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A Jordan Curve Theorem with Respect to Certain Closure Operations on the Digital Plane

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**Abstract**

The closure operations on Z *×* Z introduced and studied in the paper generalize the Khalimsky topology, which is commonly used as a basic topological structure in digital topology nowadays. By proving a digital analogy of the Jordan curve theorem for these closure operations, we show that they are also suitable for solving problems of digital topology. We mention some advantages of the closure operations studied over the Khalimsky topology.

# Introduction

Topological structures which are more general than the usual topology occur in many branches of mathematics and are utilized in numerous applications. Especially, closure operations fulfilling only some of the Kuratowski closure axioms proved to provide a suitable framework for various approaches to dig- ital topology - see [16]. The closure operations employed in the present paper are obtained from the Kuratowski closure operations by omitting the axioms of idempotency and additivity (but retaining the axiom of monotony). We will use the advantage of the closure spaces over the usual topological spaces which consists in a better behaviour of connectedness with respect to quotient maps.

The paper is a continuation of the work [15] where closure operations associated with *α*-ary relations (*α >* 1 an ordinal) were studied with a special emphasis upon those defined on the set of integers Z. For the convenience of the reader, the relevant material from [15] is repeated without proofs, which

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makes our exposition self-contained. For each natural number *n >* 1 we define a closure operation on Z *×* Z which is obtained as a product of two copies of a closure operation on Z associated with a special *n*-ary relation on Z. In the particular case *n* = 2 we get the Khalimsky topology. The closure operations defined are studied and, as the main result, an analogy of the Jordan curve theorem is formulated and proved for them. (Recall that the classical Jordan curve theorem states that a simple closed curve in the real plane separates the

plane into precisely two components). It means that these closure operations can be used, as an alternative to the Khalimsky topology, for solving problems of computer graphics and computer image processing. We also demonstrate that using them has some advantages over using the Khalimsky topology.

# Preliminaries

By a *closure operation u* on a set *X* we mean a map *u*: exp *X →* exp *X* fulfilling *u∅* = *∅*, *A ⊆ X ⇒ A ⊆ uA*, and *A ⊆ B ⊆ X ⇒ uA ⊆ uB*. Such closure operations were studied by E. Cˇech in [1] (who called them topologies). A pair (*X, u*), where *X* is a set and *u* is a closure operation on *X*, is called a *closure space*. Given a pair *u, v* of closure operations on a set *X*, we put *u ≤ v* if *uA ⊆ vA* for each *A ⊆ X*. Clearly, *≤* is a partial order on the set of all closure operations on *X*. A closure operation *u* on a set *X* is called *additive*

or *idempotent* if *A, B ⊆ X ⇒ u*(*A ∪ B*) = *uA ∪ uB* or *A ⊆ X ⇒ uuA* = *uA*

respectively. A closure operation *u* on a set *X* which is both additive and idempotent is called a *Kuratowski closure operation* or briefly a *topology* and the pair (*X, u*) is called a *topological space*. According to [13], given a cardinal *n >* 1, a closure operation *u* on a set *X* and the closure space (*X, u*) are called an *Sn*-*closure operation* and an *Sn*-*space* respectively if the following condition is satisfied:

*A ⊆ X ⇒ uA* = *{uB*; *B ⊆ A,* card *B < n}*.

*S*2-closure operations and *S*2-spaces are called *quasi-discrete* in [2]. *S*2- topological spaces are often called *Alexandroff spaces* - see e.g. [8]. Of course, any *S*2-closure operation is additive, and any *Sm*-closure operation is an *Sn*-

closure operation whenever *m, n* are cardinals with 1 *< m < n*. Since any closure operation on a set *X* is obviously an *Sn*-closure operation for each cardinal *n* with *n >* card *X*, there exists a least cardinal *n* such that *u* is an *Sn*-closure operation. Such a cardinal is then an important invariant of

the closure operation *u*. Evidently, if *n ≤ ℵ*0, then any additive *Sn*-closure operation is an *S*2-closure operation.

We will work with some basic topological concepts naturally extended from topological spaces to closure ones. Given a closure space (*X, u*), a subset *A ⊆ X* is called *closed* if *uA* = *A*, and it is called *open* if *X − A* is closed. A closure space (*X, u*) is said to be a *subspace* of a closure space (*Y, v*) if

*X ⊆ Y* and *uA* = *vA ∩ X* for each subset *A ⊆ X*. We will speak briefly about a subspace *X* of (*Y, v*). A closure space (*X, u*) is said to be *connected*

if *∅* and *X* are the only subsets of *X* which are both closed and open. A subset *X ⊆ Y* is considered to be connected in a closure space (*Y, v*) if the subspace *X* of (*Y, v*) is connected. A maximal connected subset of a closure space is called a *component* of this space. All the basic properties of connected sets and components in topological spaces (see e.g. [3]) are preserved also in closure spaces. A closure space (*X, u*) is said to be a *T*0-*space* if for any points *x, y ∈ X* from *x ∈ u{y}* and *y ∈ u{x}* it follows that *x* = *y*, and it is called a *T* 1 -*space* if each singleteon subset of *X* is closed or open. Given closure spaces

2

(*X, u*) and (*Y, v*), a map *ϕ* : *X → Y* is said to be a *continuous map* of (*X, u*)

into (*Y, v*) if *f* (*uA*) *⊆ vf*(*A*) for each subset *A ⊆ X*. If, moreover, *ϕ* is a bijection and *ϕ−*1 : *Y → X* is a continuous map of (*Y, v*) into (*X, u*), then *ϕ* is called a *homeomorphism* of (*X, u*) onto (*Y, v*). We say that closure spaces (*X, u*) and (*Y, v*) (and the closure operations *u* and *v*) are *homeomorphic* if there exists a homeomorphism of (*X, u*) onto (*Y, v*).

If (*Xj, uj*), *j ∈ J*, is a system of closure spaces , then the closure operation

*v* on *j∈J Xj* generated by the projections *prj* : *j∈J Xj* *→ Xj*, *j ∈ J* (i.e.,

the greatest - with respect to *≤* - closure operation *v* on *j∈J Xj* such that

all projec tions *prj* : ( *j∈J Xj, v*) *→* (*X* *j, uj*), *j ∈ J*, are continuous) is given

by *vA* =

*j∈J ujprj*(*A*) whenever *A ⊆*

*j∈J Xj* .

Let (*X, u*)*,* (*Y, v*) be closure spaces and let *f* : (*X, u*) *→* (*Y, v*) be a sur-

jective map. Then f is a quotient map (i.e., *v* is the least - with respect to *≤*

- closure operation on *Y* such that the map *f* : (*X, u*) *→* (*Y, v*) is continuous) if and only if *vB* = *f* (*uf−*1(*B*)) for any *B ⊆ Y* .

Clearly, if (*X, u*)*,* (*Y, v*) are closure spaces, *f* : (*X, u*) *→* (*Y, v*) is a contin- uous map and *B* is a closed subset of (*Y, v*), then *f−*1(*B*) is closed in (*X, u*). For quotient maps the following stronger statement obviously holds:

**Lemma 1.1** *Let* (*X, u*)*,* (*Y, v*) *be closure spaces, f* : (*X, u*) *→* (*Y, v*) *a quo- tient map and B ⊆ Y a subset. Then B is closed in* (*Y, v*) *if and only if f−*1(*B*) *is closed in* (*X, u*)*.*

We will need the following

**Lemma 1.2** *Let* (*X, u*)*,* (*Y, v*) *be closure spaces, let v be idempotent and let f* : (*X, u*) *→* (*Y, v*) *be a continuous surjection. Then f is a quotient map if and only if f* (*uf−*1(*B*)) *is closed in* (*Y, v*) *for each B ⊆ Y .*

**Proof.** If *f* : (*X, u*) *→* (*Y, v*) is a quotient map, then *f* (*uf−*1(*B*))

= *vB* and, as *v* is idempotent, *f* (*uf−*1(*B*)) is closed in (*Y, v*) for each *B ⊆ Y* . Conversely, let *f* (*uf−*1(*B*)) be closed in (*Y, v*) whenever *B ⊆ Y* . Since *B ⊆ f* (*uf−*1(*B*)), we get *vB ⊆ f* (*uf−*1(*B*)). As the inverse inclusion clearly holds (because *f* is continuous), we have *f* (*uf−*1(*B*)) = *vB*. Therefore, *f* : (*X, u*) *→* (*Y, v*) is a quotient map. *✷*

**Corollary 1.3** *Let* (*X, u*)*,* (*Y, v*) *be closure spaces, let v be idempotent and let f* : (*X, u*) *→* (*Y, v*) *be a quotient map. Then the restriction f | f−*1(*B*) : *f−*1(*B*) *→ B is a quotient map for each subset B of* (*Y, v*)*.*

**Proof.** Let *B* be a subset of (*Y, v*). Clearly, *f | f−*1(*B*) : *f−*1(*B*) *→ B* is a continuous surjection. Let *C ⊆ B* be an arbitrary subset. As *f | f−*1(*C*) = *vC* and *v* is idempotent, *f* (*uf−*1(*C*)) is closed in (*Y, v*) and we clearly have *f* (*uf−*1(*C*) *∩f−*1(*B*)) = *f* (*uf−*1(*C*)) *∩B*. Consequently, *f* (*uf−*1(*C*) *∩f−*1(*B*)) is closed in the subspace *B* of (*Y, v*) (because, given a subspace *T* of a closure

space (*Z, w*) and a subset *A ⊆ Z*, *A ∩ T* is closed in the subspace *T* whenever

*A* is closed in (*Z, w*)). Thus, by Lemma 1.2, *f | f−*1(*B*) is a quotient map. *✷*

As usual, given a map *f* : *X → Y* , a *ﬁbre* of *f* is any set *f−*1(*y*) with *y ∈ Y* . If all fibres of *f* are connected, then we say that *f* has connected fibres. For closure spaces, the following analogy of a theorem known for topological spaces is valid:

**Proposition 1.4** *Let* (*X, u*)*,* (*Y, v*) *be closure spaces, let v be idempotent and let f* : (*X, u*) *→* (*Y, v*) *be a quotient map having connected ﬁbres. Then* (*X, u*) *is connected if and only if* (*Y, v*) *is connected.*

**Proof.** If (*X, u*) is connected, then (*Y, v*) is connected because *f* is continuous. Let (*Y, v*) be connected. Let *X* = *X*1 *∪ X*2 where *X*1*, X*2 are disjoint closed subsets of (*X, u*). Put *Yi* = *f* (*Xi*) for *i* = 1*,* 2. As *f* has connected fibres, *Y*1 *∩ Y*2 = *∅* (because for each connected subset *C ⊆ X* we clearly have *C ∩ X*1 = *∅* or *C ∩ X*2 = *∅*) and thus *Xi* = *f−*1(*Yi*) for *i* = 1*,* 2. Since *f* is a quotient map, we have *Y*1 *∪ Y*2 = *Y* and, by Lemma 1.1, *Y*1 and *Y*2 are closed. Therefore, *Y*1 or *Y*2 is empty, hence *X*1 or *X*2 is empty. Consequently, (*X, u*) is connected. *✷*

**Corollary 1.5** *Let* (*X, u*)*,* (*Y, v*) *be closure spaces, let v be idempotent and let f* : (*X, u*) *→* (*Y, v*) *be a quotient map having connected ﬁbres. Then a subset B ⊆ Y is connected in* (*Y, v*) *if and only if f−*1(*B*) *is connected in* (*X, u*)*.*

**Proof.** The statement follows from Corollary 1.3 and Proposition 1.4. *✷*

**Remark 1.6** Recall that for topological spaces (and topological quotient maps) the Proposition 1.4 and Corollary 1.5 are not valid in general. They are valid provided that *B* is open or closed (see e.g. [3]).

