clauses related to the second accessibility relation *S*.

Proposition 4.12 7(◆)*- (or* 7*-) deﬁnable classes are closed under unimodal gen- erated subframes.*

Proof. See [[22](#_bookmark46), Proposition 4.2.1] or [[3](#_bookmark26), Theorem 3.2.1]. Adding ◆ does not change the argument.

Definition 4.13 *F* is an *hybrid amalgamation* of { *G*k | *k* ∈ *K* } if for any (uni- modal) point-generated subframe *F*w of *F* there exists *k* ∈ *K* such that *F*w is a proper generated subframe of *G*k, i.e., *F*w is a (unimodal) generated subframe of *G*k and *F*w /=*G*k.

Proposition 4.14 7(◆)*- (or* 7*-) deﬁnable classes are closed under hybrid amal- gamations.*

Proof. See [[22](#_bookmark46), Proposition 4.2.2]. Adding ◆ does not change the argument.

With respect to ultrafilter morphic images, we can prove the following similarly to the unimodal case [[22](#_bookmark46), Proposition 4.2.6]:

Proposition 4.15 *Validity (on* 7(◆)*-frames) of* 7(◆)*-formulas is preserved under taking bimodal ultraﬁlter morphic images.*

For 7(◆), we can give the following characterization:

Theorem 4.16 *An elementary frame class* F *is deﬁnable in* 7(◆) *iff* F *is closed under taking (i) bimodal ultraﬁlter morphic images, (ii) unimodally generated sub- frames, and (iii) hybrid amalgamations.*

Proof. We will prove that the right-to-left-direction. Let Th(F) = { *ϕ* | F H *ϕ* }. It suffices to prove that, for any 7(◆)-frame *F*, *F* H Th(F) =⇒ *F* ∈ F.

Suppose that *F* H Th(F). Let us assume that *F* is unimodally point generated (Otherwise, the proof that *F* ∈ F is similar to the proof of [[22](#_bookmark46), Theorem 4.3.4], where we need the closure condition (iii). Note that Lemmas needed there hold even in 7(◆)). Let *w* be the root of *F*.

We can suppose that Prop ∪ Nom contains a proposition letter *p*X and nominal *i*x for each *X* ⊂ | *F* | and each *x* ∈ | *F* |, respectively. Let *M* = ⟨ *F, V*0 ⟩, where *V*0 is a natural valuation with *V*0(*p*X )= *X* and *V*0(*i*x) = { *x* }.

Let Δ be the set consisting of the following, for all *X, Y* ⊂ |*F*| and *x* ∈ | *F* |,

*p*|*F*|\X — ч *p*X; *p*X∩Y — *p*X∧Y ; *i*x — *p*{ x }; *p*mR(X) — 0*p*X; *p*mS (X) — ◆*p*X *.*

Let Δ*F* be the following set:

{ *i*w }∪ { 1 ··· m*ϕ* | *ϕ* ∈ Δ and *m* ∈ *ω* and i ∈ { *,* □ } for any *i* }*.*

It is easy to see that Δ*F* is satisfiable on *F* at *w* under the natural valuations. Then we can prove the following claim (for the proof, see, e.g., [[22](#_bookmark46), p.59, Claim 1]): Δ*F* is satisfiable in *G* for some *G* ∈ F.

By the claim, we may infer that ⟨ *G,V* ⟩*,v* H Δ for some valuation *V* and some *v* in *G* for some *G* ∈ F. By the construction, all nominals of the set { *i*x | *x* ∈ | *F* |} denote a point in ⟨ *G,V* ⟩ that is reachable from *v*. Thus, we can think of *V* as a valuation for the frame *G*v. In this way, we can consider the point-generated

‘submodel’ ⟨ *G*v*,V* ⟩ of ⟨ *G,V* ⟩ (For the definition of *submodel*, see, e.g., [[5](#_bookmark28), Definition 2.5]). Then we can prove that ⟨ *G*v*,V* ⟩*,x* H Δ for any *x* ∈ |*G*v| and ⟨ *G*v*,V* ⟩*,v* H *p*X for all *X* ⊂ | *F* | with *w* ∈ *X*.

Let ⟨ *G*∗*,V* ∗ ⟩ be an *ω*-saturated elementary extension of ⟨ *G*v*,V* ⟩. It follows that

v

⟨ *G*∗*,V* ∗ ⟩*,x* H Δ for any *x* ∈ |*G*∗| and ⟨ *G*∗*,V* ∗ ⟩*, v*∗ H *p*X for all *X* ⊂ | *F*w | with

v v v

*w* ∈ *X* where *v*∗ is the corresponding element to *v*, since the satisfaction relation is elementary.

Claim 4.17 *F is a bimodal ultraﬁlter morphic image of G*∗*.*

v

(Proof of Claim [4.17](#_bookmark23)) For any *s* in *G*∗, { *X* ⊂ | *F* ||⟨ *G*∗*,V* ∗ ⟩*,s* H *p*X } is an

v v

ultrafilter. This defines the mapping *f* from | *G*∗ | to | *ue F* |. We can prove that *f* is a surjective bimodal *p*-morphism and satisfies the condition of ultrafilter morphic image (For the detailed proof of these, see [[22](#_bookmark46), Claim 2 in the proof of Theorem 4.3.4]). (QED of Claim [4.17](#_bookmark23))

v

Thus we can conclude that *F* ∈ F by *G* ∈ F, by the elementariness of F and the closure conditions (i) and (ii).

Thus, we can capture the precise frame expressivity of 7(◆) in terms of frame constructions.

# Concluding Remarks

One of the referees of the present paper posed two interesting further directions. Both of them are concerned with the semantical relation (with respect to the frame definability) between the irreflexive modality (especially, ł(◆)) and nominals (es- pecially, 7). First one is a syntactical investigation of 7 to simulate (some parts of) ł(◆). For example, as the referee mentions, can we simulate ł(◆) by re- stricting nominals to occur only in specific conditions, e.g., in the scope of at most one modal operator? Second one is to generalize ł(◆) to the PDL-style language using program composition, union, and *intersection-with-inequality* (but not with the star) and to specify the scope for the idea of the irreflexive modality. These two directions will make our understanding of the irreflexive modality and nominals deeper.

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