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A Polynomial-time Approximation Scheme for the MAXSPACE Advertisement Problem

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**Abstract**

In the MAXSPACE problem, given a set of ads *A*, one wants to place a subset *A′ ⊆ A* into *K* slots *B*1*,..., BK* of size *L*. Each ad *Ai ∈ A* has a *size si* and a *frequency wi*. A schedule is feasible if the total size of ads in any slot is at most *L*, and each ad *Ai ∈ A′* appears in exactly *wi* slots. The goal is to find a feasible schedule which maximizes the sum of the space occupied by all slots. We introduce a generalization, called MAXSPACE-RD, in which each ad *Ai* also has a release date *ri ≥* 1 and a deadline *di ≤ K*, and may only appear in a slot *Bj* with *ri ≤ j ≤ di*. These parameters model situations where a subset of ads corresponds to a commercial campaign with an announcement date that may expire after some defined period. We present a polynomial-time approximation scheme for MAXSPACE-RD when *K* is bounded by a constant, i.e., for any *ε >* 0, we give a polynomial-time algorithm which returns a solution with value at least (1 *− ε*)*Opt* , where *Opt* is the optimal value. This is the best factor one can expect, since MAXSPACE

is NP-hard, even if *K* = 2.

*Keywords:* Approximation Algorithm, PTAS, Scheduling of Advertisements, MAXSPACE.

# Introduction

Many websites (such as Google, Yahoo!, Facebook and others) offer free services while displaying advertisements, or simply ads, to users. Often, each website has a single strip of fixed height which is reserved for scheduling ads, and the set of displayed ads changes on a time basis. For such websites, the advertisement is the main source of revenue. Thus, it is important to find the best way to dispose the ads in the available time and space while maximizing the revenue [[7](#_bookmark25)].

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The revenue from web advertising grew considerably in the 21st century. In 2013, the total revenue was US$42.78 billion, an increase of 17% from the previous year. It is estimated that the U.S. web advertising reached US$77 billion in 2016, and comprised 35% of all advertising spending, overtaking television advertising [[7](#_bookmark25)]. In 2016, ads in banners comprised 31,4% of internet advertising (considering banners and mobile platforms), which represents a revenue of US$22.7 billion [[8](#_bookmark26)]. Web advertising has created a multi-billionaire industry where algorithms for scheduling advertisements play an important role.

We consider the class of Scheduling of Advertisements Problems introduced by [Adler et al.](#_bookmark19) [[1](#_bookmark19)], where, given a set *A* = *{A*1*, A*2*,..., An}* of advertisements, the goal is to schedule a subset *Aj ⊆A* into a banner in *K* equal time-intervals. The set of ads scheduled to a particular time interval *j*, 1 *≤ j ≤ K*, is represented by a set of ads *Bj ⊆ Aj*, which is called a *slot*. Each ad *Ai* has a *size si* and a *frequency wi* associated with it. The size *si* represents the amount of space *Ai* occupies in a slot and the frequency *wi ≤ K* represents the number of slots which should contain a copy of *Ai*. An ad *Ai* can be displayed at most once in a slot and *Ai* is said to be *scheduled* if *wi* copies of *Ai* appear in slots with at most one copy per slot [[1,](#_bookmark19) [4](#_bookmark20)].

The main problems in that class are MINSPACE and MAXSPACE. In MINSPACE, all the ads are to be scheduled in the slots, and the goal is to minimize the height of the highest slot. In MAXSPACE, an upper bound *L* is specified which represents the size of each slot. A feasible solution for this problem is a schedule of a subset *Aj ⊆A* into slots *B*1*, B*2*,..., BK*, such that each *Ai ∈ Aj* is scheduled and the fullness of any slot does not exceed the upper bound *L*, that is, for each slot *B* , *s ≤ L*. The goal of MAXSPACE is to maximize the fullness of the slots, defined by *Ai∈A′ siwi*. Both of these problems are strongly NP-hard [[1,](#_bookmark19) [4](#_bookmark20)]. [Dawande et al.](#_bookmark20) [[4](#_bookmark20)] define three special cases of MAXS- PACE: MAX*w*, MAX*K|w* and MAX*s*. In MAX*w*, every ad has the same frequency

*j* Σ*Ai∈Bj i*

[Σ](#_bookmark20)

*w*. In MAX*K|w* , every ad has the same frequency *w* and the number of slots *K* is a multiple of *w*. And, in MAX*s*, every ad has the same size *s*. In an analogous way, they define three special cases of MINSPACE: MIN*w*, MIN*K|w* and MIN*s*.