From now on, *n* will denote a natural number with *n >* 1. As natural num- bers are understand to be finite cardinals, we always have *n* = *{*0*,* 1*, ..., n−* 1*}*. Given a set *X*, we denote by *Xn* the set of all maps of *n* into *X*, i.e., ordered *n*-touples (*xi| i < n*) consisting of elements of *X*. Ordered *n*-touples will be sometimes referred to as finite sequences. Any subset *R ⊆ Xn* is called an

*n*-*ary relation* on *X* and the pair (*X, R*) is called an *n*-*ary relational system*. An *n*-ary relation on a set *X* is said to be *reflexive* if it contains all constant

*n*-touples consisting of elements of the set *X*. Given *n*-ary relational systems (*X, R*) and (*Y, S*), a map *ϕ* : *X → Y* is called a *homomorphism* of (*X, R*) into (*Y, S*) if the implication (*xi| i < n*) *∈ R ⇒* (*ϕ*(*xi*)*| i < n*) *∈ S* is valid. If *Rj* is an *n*-ary relation on a set *Xj* for each *j ∈ J*, then the *product* of the system *Rj*,

*j ∈ J*, is the *n*-ary relation

*j∈J*

*Rj* on the cartesian product

*j∈J*

*Xj* gener-

ated by the projections *prj* : *j∈J Xj → Xj*, *j ∈* *J* (i.e., the greatest - with

respect to the set inclusion - *n*-ary relation *R* on

*j∈J Xj* such that all pro-

jections *prj* : ( *j∈J Xj, R*) *→* (*Xj, Rj*), *j ∈ J*, are homomorphisms). Clearly,

*j∈J Rj* = *{*(*xi| i < n*) *∈* ( *j∈J Xj*)*n*; (*prj*(*xi*)*| i < n*) *∈ Rj* for each *j ∈ J}*.

Analogously to the cartesian product of sets, also the product of a pair *R, S*

of *n*-ary relations will be usually denoted by *R × S*.

Let (*X, R*)*,* (*Y, S*) be *n*-ary relational systems and *f* : (*X, R*) *→* (*Y, S*) a surjective map. Then *f* is a quotient map (i.e., *S* is the least - with respect to the set inclusion - *n*-ary relation on *Y* such that *f* : (*X, R*) *→* (*Y, S*) is a homomorphism) if and only if, whenever (*yi| i < n*) *∈ Y n*, we have

(*yi| i < n*) *∈ S ⇔ ∀i < n ∃xi ∈ f−*1(*yi*) : (*xi| i < n*) *∈ R*. It is obvious that

relational systems are quotient-productive, i.e., if (*Xi, Ri*)*,* (*Yi, Si*) are *n*-ary relational systems a nd *fi* : (*X* *i, Ri*) *→* (*Y, Si*) is a quotient map for each *i ∈ I*,

then also the map

*i∈I fi* :

*i∈I* (*Xi, Ri*) *→*

*i∈I* (*Yi, Si*) is quotient.

# Closure operations associated with relations

Most results and definitions of this paragraph are taken from [15] (where all proofs not presented here can be found). Let *X* be a set and *R* an *n*-ary

relation on *X*. Then we define a map *uR*: exp *X →* exp *X* as follows:

*uRA* = *A ∪ {x ∈ X*; there exist (*xi| i < n*) *∈ R* and *i*0*,* 0 *< i*0 *< n,* such that *x* = *xi*0 and *xi ∈ A* for all *i < i*0*}*.

Clearly, *uR* is a closure operation on *X*. It can be shown that *uR* is idem- potent if and only if (*X, uR*) is an Alexandroff topological space. Of course, *uR* is not additive in general, but, on the othere hand, the union of a system of closed subsets of (*X, uR*) is a closed subset of (*X, uR*).

Obviously, we have:

**Proposition 2.1** *For any n-ary relation R on a set X,* (*X, uR*) *is an Sn- space.*

We will need the following assertion:

**Proposition 2.2** *Let* (*X, R*)*,* (*Y, S*) *be n-ary relational systems and let f* : (*X, R*) *→* (*Y, S*) *be a quotient map. Then f* : (*X, uR*) *→* (*Y, uS*) *is a quotient map.*

**Proof.** Let *B ⊆ Y* , *y ∈ Y* and suppose that *y ∈ uSB*. Then there exist (*yi| i < n*) *∈ S* and a natural number *i*0, 0 *< i*0 *< n*, such that *y* = *yi*0 and *yi ∈ B* for all *i < i*0. Further, for each *i < n* there exists *xi ∈ f−*1(*yi*) such that (*xi| i < n*) *∈ R*. Now we have *y* = *f* (*xi*0 ) and *xi*0 *∈ uR{xi| i < i*0*} ⊆ uRf−*1(*B*). Hence *y ∈ f* (*uRf−*1(*B*)).

Conversely, suppose that *y ∈ f* (*uRf−*1(*B*)). Then there is *x ∈ uRf−*1(*B*) such that *f* (*x*) = *y*. Next, there exist (*xi| i < n*) *∈ R* and a natural number

*i*0, 0 *< i*0 *< n*, such that *x* = *xi* and *xi ∈ f−*1(*B*) for each *i < i*0. Since (*f* (*xi*)*| i < n*) *∈ S*, we have *y* = *f* (*xi*0 ) *∈ uS{f* (*xi*); *i < i*0*} ⊆ uSB* (because

0

*{f* (*xi*); *i < i*0*}⊆ B*). Thus, *vB* = *f* (*uf−*1(*B*)) and the proof is complete. *✷*

**Definition 2.3** An *n*-ary relation *R* on a set *X* is said to be *terse* provided that it is reflexive and fulfils the following condition:

If (*xi| i < n*) *∈ R*, (*yi| i < n*) *∈ R*, and there are natural numbers *i*0*, i*1 *< n*, *i*0 */*= *i*1, such that *x*0 = *yi*0 and *x*1 = *yi*1 , then (*xi| i < n*) = (*yi| i < n*).

Thus, an *n*-ary relation *R* is terse if and only if *R* is reflexive and any nonconstant *n*-touple (*xi| i < n*) *∈ R* is injective and is the only *n*-touple belonging to *R* which contains the elements *x*0 and *x*1. So, a binary relation is terse if and only if it is reflexive and antisymmetric.

**Proposition 2.4** *Let R be a terse n-ary relation on a set X. Then* (*X, uR*) *is a T*0*-space and R is a minimal element (with respect to the set inclusion) of the set of all reflexive α-ary relations S on X fulﬁlling uR* = *uS.*

**Proposition 2.5** *For terse n-ary relations the correspondence R '→ uR is one-to-one.*

**Definition 2.6** A closure operation *u* on a set *X* and the closure space (*X, u*) are called an *S∗*-*closure operation* and and *S∗*-*space* respectively if there is a

*n n*

terse *n*-ary relation *R* on *X* such that *u* = *uR*. This (unique) relation *R* will

be denoted by *Ru*.

Clearly, the (relational) product of a system of terse *n*-ary relations is a terse *n*-ary relation. This fact, together with Proposition 2.5, enables us to define:

**Definition 2.7** Let (*Xj, uj*), *j ∈ J*, be a system of *S∗*-spaces. By the *product*

of this system we understand the *S∗*-space (

*n*

*n*

*j∈J Xj, uR*) where *R* =

*j∈J Ruj* .

The product of a pair (*X, u*)*,* (*Y, v*) of *S∗*-spaces will be denoted by (*X, u*)*×*

*n*

(*Y, v*).

**Remark 2.8** Let (*Xj, uj*), *j ∈ J*, be a system of *S∗*-spaces and let *v* be the

closure operation on

*n*

*j∈J Xj* generated by the projections *prj* :

*j∈J Xj →*

*Xj*, *j ∈ J*. If (

*j∈J Xj, u*) is the product of the system defined in 2.7, then it

can easily be seen that *u ≤ v*. The equality *u* = *v* is valid, in general, only for

*n* = 2.

**Definition 2.9** Let *R* be an *n*-ary relation on a set *X* and let *p* be a natural number with 1 *< p ≤ n*. An ordered *p*-touple (*yi| i < p*) of points of *X* is called a *connected element* in (*X, uR*) if there is an *n*-touple (*xi| i < n*) *∈ R* such that *yi* = *xi* for all *i < p* or *yi* = *xp−*1*−i* for all *i < p*.

Clearly, each connected element is a connected set. We will need the following observation which immediately follows from Definition 2.9:

**Lemma 2.10** *Let* (*yi| i < p*) *be a connected element in* (*X, uR*) *and let* (*xi| i < n*) *∈ R be an n-touple with yi* = *xi for all i < p or yi* = *xp−*1*−i for all i < p. If i*0 *< p is a natural number, then either* (*yi*0*−i| i ≤ i*0) *or* (*yi| i*0 *≤ i < p*) *is a connected element in* (*X, uR*) *with the ﬁrst member yi*0 *and the last one x*0*.*

**Definition 2.11** Let *R* be an *n*-ary relation on a set *X*. A finite nonempty sequence *C* = (*xi| i < m*) of points of *X* is called a *path* in (*X, uR*) if, whenever *m >* 1, there is a finite increasing sequence (*jk| k < p*) of natural numbers with *j*0 = 0 and *jp−*1 = *m −* 1 such that (*xj| jk ≤ j ≤ jk*+1) is a connected element in (*X, uR*) for each natural number *k < p −* 1.

Clearly, each path is a connected set and each connected element is a path. If (*xi| i < m*) is a path, then also its *inversion*, i.e., the sequence (*yi| i < m*) where *yi* = *xm−*1*−i* for all *i < m*, is a path. Further, if (*xi| i < m*), (*yi| i < p*) are paths such that *xm−*1 = *y*0, then also their *union*, i.e., the sequence (*zi| i < m* + *p −* 1) where *zi* = *xi* for all *i < n* and *zi* = *yi−m*+1 for all *i* with *m ≤ i < p*, is a path.

**Theorem 2.12** *Let R be an n-ary relation on a set X and A ⊆ X be a subset. Then A is connected in* (*X, uR*) *if and only if any two points of A can be joined by a path in* (*X, uR*) *contained in A.*

**Lemma 2.13** *Let* (*X*0*, R*0)*,* (*X*1*, R*1) *be reflexive n-ary relational systems and let* (*yj| i < pj*) *be a connected element in* (*Xj, uR* ) *for each j* = 0*,* 1*. Then*

*i j*

*{y*0; *i < p*0*}× {y*1; *i < p*1*} is a connected set in* (*X*0*, uR* ) *×* (*X*1*, uR* )*.*

*i i* 0 1

**Proof.** For each *j* = 0*,* 1, there is an *n*-touple (*xj| i < n*) *∈ Rj* such that

*i*

*i*

= *x*

*i*

*j* = *xj*

*y*

*i*

*i*

for all *i < pj* or *yj*

*j pj−*1*−i*

for all *i < pj*. Let *y ∈ {y*0; *i < p*0*}×*

*{y*1; *i < p*1*}* be an arbitrary element. Then there are natural numbers *i*0*, i*1

*i*

with *i*0 *< p*0*, i*1 *< p*1 such that *y* = (*y*0 *, y*1 ). By Lemma 2.10, (*y*0 *| i ≤ i*0)

*i*0 *i*1 *i*0*−i*

or (*y*0*| i*0 *≤ i < p*0) is a connected element in (*X*0*, uR* ) with the first member

*i* 0

*y*0 and the last one *x*0. Denote this connected element by (*z*0*| i < q*0) and put

*i*0 0 *i*

*C*0 = ((*z*0*, y*1 )*| i < q*0). As (*X*1*, R*1) is reflexive, *C*0 is a connected element in

*i i*1

(*X*0*, uR* ) *×* (*X*1*, uR* ) whose last member equals (*x*0*, y*1 ). Further, by Lemma

0 1 0 *i*1

2.10, (*y*1 *| i ≤ i*1) or (*y*1*| i*1 *≤ i < p*1) is a connected element in (*X*1*, uR* ) with

*i*1*−i i* 1

the first member *y*1 and the last one *x*1. Denote this connected element by

0

0

(*z*1*| i < q*1) and put *C*1 = ((*z*0 *, z*1)*| i < q*1). As (*X*0*, R*0) is reflexive, *C*1 is a

*i q*0*−*1 *i*

connected element in (*X*0*, uR* )*×*(*X*1*, uR* ) whose first member equals (*x*0*, y*1 ).

0 1 0 *i*1

Now, the union of the connected elements *C*1 and *C*2 is a connected element

in (*X*0*, uR* ) *×* (*X*1*, uR* ) with the first member (*y*0 *, y*1 ) = *y* and the last one

0 1 *i*0 *i*1

(*x*0*, x*1). We have shown that any point *y ∈ {y*0; *i < p*0*}× {y*1; *i < p*1*}* can

0 1 *i* *i*

be joined with the point (*x*0*, x*1) *∈ {y*0; *i < p*0*}× {y*1; *i < p*1*}* by a path

0 0 *i* *i*

contained in *{y*0; *i < p*0*}× {y*1; *i < p*1*}*. Now the statement follows from

*i* *i*

Theorem 2.12. *✷*

**Lemma 2.14** *Let* (*X*0*, R*0)*,* (*X*1*, R*1) *be reflexive n-ary relational systems and let Cj* = (*xj| i < pj*) *be a path in* (*Xj, uR* ) *for each j* = 0*,* 1*. Then {x*0; *i <*

*i j* *i*

*p*0*}× {x*1; *i < p*1*} is a connected set in* (*X*0*, uR* ) *×* (*X*1*, uR* )*.*

*i* 0 1

**Proof.** If *C*0 or *C*1 contains only one member, then the assertion is trivial because (*X*0*, R*0) and (*X*1*, R*1) are reflexive. So, suppose that both *C*0 and *C*1 contain more than one member. By Definition 2.11., for each *j* = 0*,* 1 there is a finite increasing sequence (*ij | k < qj*) of natural numbers with

*k*

*ij* = 0 and *ij* = *pj −* 1 such that (*xj| ij ≤ i ≤ ij*

) is a connected

0 *qj−*1 *i k k*+1

element in (*Xj, uRj* ) for each natural number *k < qj −* 1. For each *j* = 0*,* 1,

putting *Cj* = *{xj*; *ij*

*≤ i ≤ ij }*, we get *{xj*; *i < pj}* =

*Cj*. Thus,

*k i k*

*k*+1 *i*

*k<qj−*1 *k*

*{x*0; *i < p*0*}× {x*1; *i < p*1*}* =

(*C*0 *× C*1 ) where *C*0

*× C*1

*i i k*0*<q*0*−*1

*k*1*<q*1*−*1 *k*0 *k*1

*k*0 *k*1

is connected in (*X*0*, uR*0 ) *×* (*X*1*, uR*1 ) for any *k*0 *< q*0 *−* 1 and any *k*1 *< q*1 *−* 1

by Lemma 2.13. Thus, whenever *k*0 *< q*0 *−* 1, (*C*0 *× C*1 *| k*1 *< qm−*1 *−* 1) is

*k*0

*k*1

a finite sequence of connected sets such that any two consecutive members of

(*C*

it have nonempty intersection. Therefore, the set *Dk*0

=

*k*1*<q*1*−* 1

0 *× C*1 )

is a connected set. We have *{x*0; *i < p*0*}× {x*1; *i < p*1*}* =

*Dk* .

*i i k*0*<q*0*−*1 0

*k*1

*k*0

This proves the statement because any two consecutive members of the finite

sequence (*Dk*0 *| k*0 *< q*0 *−* 1) have nonempty intersection. *✷*

**Theorem 2.15** *Let* (*X*0*, R*0)*,* (*X*1*, R*1) *be reflexive n-ary relational systems and let Yj ⊆ Xj be a subset for each j* = 0*,* 1*. Then Yj is connected in* (*Xj, uRj* ) *for each j* = 0*,* 1 *if and only if Y*0 *× Y*1 *is connected in* (*X*0*, uR*0 ) *×* (*X*1*, uR*1 )*.*

**Proof.** Let *Yj* be connected in (*Xj, uRj* ) for each *j* = 0*,* 1 and let (*x*0*, x*1)*,* (*y*0*, y*1) *∈ Y*0 *× Y*1 be arbitrary points. Then, for each *j* = 0*,* 1, there is a path (*zj| i < pj*) in (*X*0*, uR* ) *×* (*X*1*, uR* ) joining the points *xj* and *yj* which

*i* 0 1

is contained in *Yj*. As *{z*0; *i < p*0*} × {z*1; *i < p*1*}* contains the points

*i* *i*

(*x*0*, x*1)*,* (*y*0*, y*1) and is connected in (*X*0*, uR*0 ) *×* (*X*1*, uR*1 ) by Lemma 2.14,

there is a path in (*X*0*, uR*0 ) *×* (*X*1*, uR*1 ) joining the points (*x*0*, x*1) and (*y*0*, y*1)

which is contained in *{z*0; *i < p*0*}× {z*1; *i < p*1*}*. Thus, *Y*0 *× Y*1 is connected

*i* *i*

in (*X*0*, uR* ) *×* (*X*1*, uR* ) because *{z*0*| i < p*0*}× {z*1*| i < p*1*}⊆ Y*0 *× Y*1.

0 1 *i* *i*

Conversely, let *Y*0 *× Y*1 be connected in (*X*0*, uR*0 ) *×* (*X*1*, uR*1 ) and let *v* denote the closure operation on *X*0 *× X*1 generated by the projections *prj* : *X*0 *× X*1 *→ Xj*, *j* = 0*,* 1. By Remark 2.8, *u ≤ v*. Thus, as the projections *prj* : (*X*0 *× X*1*, v*) *→* (*Xj, uRj* ), *j* = 0*,* 1, are continuous, also the projections *prj* : (*X*0*, uR*0 )*×*(*X*1*, uR*1 ) *→* (*Xj, uRj* ), *j* = 0*,* 1, are continuous. Consequently, *Yj* = *prj*(*Y*0 *× Y*1) is connected in (*Xj, uRj* ) for each *j* = 0*,* 1. *✷*

# *n*-ary digital plane

We denote by Z the set of integers and define an *n*-ary relation *Rn* on Z as follows:

*Rn* = *{*(*xi| i < n*) *∈* Z*n*; (*xi| i < n*) is constant or there exists an odd number *l ∈* Z fulfilling either *xi* = *l*(*n −* 1) + *i* for all *i < n* or *xi* = *l*(*n −* 1) *− i* for all *i < n}*.

The relation *Rn* is demonstrated on the following picture where the non- constant *n*-touples of *Rn* are represented as arrows (oriented from the first to the last members of the sequences):

... ✲✛

✲✛ ✲✛

...

-3(n-1) -2(n-1) -(n-1) 0 n-1 2(n-1) 3(n-1)

*∗*

It is evident that *Rn* is terse so that *uRn* is an *S* -closure operation on

*n*

Z. Instead of *uRn* we will write briefly *un*. The closure operation *u*2 coincides with the Khalimsky topology on Z generated by the subbase *{{*2*k−* 1*,* 2*k,* 2*k* +

1*}*; *k ∈* Z*}* - cf. [8]. The relation *R−*1

2

is nothing else than the so-called

*specialization order* of *u*2. Clearly, *un* is additive if and only if *n* = 2.

By [15], we have:

**Proposition 3.1** (Z*, un*) *is a connected S∗-space in which the points l*(*n−* 1)*, l ∈* Z *odd, are open, while all the other points are closed (so that un is a T* 1 *-closure operation).*

*n*

2

Let *R*˜*n* be the *n*-ary relation on Z given as follows:

*R*˜*n* = *{*(*xi| i < n*) *∈* Z*n*; (*xi| i < n*) is a constant *n*-touple or *xn−*1 *∈* Z is an even number with either *xi* = *xn−*1 *−*1 for all *i < n−*1 or *xi* = *xn−*1 +1 for all *i < n −* 1*}.*

Further, let *fn* : Z *→* Z be the surjection defined by

*fn*(*l*(*n −* 1)) = *l* for each even number *l ∈* Z*,*

*fn*(*l*(*n −* 1) + *i*) = *l* for each odd number *l ∈* Z and each *i ∈* Z with

*|i| < n−*1.

**Theorem 3.2** *fn* : (Z*, Rn*) *→* (Z*, R*˜*n*) *it a quotient map.*

**Proof.** Let (*yi| i < n*) *∈ R*˜*n*. If (*yi| i < n*) is constant, say *yi* = *y* for all *i < n*, we can choose an arbitrary element *x ∈ f−*1(*y*). Then, putting *xi* = *x* for all *i < n*, we get (*xi| i < n*) *∈ Rn*. Let (*yi| i < n*) be not constant. Then *yn−*1 *∈* Z is even and *yi* = *yn−*1 *−* 1 for all *i < n −* 1 or *yi* = *yn−*1 + 1 for all *i < n −* 1. Suppose that *yi* = *yn−*1 *−* 1 for all *i < n −* 1 and put *xi* = (*yn−*1 *−* 1)(*n −* 1) + *i* for all *i < n*. Then (*xi| i < n*) *∈ Rn* and we have *f−*1(*yn−*1) = *{yn−*1(*n −* 1)*}* = *{xn−*1*}*, and *xi ∈ {*(*yn−*1 *−* 1)(*n −* 1) + *i*; *i <*

*n*

*n*

*n} ⊆ f−*1(*yn−*1 *−* 1) = *f−*1(*yi*) for any *i < n −* 1. Further, suppose that

*n n*

*yi* = *yn−*1 + 1 for all *i < n −* 1 and put *xi* = (*yn−*1 + 1)(*n −* 1) *− i*. Again,

(*xi| i < n*) *∈ Rn* and we have *f−*1(*yn−*1) = *{yn−*1(*n −* 1)*}* = *{xn−*1*}*, and

*n*

*xi ∈ {*(*yn−*1 + 1)(*n −* 1) *− i*; *i < n} ⊆ f−*1(*yn−*1 + 1) = *f−*1(*yi*) for any

*n n*

*i < n −* 1.

Conversely, let (*yi| i < n*) *∈* Z*n* and for each *i < n* let there exist *xi ∈ f−*1(*yi*) such that (*xi| i < n*) *∈ Rn*. If (*yi| i < n*) is constant, then clearly (*yi| i < n*) *∈ R*˜*n*. Let (*yi| i < n*) be not constant. Then there exists an even number *l* with *xn−*1 = *l*(*n −* 1). Thus, *yn−*1 = *l*. Since *xi* = (*l −* 1)(*n −* 1) + *i* for all *i < n −* 1 or *xi* = (*l* + 1)(*n −* 1) *− i* for all *i < n −* 1, we have *yi* = *l −* 1 for all *i < n −* 1 or *yi* = *l* + 1 for all *i < n −* 1. Therefore, (*yi| i < n*) *∈ R*˜*n*. The proof is complete. *✷*

*n*

**Theorem 3.3** *u* e

*R*

*n*

= *u*2*.*

**Proof.** Let *A ⊆* Z and *x ∈ u* e

*R*

*n*

*A*. If *x ∈ A*, then *x ∈ u*2*A*. Suppose that

*x ∈/ A*. Then there exist (*xi| i < n*) *∈ R*˜*n* and *i*0, 0 *< i*0 *< n*, such that

*x* = *xi*0 and *xi ∈ A* for all *i < i*0. Since *xn−*1 is even and *xn−*2 = *xn−*1 *−* 1 or

*xn−*2 = *xn−*1 + 1, we have *xn−*2*R*2*xn−*1. As clearly *i*0 = *n −* 1, we get *x ∈ u*2*A*.

We have shown that *u* e

*R*

*n*

*≤ u*2.

Conversely, let *x ∈ u*2*A*. If *x ∈ A*, then *x ∈ u* e *A*. Suppose that *x ∈/ A*.

*Rn*

Then there exists *y ∈ A* such that *x ∈ u*2*{y}*. Consequently, *yR*2*x* and thus *x*

is even and *y* = *x −* 1 or *y* = *x* + 1. Now, putting *xi* = *y* for all *i < n −* 1 and

*xn−*1 = *x*, we get (*xi| i < n*) *∈ R*˜*n* and *xi ∈ A* for each *i < n −* 1. Therefore

*x ∈ u* e

*R*

*Rn*

*n*

*A*. We have shown that *u*2 *≤ u* e

which completes the proof. *✷*

**Corollary 3.4** *fn* : (Z*, un*) *→* (Z*, u*2) *is a quotient map.*

**Proof.** The statement follows from Theorems 3.2, 3.3 and Proposition 2.2.*✷*

We put (Z2*, vn*) = (Z*, un*) *×* (Z*, un*), so that *vn* = *uR*

*n*

*×Rn*

. The closure

space (Z2*, vn*) will be called the *n*-*ary digital plane*. Clearly, the product (Z*, u*2)*×*(Z*, u*2) coincides with the usual topological product and so the binary digital plane coincides with the Khalimsky plane (cf. [8]). As a consequence of 3.1 we get:

**Proposition 3.5** (Z2*, vn*) *is a connected S∗-space in which a point z* = (*z*1*, z*2)

*n*

*∈* Z2 *is open if and only if z*1 = *k*(*n −* 1) *and z*2 = *l*(*n −* 1) *where k, l ∈* Z *are odd, and it is closed if and only if z*1 */*= *k*(*n −* 1) *for all odd numbers k ∈* Z *and z*2 */*= *l*(*n −* 1) *for all odd numbers l ∈* Z*.*

We denote by *Fn* : (Z2*, vn*) *→* (Z2*, v*2) the map *Fn* = *fn × fn*. Clearly, if

*k, l ∈* Z, then *F−*1 equals the singleton *{*(*k*(*n −* 1)*, l*(*n −* 1))*}* whenever *k, l*

*n*

are even, the ”square” *{*(*x, y*) *∈* Z2; *|x − k| < n −* 1*, |y − l| < n −* 1*}* of (2*n −* 3)2 points whenever *k, l* are odd, the ”abscissa” *{*(*x, y*) *∈* Z2; *|x − k| < n −* 1*, y* = *l}* of 2*n −* 3 points whenever *k* is odd and *l* is even, and the ”abscissa” *{*(*x, y*) *∈* Z2; *x* = *k, |y − l| < n −* 1*}* of 2*n −* 3 points whenever *k* is even and *l* is odd.

**Theorem 3.6** *Fn* : (Z2*, vn*) *→* (Z2*, v*2) *is a quotient map with connected ﬁ- bres.*

**Proof.** *Fn* is a quotient map because of Theorems 3.2, 3.3, Proposition 2.2, Definition 2.7 and the fact that relational systems are quotient-productive. Let (*x, y*) *∈* Z2 be a point. Then *F−*1(*x, y*) = *f−*1(*x*) *× f−*1(*y*) and both the

*n n n*

sets *f−*1(*x*) and *f−*1(*y*) are connected in (Z*, vn*) (as each of them is a singleton

*n n*

or the union of two connected elements in (Z*, vn*) having a common member). Thus, *F−*1(*x, y*) is connected in (Z2*, vn*) by Theorem 2.15. *✷*

*n*

Let *J ⊆* Z2 be a subset and *z ∈ J* a point. Then we put

*Jz* = *{*(*zi| i < n*) *∈ Rn × Rn*; (*zi| i < n*) is a non-constant *n*-touple with *z ∈ {zi*; *i < n}⊆ J}*.

**Definition 3.7** A *simple closed curve* in (Z2*, vn*) is a finite connected set

*J ⊆* Z2 that satisfies the following two conditions:

1. For any point *z ∈ J* there exists an *n*-touple (*zi| i < n*) *∈ Jz* with

*z*0 = (*k*(*n −* 1)*, l*(*n −* 1)), *k, l ∈* Z.

1. If *z ∈ J* is a point with *z* = (*k*(*n −* 1)*, l*(*n −* 1)), *k, l ∈* Z, then card *Jz* = 2.

For *n >* 2, simple closed curves in the *n*-ary digital plane are precisely circles in the following graph whose vertices are points of Z2 (but only the points (*k*(*n −* 1)*, l*(*n −* 1)), *k, l ∈* Z, are marked out and thus, on any edge connecting a pair of points there are another (*n −* 2) points):

4(n-1) .❅

. /.❅

. /.❅

. /.

❅ / ❅ / ❅ /

3(n-1) .

❅ /

/❅. .

❅ /

/❅. .

❅ /

/❅. .

/ ❅ / ❅ / ❅

2(n-1) ./

❅

. ❅./

/❅

. ❅./

/❅

. ❅.

/

❅ / ❅ / ❅ /

n-1 .

❅ /

/❅. .

❅ /

/❅. .

❅ /

/❅. .

/ ❅ / ❅ / ❅

./ . ❅./ . ❅./ . ❅.

0 n-1 2(n-1)3(n-1)4(n-1)5(n-1)6(n-1)

Note that for *n* = 2 the condition (1) of 3.7 is always satisfied and the simple closed curves with at least four points coincide with the COTS-Jordan curves from [4]. So, by [4] we have

**Theorem 3.8** *Any simple closed curve J in* (Z2*, v*2) *having at liest four points separates* (Z2*, v*2) *into precisely two components (i.e., the subspace* Z2 *− J of* (Z2*, v*2) *consists of precisely two components).*

Corollary 1.5 and Theorems 3.6 and 3.8 result in

**Theorem 3.9** *Let J ⊆* Z2 *be a set such that Fn*(*J*) *is a simple closed curve in* (Z2*, vn*) *having at least four points. If F−*1(*Fn*(*J*)) = *J, then J separates*

*n*

(Z2*, vn*) *into precisely two components.*

The sets *J* from Theorem 3.9 need not be simple closed curves in (Z2*, vn*) in general and so this Theorem is not a satisfactory analogy of the classical Jordan curve theorem. In what follows we will give a criterion under which a given simple closed curve in the space (Z2*, vn*) separates this space into precisely two components.

**Definition 3.10** A simple closed curve *J* in (Z2*, vn*) is said to be a *Jordan curve* provided that for any point *z ∈ J* with *z* = (*k*(*n −* 1)*, l*(*n −* 1)), *k, l ∈* Z odd, from *{*((*xi, yi*)*| i < n*)*,* ((*x' , y'*)*| i < n*)*}* = *Jz* it follows that *|xn−*1 *−*

*'*

*x*

*n−*1

*|* = *|yn−*1 *− y'*

*i i*

*|* = 2(*n −* 1).

*n−*1

Clearly, if (*zi| i < n*) *∈ Rn × Rn* and *z* = (*k*(*n −* 1)*, l*(*n −* 1)) *∈* Z2 where *k, l ∈* Z are odd, then from *z ∈ {zi*; *i < n}* it follows that *z* = *z*0. Thus, the condition in Definition 3.10 means that *J* can turn only at the points (*k*(*n −* 1)*, l*(*n −* 1)) for which *k, l ∈* Z are even. It immediately follows that *Fn*(*J*) has at least six points for any Jordan curve *J* in (Z2*, vn*). For *n >* 2, Jordan curves are nothing else than circles in the following graph whose vertices are points of Z2 (but only the points (*k*(*n −* 1)*, l*(*n −* 1)) for even *k, l ∈* Z are marked out and thus, on any edge connecting a pair of points there are another 2*n −* 1 points):

6(n-1) .

|  |  |  |
| --- | --- | --- |
| .❅ /  ❅/  ./ ❅ | .❅ /  ❅/  ./ ❅ | .❅ /  ❅/  ./ ❅ |
| ❅ /  ❅/  ./ ❅ | ❅ /  ❅/  ./ ❅ | ❅ /  ❅/  ./ ❅ |
| ❅ /  ❅/  ./ ❅ | ❅ /  ❅/  ./ ❅ | ❅ /  ❅/  ./ ❅ |

4(n-1) .

2(n-1) .

.

0 2(n-1)4(n-1)6(n-1)

**Theorem 3.11** *Let J be a simple closed curve in* (Z2*, vn*) *and let for any k, l ∈* Z *from* (*k*(*n −* 1)*, l*(*n −* 1)) *∈ J it follows that k or l is even. Then J is a Jordan curve in* (Z2*, vn*) *which separates* (*Z*2*, vn*) *into precisely two components.*

**Proof.** Clearly, *J* is a Jordan curve in (Z2*, vn*), *Fn*(*J*) is a Jordan curve in (Z2*, v*2) having at least eight points and *F−*1(*Fn*(*J*)) = *J*. Thus, the statement follows from Corollary 1.5 and Theorems 3.6 and 3.8. *✷*

*n*

The Jordan curves from Theorem 3.11 are said to be *rectangular* because they are just the circles in the following graph:

6(n-1) .

.

.

.

.

.

.

.

.

.

.

.

.

4(n-1) .

2(n-1) .

.

0 2(n-1)4(n-1)6(n-1)

**Definition 3.12** Let *n >* 2 and let *k, l ∈* Z be odd numbers. Then we set:

*A*1(*k, l*) = *F−*1(*{*(*k −* 1*,l −* 1)*,* (*k, l −* 1)*,* (*k* + 1*,l −* 1)*}*),

*n*

*A*2(*k, l*) = *F−*1(*{*(*k −* 1*,l* + 1)*,* (*k, l* + 1)*,* (*k* + 1*,l* + 1)*}*),

*n*

*B*1(*k, l*) = *F−*1(*{*(*k −* 1*,l −* 1)*,* (*k −* 1*, l*)*,* (*k −* 1*,l* + 1)*}*),

*n*

*B*2(*k, l*) = *F−*1(*{*(*k* + 1*,l −* 1)*,* (*k* + 1*, l*)*,* (*k* + 1*,l* + 1)*}*),

*n*

∆1(*k, l*) = *{*(*x, y*) *∈* Z2; *|x − k*(*n −* 1)*| < n −* 1*,y* = *l*(*n −* 1) *− x* + *k*(*n −* 1)*}*,

∆2(*k, l*) = *{*(*x, y*) *∈* Z2; *|x − k*(*n −* 1)*| < n −* 1*,y* = *l*(*n −* 1) + *x − k*(*n −* 1)*}*,

*T*1(*k, l*) = *{*(*x, y*) *∈* Z2; *|x − k*(*n −* 1)*| < n −* 1*,* (*l −* 1)(*n −* 1) *< y < l*(*n −* 1) *−*

*x* + *k*(*n −* 1)*}*,

*T*2(*k, l*) = *{*(*x, y*) *∈* Z2; *|x − k*(*n −* 1)*| < n −* 1*, l*(*n −* 1) + *x − k*(*n −* 1) *< y <*

(*l* + 1)(*n −* 1)*}*,

*T*3(*k, l*) = *{*(*x, y*) *∈* Z2; *|x − k*(*n −* 1)*| < n −* 1*,* (*l −* 1)(*n −* 1) *< y < l*(*n −* 1) +

*x − k*(*n −* 1)*}*,

*T*4(*k, l*) = *{*(*x, y*) *∈* Z2; *|x − k*(*n −* 1)*| < n −* 1*, l*(*n −* 1) *− x* + *k*(*n −* 1) *< y <*

(*l* + 1)(*n −* 1)*}*.

Graphically, the sets *A*1(*k, l*), *A*2(*k, l*), *B*1(*k, l*) and *B*2(*k, l*) are the lower, upper, left and right sides of the square *F−*1(*k, l*) respectively which do not meet this square. The sets ∆1(*k, l*) and ∆2(*k, l*) are the increasing and decreas- ing diagonals of the square *F−*1 (which are contained in the square). Finally, *T*1(*k, l*) and *T*4(*k, l*) are the left lower and right upper triangles respectively

*n*

*n*

obtained by separating the square *F−*1 by the decreasing diagonal ∆1(*k, l*)

*n*

(which are included in the square but do not meet the diagonal). Similarly,

*T*2(*k, l*) and *T*3(*k, l*) are the left upper and right lower triangles respectively

obtained by separating the square *F−*1 by the increasing diagonal ∆2(*k, l*)

*n*

(which are included in the square but do not meet the diagonal).

**Lemma 3.13** *Let n >* 2 *and let k, l be odd numbers. Then each of the ten sets introduced in Denotation 3.12 is connected in* (Z2*, vn*)*.*

**Proof.** The sets *A*1(*k, l*), *A*2(*k, l*), *B*1(*k, l*) and *B*2(*k, l*) are inverse images under *Fn* of connected sets in (Z2*, v*2). Thus, they are connected by Corollary

1.5 and Theorem 3.6. It is evident that also the sets ∆1(*k, l*) and ∆2(*k, l*) are

connected because each of them is the union of a pair of connected elements having a common member. Now, consider the set *T*1(*k, l*). Then, for each *x ∈* Z with (*k −* 1)(*n −* 1) *< x < k*(*n −* 1), the set *Cx* = *{*(*x, y*) *∈* Z2; (*l −* 1)(*n −* 1) *< y < l*(*n −* 1) *− x* + *k*(*n −* 1)*}* is connected (as it is formed by

a connected element in the case *x* = *k*(*n −* 1) *−* 1 or is a union of a pair of connected elements having a common member otherwise). Similarly, for each *y ∈* Z with (*l −* 1)(*n −* 1) *< y < l*(*n −* 1), the set *Dy* = *{*(*x, y*) *∈* Z2; (*k −* 1)(*n −* 1) *< x < l*(*n −* 1) *− y* + *k*(*n −* 1)*}* is connected (as it is formed by a connected element in the case *y* = *l*(*n −* 1) *−* 1 or is a union of

a pair of connected elements having a common member otherwise). Clearly,

*T*1(*k, l*) = *{Cx*; (*k −* 1)(*n −* 1) *< x < k*(*n −* 1)*}∪ {Dy*; (*l −* 1)(*n −* 1) *<*

*y < l*(*n −* 1)*}*. For each natural number *i < n −* 2 put *C'* = *C*(*k−*1)(*n−*1)+1+*i*,

*i*

*D'* = *D*(*l−*1)(*n−*1)+1+*i* and *Ei* = *C' ∪ D'*. Then (*C'| i < n −* 2) = (*Cx|* (*k −*

*i i i* *i*

1)(*n −* 1) *< x < k*(*n −* 1)) and (*D'| i < n −* 2) = (*Dy|* (*l −* 1)(*n −* 1) *<*

*i*

*y < l*(*n −* 1)). Thus, *T*1(*k, l*) = *i<n−*2 *Ei* where *Ei* is a connected set for

each *i < n −* 2 (because *C' ∩ D'* = *{*(*x*(*k−*1)(*n−*1)+1+*i, y*(*l−*1)(*n−*1)+1+*i*)*}*). Since

*i* *i*

*D' ∩C'* = *{*(*x*(*k−*1)(*n−*1)+1*, y*(*l−*1)(*n−*1)+1+*i*)*}* for each *i < n−*2, we have *Ei ∩E*0 */*=

*i* 0

*∅* for each *i < n−*2. Consequently, *T*1(*k, l*) is connected in (Z2*, vn*). For *T*2(*k, l*), *T*3(*k, l*) and *T*4(*k, l*) the proofs are analogous. *✷*

We will need the following, quite obvious statement:

**Lemma 3.14** *Let J be a simple closed curve in* (Z2*, v*2) *having at liest four points and let k, l ∈* Z *be odd numbers with* (*k, l*) *∈ J. If* (*k −* 1*,l −* 1) *∈ J and* (*k* + 1*,l* + 1) *∈ J, then the points* (*k −* 1*, l*) *and* (*k, l* + 1) *belong to one component of* Z2 *− J while the points* (*k, l −* 1) *and* (*k* + 1*, l*) *belong to the other. Similarly, if* (*k −* 1*,l* + 1) *∈ J and* (*k* + 1*,l −* 1) *∈ J, then the points* (*k, l −* 1) *and* (*k −* 1*, l*) *belong to one component of* Z2 *− J while the points* (*k, l* + 1) *and* (*k* + 1*, l*) *belong to the other.*

**Theorem 3.15** *Let J be a Jordan curve in* (Z2*, vn*) *satisfying the condition that, whenever there are odd numbers k, l ∈* Z *such that* (*k*(*n−*1)*, l*(*n−*1)) *∈ J, J contains just two of the four points* ((*k ±* 1)(*n −* 1)*,* (*l ±* 1)(*n −* 1))*. Then J separates* (Z2*, vn*) *into precisely two components.*

**Proof.** Clearly, the condition from Theorem 3.15 is necessary and sufficient for *Fn*(*J*) to be a simple closed curve in (Z2*, v*2) (having at least six points). For *n* = 2 the statement immediately follows from Theorem 3.8. Suppose that *n >* 2. If (*k*(*n −* 1)*, l*(*n −* 1)) *∈/ J* whenever *k, l ∈* Z are odd, then *J* is rectangular and the statement follows from Theorem 3.11. So, let there be odd numbers *k, l ∈* Z with (*k*(*n −* 1)*, l*(*n −* 1)) *∈ J* and put *J'* = *F−*1(*Fn*(*J*)). By Corollary 1.5 and Theorems 3.6 and 3.8, *J'* separates (Z2*, vn*) into precisely two components and *J ⊆ J'*, *J /*= *J'*. Clearly, *J* contains just one of the sets

*n*

∆1(*k, l*), ∆2(*k, l*). We will assume, without loss of generality, that ∆1(*k, l*) *⊆*

*J*. Then *F−*1(*k −* 1*,l* + 1) *∪ F−*1(*k* + 1*,l −* 1) *⊆ J*, thus (*k −* 1*,l* + 1) *∈*

*n n*

*Fn*(*J*) and (*k* + 1*,l −* 1) *∈ Fn*(*J*). Clearly, for any (*x, y*) *∈* ∆1(*k, l*) we have

*F−*1(*Fn*(*x, y*)) = *T*1(*k, l*)*∪*∆1(*k, l*)*∪T*4(*k, l*). The condition from Theorem 3.15 implies that *F−*1(*k−*1*, l−*1)*∩J* = *∅* and *F−*1(*k*+1*, l*+1)*∩J* = *∅*. Consequently,

*n*

*n n*

*F−*1(*k, l−* 1) *∩J* = *F−*1(*k, l* +1) *∩J* = *F−*1(*k−* 1*, l*) *∩J* = *F−*1(*k* +1*, l*) *∩J* = *∅*.

*n n n n*

Put *z*1 = (*k*(*n −* 1) *−* 1*,* (*l −* 1)(*n −* 1)) and *z*2 = (*k*(*n −* 1)+ 1*,* (*l* + 1)(*n −* 1)).

Then *z*1 *∈ F−*1(*k, l −* 1) and *z*2 *∈ F−*1(*k, l* + 1), hence *z*1 *∈/ J'* and *z*2 *∈/ J'*. Let

*n n*

*E*1*, E*2 be the components of Z2 *− J* with *z*1 *∈ E*1 and *z*2 *∈ E*2. By Corollary 1.5, Theorem 3.6 and Lemma 3.14, *E*1 and *E*2 are different. Clearly, we have *z*1 *∈ vnT*1(*k, l*) and *z*2 *∈ vnT*4(*k, l*). Thus, as *T*1(*k, l*) and *T*4(*k, l*) are connected by Lemma 3.13, the sets *E*1 *∪ T*1(*k, l*) and *E*2 *∪ T*4(*k, l*) are connected in the subspace Z2*−J*1 of (Z2*, vn*) where *J*1 = *J'−*(*T*1(*k, l*)*∪T*4(*k, l*)). Obviously, *E*1*∪ T*1(*k, l*) and *E*2 *∪T*4(*k, l*) are disjoint and *E*1 *∪T*1(*k, l*) *∪E*2 *∪T*4(*k, l*) = Z2 *−J*1. As Z2 *−J*1 is not connected (because *Fn*(Z2 *−J*1) = Z2 *−Fn*(*J*1) = Z2 *−Fn*(*J*) is not connected), the sets *E*1 *∪ T*1(*k, l*) and *E*2 *∪ T*4(*k, l*) are components of Z2 *− J*1. Consequently, *J*1 separates (Z2*, vn*) into precisely two components. Now, if there are odd numbers *k*1*, l*1 *∈* Z such that (*k*1(*n −* 1)*, l*1(*n −* 1)) *∈ J*, in the next step we repeat the previous considerations for *k*1*, l*1 and *J*1 instead

of *k, l* and *J*. By this way we obtain a set *J*2 which separates (Z2*, vn*) into precisely two components. After a finite number *m* of steps we will get a set *Jm* which separates (Z2*, vn*) into precisely two components and contains no point (*k*(*n −* 1)*, l*(*n −* 1)) with *k, l ∈* Z odd. Clearly, *Jm* = *J* which proves the

statement. *✷*

The Jordan curves *J* in (Z2*, vn*) satisfying the condition from Theorem

3.15 are said to be *without acutes*. Thus, every rectangular Jordan curve is without acutes.

Whenever *n >* 2, each Jordan curve J in (Z2*, vn*) clearly satisfies card *J ≥* 6(*n −* 1). The shortest Jordan curves *J* in (Z2*, vn*), i.e., those satisfying card *J* = 6(*n −* 1), are said to be *elementary*. Of course, if *J* is an elementary Jordan curve in (Z2*, vn*), then there are odd numbers *k, l ∈* Z such that *J* is the union of one of the sets *A*1(*k, l*)*, A*2(*k, l*), one of the sets *B*1(*k, l*)*, B*2(*k, l*), and one of the sets ∆1(*k, l*)*,* ∆1(*k, l*).

**Theorem 3.16** *Let n >* 2 *and let J be an elementary Jordan curve in*

(Z2*, vn*)*. Then J separates* (Z2*, vn*) *into precisely two components.*

**Proof.** We will suppose, without loss of generality, that *J* = *A*1(*k, l*) *∪ B*1(*k, l*) *∪* ∆1(*k, l*). Put *J'* = *A*1(*k, l*) *∪ B*1(*k, l*) *∪ A*2(*k, l*) *∪ B*2(*k, l*). Then *Fn*(*J'*) = *{*(*x, y*) *∈* Z2; max*{|x − k|, |y − l|}* = 1*}* is a simple closed curve in (Z2*, v*2) with card *Fn*(*J'*) = 8. Thus, *Fn*(*J'*) separates (Z2*, v*2) into pre- cisely two components by Theorem 3.8. As *J'* = *F−*1(*Fn*(*J'*)), *J'* sepa- rates (Z2*, vn*) into precisely two components by Corollary 1.5 and Theorem

*n*

3.6. Clearly, one of these components is the set *C* = *F−*1(*k, l*) = *T*1(*k, l*) *∪ T*4(*k, l*) *∪* ∆1(*k, l*), hence the other is the set Z2 *−* (*C ∪ J'*). By Lemma 3.13, *T*1(*k, l*) is connected in (Z2*, vn*) and thus it is connected also in the subspace Z2 *− J* of (Z2*, vn*) because *T*1(*k, l*) *⊆* Z2 *− J*. We have Z2 *−* (*J ∪ T*1(*k, l*)) = (Z2 *−* (*C ∪ J'*)) *∪* (*A*2(*k, l*) *− {z*1*}*) *∪* (*B*2(*k, l*) *− {z*2*}*) *∪ T*4(*k, l*)

*n*

where *{z*1*}* = *F−*1(*k −* 1*,l* + 1) and *{z*2*}* = *F−*1(*k* + 1*,l −* 1). Clearly,

*n n*

Z2*−*(*C∪J'*) *⊆* Z2*−J* and (*A*2(*k, l*)*−{z*1*}*)*∪*(*B*2(*k, l*)*−{z*2*}*) *⊆ vn*(Z2*−*(*C∪J'*)).

Thus, (Z2*−*(*C∪J'*))*∪*(*A*2(*k, l*)*−{z*1*}*)*∪*(*B*2(*k, l*)*−{z*2*}*) is connected in Z2*−J*.

Let *z*0 = ((*k* + 1)(*n−* 1)*, l*(*n−* 1) + 1). Then *z*0 *∈* (*B*2(*k, l*) *− {z*2*}*) *∩ vnT*4(*k, l*). Consequently, as *T*4(*k, l*) is connected in Z2 *− J* by Lemma 3.13, the set Z2 *−* (*J ∪T*1(*k, l*)) is connected in Z2 *−J* too. As any path in (Z2*, vn*) connect- ing a point of *T*1(*k, l*) with another one of Z2 *−* (*J ∪ T*1(*k, l*)) clearly contains a point of *J*, the sets Z2 *−* (*J ∪ T*1(*k, l*)) and *T*1(*k, l*) are components of the subspace Z2 *− J* of (Z2*, vn*). *✷*

**Lemma 3.17** *Let n >* 2 *and let k, l ∈* Z *be odd numbers. Put A* = *T*1(*k, l*) *∪ F−*1(*{*(*k −* 1*, l*)*,* (*k −* 1*,l* + 1)*}*) *∪* ∆1(*k, l*) *and W* = *T*4(*k −* 2*, l*) *∪ F−*1(*k −* 2*,l* +

*n n*

1) *∪T*3(*k−* 1*,l−* 1) *∪F−*1(*k−* 1*,l* + 2) *∪T*1(*k, l* + 2) *∪F−*1(*k, l* + 1) *∪T*4(*k, l*)*. Let*

*n n*

*B ⊇ A be a connected set in* (Z2*, vn*) *such that B ∩ F−*1(*{*(*k −* 1*,l −* 1)*,* (*k, l −*

*n*

1)*,* (*k*+1*, l−*1)*}*) = *∅ and W ⊆ B. If F−*1(*k*+1*, l*) *⊆ B or F−*1(*k*+1*, l*)*∩B* = *∅,*

*n n*

*then B − A is a connected set in* (Z2*, vn*)*.*

**Proof.** Note that *A* and *W* are disjoint. The sets *F−*1(*k −* 2*,l* + 1), *F−*1(*k −*

*n n*

1*,l*+2), *F−*1(*k, l*+1) and *F−*1(*k*+1*, l*) are connected because *Fn* has connected

*n n*

fibres. By Lemma 3.13, also the sets *T*4(*k −* 2*, l*), *T*3(*k −* 1*,l −* 1), *T*1(*k, l* + 2)

and *T*4(*k, l*) are connected. We clearly have ((*k−* 1)(*n−* 1) *−* 1*,* (*l* +1)(*n−* 1)) *∈ F−*1(*k −* 2*,l* + 1) *∩vnT*4(*k −* 2*, l*) *∩vnT*3(*k −* 1*,l −* 1), ((*k −* 1)(*n−* 1)*,* (*l* + 1)(*n−* 1)+ 1) *∈ F−*1(*k −* 1*,l* + 2) *∩vnT*3(*k −* 1*,l −* 1) *∩vnT*1(*k, l* + 2), ((*k −* 1)(*n−* 1) +

*n*

*n*

1*,* (*l*+1)(*n−*1)) *∈ F−*1(*k, l*+1)*∩vnT*1(*k, l*+2) and (*k*(*n−*1)+1*,* (*l*+1)(*n−*1)) *∈*

*n*

*F−*1(*k, l* + 1) *∩ vnT*4(*k, l*). Consequently, *W* is a connected set. Further, we clearly have ((*k −* 1)(*n −* 1) *−* 1*,* (*l −* 1)(*n −* 1) + 1) *∈ vnT*4(*k −* 2*, l*) *⊆ vnW* . Therefore, *W ∪ {*((*k −* 1)(*n −* 1) *−* 1*,* (*l −* 1)(*n −* 1) + 1)*}* is connected. As *F−*1(*k* + 1*,l* + 1) *∈ vnF−*1(*k, l* + 1) *∈ vnW* , also *W ∪F−*1(*k* + 1*, l*) is connected.

*n*

*n n n*

We will show that for any path connecting two different points of *B − A* which is contained in *B* and meets *A* there is a path connecting these points which is contained in *B − A*. To this end, let *C* = (*zi| i < m*) be a path

contained in *B* such that *z*0*, zm−*1 *∈ B − A* and *{zi*; *i < m}∩ A /*= *∅*. Then there is a finite increasing sequence (*jk| k < p*) of natural numbers with *j*0 = 0 and *jp−*1 = *m −* 1 such that (*zj| jk ≤ j ≤ jk*+1) is a connected element for each *k < p −* 1. Let *k*1 *< p −* 1 be the least natural number having the

property that the connected element *Ck*1 = (*zi| jk*1 *≤ i ≤ jk*1+1) meets *A* and let *k*2 *< p −* 1 be the greatest natural number having the property that the

connected element *Ck*2 = (*zi| jk*2 *≤ i ≤ jk*2+1) meets *A*. Then *zjk*

1

*∈ B − A*,

*zjk* +1 *∈ B − A* and both *Ck*1 and *Ck*2 meet *A − T*1(*k, l*).

2

First, suppose that *Ck*1 *∩* ∆1(*k, l*) = *∅* = *Ck*2 *∩* ∆1(*k, l*). Then it is evident

that *Ck*1 *∈ Rn × Rn*, *Ck*1 *∩ A* = *{zjk* +1 *}* and *zjk* +1*−*1 *∈ W ∪ E* where *E* = *∅* if

1 1

((*k−*1)(*n−*1)*−*1*,* (*l−*1)(*n−*1)+1) *∈/ B* and *E* = *{*((*k−*1)(*n−*1)*−*1*,* (*l−*1)(*n−*

1) + 1)*}* if ((*k −* 1)(*n −* 1) *−* 1*,* (*l −* 1)(*n −* 1) + 1) *∈ B*. Similarly, denoting the

inversion of *Ck*

by *C−* , we get *C− ∈ Rn × Rn*, *Ck ∩ A* = *{zj }* and *zj*

+1 *∈*

2 *k*2 *k*2 2 *k*2 *k*2

*W ∪ E*. Thus, *C'* = (*zi| jk ≤ i < jk* +1) and *C'* = (*zi| jk < i ≤ jk* +1) are

*k*1 1 1 *k*2 2 2

connected elements contained in *B − A*. As *W ∪ E* is connected, there is a

path *C'* connecting *zj*

*−*1 and *zj*

+1 which is contained in *W ∪ E ⊆ B − A*.

*k*1+1 *'' k*2

*' ' '*

Consequently, the union *C*

of the paths *Ck*1 , *C*

and *Ck*2 is a path connecting

*zjk* and *zjk* +1 which is contained in *B − A*. Now, the union of the paths

1

(*zi| i ≤ j*

2

*k*1 ),

*C''* and (*zi| j*

*k*2+1

*≤ i < m*) is a path connecting *z*0

and *zm−*1

which is contained in *B − A*.

Next, let *Ck*1 *∩* ∆1(*k, l*) */*= *∅* or *Ck*2 *∩* ∆1(*k, l*) */*= *∅*. We can suppose, without loss of generality, that *Ck*1 *∩* ∆1(*k, l*) */*= *∅* (because otherwise we

can replace *C* by its inversion). Then *zjk*

1

*∈ W ∪ G ∪ H* where *G* = *∅* if

*F−*1(*k* + 1*,l* + 1) *∩ B* = *∅* and *G* = *F−*1(*k* + 1*,l* + 1) if *F−*1(*k* + 1*,l* + 1) *⊆ B*,

*n n n*

and *H* = *∅* if *F−*1(*k* + 1*, l*) *∩B* = *∅* and *H* = *F−*1(*k* + 1*, l*) if *F−*1(*k* + 1*, l*) *⊆ B*.

*n*

First, suppose that *Ck*2

*n*

*∩* ∆1(*k, l*) = *∅*. Then *C'*

*k*2

= (*zi| jk*2

*n*

*< i ≤ jk*2+1) is a

connected element with *zjk* +1 *∈ W ∪E* which is contained in *B−A* and whose

inversion belongs to *Rn*

*k*

2

*× Rn*

. Thus, there is a path *C'* connecting *zj*

and

*zjk* +1

1

which is contained in *W ∪ E ∪ G ∪ H ⊆ B − A*. Hence, the union *C*

*''* of

2

the paths *C'*

and *C'* is a path connecting *zj* and *zj* which is contained in

*B−A*. Now, the union of the paths (*zi| i ≤ jk* ), *C''* and (*zi| jk* +1 *≤ i ≤ m*) is

*k*2 *k*1 *k*2+1

1 2

a path connecting *z*0 and *zm−*1 which is contained in *B − A*. Finally, suppose

that *Ck*2 *∩* ∆1(*k, l*) */*= *∅*. Then *zjk*2+1 *∈ W ∪ G ∪ H*, thus there is a path

*C'* connecting *zj* and *zj* which is contained in *W ∪ G ∪ H ⊆ B − A*.

*k*1

*k* +1

2

*'*

Therefore, the union of the paths (*zi| i ≤ jk*1 ), *C* and (*zi| jk*2+1 *≤ i < m*)

is a path connecting *z*0 and *zm−*1 which is contained in *B − A*. The proof is

complete. *✷*

**Remark 3.18** Of course, statements analogical to Lemma 3.17 are valid also for *A* = *T*4(*k, l*) *∪ F−*1(*{*(*k* + 1*,l −* 1)*,* (*k* + 1*, l*)*}*) *∪* ∆1(*k, l*), *A* = *T*2(*k, l*) *∪*

*n*

*F−*1(*{*(*k−*1*,l−*1)*,* (*k−*1*, l*)*}*)*∪*∆2(*k, l*), and *A* = *T*3(*k, l*)*∪F−*1(*{*(*k* +1*, l*)*,* (*k* +

*n n*

1*,l* + 1)*}*) *∪* ∆2(*k, l*).

The following digital analogy of the classical Jordan curve theorem shows that, for any *n >* 1, the *n*-ary digital plane provides a structure suitable for solving problems of digital image processing:

**Theorem 3.19** *Let J be a Jordan curve in* (Z2*, vn*)*. Then J separates* (Z2*, vn*)

*into precisely two components.*

**Proof.** For *n* = 2 the statement follows from Theorem 3.8. Let *n >* 2. If

*J* is without acutes or elementary, then the statement follows from Theorem

3.15 or 3.16 respectively. Suppose that *J* is neither without acutes nor ele- mentary. Then there are odd numbers *k*1*, l*1 *∈* Z such that ∆1(*k*1*, l*1) *⊆ J* or

∆2(*k*1*, l*1) *⊆ J*. We will suppose, without loss of generality, that ∆1(*k*1*, l*1) *⊆ J* (so that *J ∩* ∆2(*k*1*, l*1) = *∅*). Clearly, *J* contains at least one of the sets *A*1(*k*1*, l*1)*, A*2(*k*1*, l*1)*, B*1(*k*1*, l*1)*, B*2(*k*1*, l*1). We will suppose, without loss of generality, that *B*1(*k*1*, l*1) *⊆ J*. Obviously, *A*2(*k*1*, l*1) is not a subset of *J* and, as *J* is not elementary, neither *A*1(*k*1*, l*1) is a subset of *J*. Put *J*1 = (*J −* (*B*1(*k*1*, l*1) *∪* ∆1(*k*1*, l*1))) *∪ A*1(*k*1*, l*1). Clearly, *J*1 is a Jordan curve in (Z2*, vn*). Now, if *J*1 is neither withou acutes nor elementary, there are odd numbers *k*2*, l*2 *∈* Z such that ∆1(*k*2*, l*2) *⊆ J* or ∆2(*k*2*, l*2) *⊆ J*. In the next step, we apply the previous considerations to *J*1 and (*k*2*, l*2) to ob- tain a new Jordan curve *J*2 = (*J*1 *−* (*B*1(*k*2*, l*2) *∪* ∆1(*k*2*, l*2))) *∪ A*1(*k*2*, l*2),

and so on. After a finite number *m* of steps we get a Jordan curve *Jm* = (*Jm−*1 *−* (*B*1(*km, lm*) *∪* ∆1(*km, lm*))) *∪ A*1(*km, lm*) which is without acutes or elementary.

By Theorem 3.15 or 3.16, *Jm* separates (Z2*, vn*) into precisely two compo- nents. Denote these components by *Dm* and *Em*. As *T*1(*km, lm*) is connected (by Lemma 3.13) and *T*1(*km, lm*) *∩ Jm* = *∅*, *T*1(*km, lm*) is contained in just one of the components *Dm, Em*. We will suppose, without loss of generality, that *T*1(*km, lm*) *⊆ Em*. Clearly, we have *Jm−*1 = (*Jm − A*1(*km, lm*)) *∪ B*1(*km, lm*) *∪ F−*1(*km*+1*, lm−*1). Put *Dm−*1 = *Dm∪F−*1(*km, lm−*1)*∪T*1(*km, lm*) and *Em−*1 =

*n n*

*Em −* (*T*1(*km, lm*) *∪ F−*1(*{*(*km −* 1*, lm*)*,* (*km −* 1*, lm* + 1)*}*) *∪* ∆1(*km, lm*)). Then

*n*

*Dm−*1 *∩Em−*1 = *∅* and *Dm−*1 *∪Em−*1 = Z2 *−Jm−*1. Evidently, *T*2(*km, lm −* 2) *⊆ Dm* or *T*4(*km, lm −* 1) *⊆ Dm*. We will suppose, without loss of generality, that *T*2(*km, lm −*2) *⊆ Dm*. Then (*km*(*n−*1)*−*1*,* (*lm −*1)(*n−*1)) *∈ vnT*2(*km, lm −*2) *⊆ vnDm* and (*km*(*n −* 1) *−* 1*,* (*lm −* 1)(*n −* 1)) *∈ F−*1(*km, lm −* 1). As *Fn* has con- nected fibres, *F−*1(*km, lm −* 1) is connected and thus *Dm ∪ F−*1(*km, lm −* 1)

*n*

*n n*

is connected too. Further, *T*1(*km, lm*) is connected by Lemma 3.13 and hence *T*1(*km, lm*) *∪F−*1(*km, lm −* 1) is connected (because (*km*(*n−* 1) *−* 1*,* (*lm −* 1)(*n−* 1)) *∈ vnT*1(*km, lm*)). Consequently, *Dm−*1 is connected. From Lemma 3.17 it follows that also *Em−*1 is connected.

*n*

Let (*zi| i < n*) *∈ Rn×Rn* be an *n*-touple with *z*0 *∈ Dm−*1 and *zi*0 *∈ Em−*1 for some *i*0*,* 0 *< i*0 *< n*. If *z*0 *∈ Dm*, then there is *i*1*,* 0 *< i*1 *< i*0, such that *zi*1 *∈ Jm − A*1(*km, lm*) *⊆ Jm−*1 because *Dm* is closed in Z2 *− Jm* and any *n*-touple of *Rn×Rn* whose first member does not belong to *A*1(*km, lm*) can meet *A*1(*km, lm*) in the last member only. Let *z*0 *∈/ Dm*, i.e., let *z*0 *∈ F−*1(*km, lm−*1)*∪T*1(*km, lm*). Then *z*0 *∈ {*(*x, y*) *∈* Z2; (*km −* 1)(*n−* 1) *< x < km −* 1*, y* = *lm −* 1*}∪ {*(*x, y*) *∈* Z2; *x* = *km*(*n−*1)*,* (*lm −*1)(*n−*1) *< y < lm −*1*}*. Thus, there is *i*1*,* 0 *< i*1 *< i*0,

*n*

such that *zi*1 *∈* ∆1(*km, lm*) *⊆ Jm−*1. Consequently, *Dm−*1 is closed in Z *−Jm−*1.

2

Similarly, let (*zi| i < n*) *∈ Rn × Rn* be an *n*-touple with *z*0 *∈ Em−*1 and

*zi*0 *∈ Dm−*1 for some *i*0*,* 0 *< i*0 *< n*. If *zi*0 *∈ Dm*, then there is *i*1*,* 0 *< i*1 *< i*0,

such that *zi*1 *∈ Jm − A*1(*km, lm*) *⊆ Jm−*1 because *Dm* is closed in Z *− Jm* and

2

any *n*-touple of *Rn × Rn* whose first member does not belong to *A*1(*km, lm*)

can meet *A*1(*km, lm*) in the last member only. Let *zi*0 *∈/ Dm*. Then *z*0 *∈*

*{*(*x, y*) *∈* Z2; *x* = *km*(*n −* 1)*, lm*(*n −* 1) *< y <* (*lm* + 1)(*n −* 1) *−* 1*}∪ {*(*x, y*) *∈* Z2; *km*(*n −* 1) *< x <* (*km* + 1)(*n −* 1) *−* 1*, y* = *lm*(*n −* 1)*}*. But then there is *i*1*,* 0 *< i*1 *< i*0, such that *zi*1 *∈* ∆1(*km, lm*) *⊆ Jm−*1. Consequently, *Dm−*1 is closed in Z2 *− Jm−*1.

As both *Dm−*1 and *Em−*1 are closed in Z2 *− Jm−*1, they are components of Z2 *− Jm−*1, i.e., *Jm−*1 separates (Z2*, vn*) into precisely two components. If *m >* 1, in the next step we apply the previous considerations to *Jm−*1 instead of *Jm* and show that *Jm−*2 separates (Z2*, vn*) into precisely two components *Dm−*2 and *Em−*2. Afer *m* steps we get a set *J*0 which separates (Z2*, vn*) into precisely two components *D*0 and *E*0. As obviously *J*0 = *J*, the proof is

complete. *✷*

If *n >* 2, then Jordan curves in the *n*-ary digital plane can turn (at the points (*k*(*n −* 1)*, l*(*n −* 1)) with *k, l ∈* Z even) at all angles *pπ* , *p* = 1*,* 2*,* 3. But this is not true for *n* = 2 because COTS-Jordan curves in the Khalimsky plane cannot turn at the acute *π* (and at the so-called *mixed points* of Z2, i.e., the points one coordinate of which is even and the second is odd, they can never turn). So, it can be useful to work with an *n*-ary digital plane where *n >* 2. Especially, for *n* = 3 the graph placed after Definition 3.10 has the following form:

4

4

12 . . . . . . . . . . . . .

|  |  |  |
| --- | --- | --- |
| .❅. . /.  . .❅/. .  ../. . ❅.  . . . | .❅. . /.  . .❅/. .  . . . .  ./. . ❅. | .❅. . /.  . .❅/. .  . . . .  ./. . ❅. |
| .❅. . /.  . .❅/. .  ../. . ❅.  . . . | .❅. . /.  . .❅/. .  . . . .  ./. . ❅. | .❅. . /.  . .❅/. .  . . . .  ./. . ❅. |
| .❅. . /.  . .❅/. .  ../. . ❅.  . . . | .❅. . /.  . .❅/. .  . . . .  ./. . ❅. | .❅. . /.  . .❅/. .  . . . .  ./. . ❅. |

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10 .

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8 .

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6 .

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4 .

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2 .

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0 2 4 6 8 10 12

**Example 3.20** Consider the following (digital picture of a) triangle:

E =(4,4)

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. .

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. .

. . . . . . . . .

(0,0)= A B C D =(0,0)

While in the ternary digital plane the triangle ADE is a Jordan curve, in the Khalimsky plane it is not even a COTS-Jordan curve. In order that this triangle be a COTS-Jordan curve in the Khalimsky plane, we have to delete the points A,B,C and D (and then it will become even a Jordan curve). But this will lead to a deformation of the triangle.

**Theorem 3.21** *Let J be a Jordan curve in* (Z2*, v*2) *and let D, E be the components of* Z2 *− J. Then both D ∪ J and E ∪ J are connected subsets of* (Z2*, v*2)*.*

**Proof.** First, suppose that *J* is rectangular. Then there is a point *z* = (*k*(*n −* 1)+1*, l*(*n−*1)) with *k, l ∈* Z even such that *z ∈ J*. Put *A*1 = *{*(*k*(*n−*1)+1*,* (*l*+ 1)(*n−* 1) *− i*); 0 *≤ i < n−* 1*}* and *A*2 = *{*(*k*(*n−* 1)+ 1*,* (*l −* 1)(*n−* 1) + *i*); 0 *≤*

*i < n −* 1*}*. Clearly, *A*1 and *A*2 are contained in different components of Z2 *−J*. We can suppose, without loss of generality, that *A*1 *⊆ D* and *A*2 *⊆ E*. As we obviously have *z ∈ vnA*1 *∪ vnA*2, *D ∪ {z}* and *E ∪ {z}* are connected. Consequently, *D ∪ J* and *E ∪ J* are connected.

Further, suppose that *J* is not rectangular. Then there is a point *z* = (*k*(*n −* 1)*, l*(*n −* 1)) with *k, l ∈* Z odd such that *z ∈ J*. Put *z*1 = (*k*(*n −* 1) *−* 1*, l*(*n −* 1)) and *z*2 = (*k*(*n −* 1) + 1*, l*(*n −* 1)). Clearly, *z*1 and *z*2 belong to different components of Z2 *− J*. We can suppose, without loss of generality, that *z*1 *∈ D* and *z*2 *∈ E*. Obviously, we have *{z*1*, z*2*}⊆ vn{z}* and thus *D∪{z}* and *E ∪ {z}* are connected. Consequently, *D ∪ J* and *E ∪ J* are connected.

The proof is complete. *✷*

**Remark 3.22** If J is a COTS-Jordan curve in (Z2*, v*2) and *D, E* are the components of Z2 *− J*, then *D ∪ {z}* and *E ∪ {z}* are connected subsets of (Z2*, v*2) for each point *z ∈ J*. For a Jordan curve *J* in (Z2*, vn*), *n >* 2, thit is not true in general (it is true, however, whenever J is rectangular). Indeed, if *n >* 2 and *J* is an elementary Jordan curve in (Z2*, vn*) such that, say, ∆1(*k, l*) *⊆ J* and ((*k −* 1)(*n −* 1)*,* (*k* + 1)(*n −* 1)) *∈ J* where *k, l ∈* Z are odd numbers, and if *D* is the finite component of Z2 *− J* and *z* = ((*k −* 1)(*n−* 1)*,* (*k* + 1)(*n−* 1)), then *D ∪ {z}* is not connected.

As a concluding remark let us note that the *n*-ary digital planes with *n >* 2 can be useful also in the cases when we want to work with a topological struc- ture of Z2 that is more ”continuous” than the Alexandroff topology provided by the Khalimsky topology.

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