[Adler et al.](#_bookmark19) [[1](#_bookmark19)] present a 1 -approximation called SUBSET-LSLF for MAXS- PACE when the ad sizes form a sequence *s*1 *> s*2 *> s*3 *> . . .*, such that for all *i*, *si* is a multiple of *si*+1. [Dawande et al.](#_bookmark20) [[4](#_bookmark20)] present three approximation algorithms, a ( 1 + 1 )-approximation for MAXSPACE, a 1 -approximation for MAX*w* and a 1 -

2

4 4*K* 3 2

approximation for MAX*K|w* . [Freund and Naor](#_bookmark24) [[6](#_bookmark24)] proposed a ( 1 *− ε*)-approximation for MAXSPACE and a ( 1 *− ε*)-approximation for the special case in which the size of ads are in the interval [*L/*2*, L*].

3

2

[Adler et al.](#_bookmark19) [[1](#_bookmark19)] present a 2-approximation called *Largest-Size Least-Full* (LSLF) for MINSPACE. The algorithm LSLF is also a ( 4 *−*  1 )-approximation to

3 3*K/w*

MIN*K|w* [[4](#_bookmark20)]. [Dawande et al.](#_bookmark20) [[4](#_bookmark20)] present a 2-approximation for MINSPACE using

*LP Rouding*, and [Dean and Goemans](#_bookmark21) [[5](#_bookmark21)] present a 4 -approximation for MINSPACE

3

using Graham’s algorithm for schedule.

In practice, the time interval relative to each slot in scheduling advertising can represent minutes, seconds or long periods, such as days and weeks. Often, one

considers the idea of *release dates* and *deadlines*. An ad has a release date that indicates the beginning of its advertising campaign. Analogously, the deadline of an ad indicates the end of its advertising campaign. For example, ads for Christmas must be scheduled before December, 25th.

We introduce a MAXSPACE generalization called MAXSPACE-RD in which each ad *Ai* has two additional parameters, a release date *ri ≥* 1 and a dead- line *di ≤ K*. The release date of ad *Ai* represents the first slot where a copy of *Ai* can be scheduled, that is, a copy of *Ai* cannot be scheduled in a slot *Bj* with *j < ri*. Similarly, the deadline of an ad *Ai* represents the last slot where we can schedule a copy of *Ai*, thus *Ai* cannot be scheduled in a slot *Bj* with *j > di*. We assume that the frequency of each ad *Ai* is compatible with its release date and deadline, that is, *di − ri* +1 *≥ wi*.

Let Π be a maximization problem. A family of algorithms *{Hε}* is a *Polynomial- Time Approximation Scheme* (PTAS) for Π if, for every constant *ε >* 0, *Hε* is a (1 *− ε*)-approximation for Π [[9](#_bookmark27)]. A *Fully Polynomial-Time Approximation Scheme* (FPTAS) is a PTAS whose running time is also polynomial in 1*/ε*. In this work, we present a PTAS to MAXSPACE-RD when the number of slots is a constant. This approximation is the best one can expect, since MAXSPACE is NP-hard even when the number of slots is 2 [[1,](#_bookmark19) [4](#_bookmark20)].

In Section [2](#_bookmark1) we define the notation and concepts used in this work. In Section [3](#_bookmark4) we present an algorithm to schedule small ads and in Section [4](#_bookmark14) we present a PTAS to the whole set of ads. In Section [5](#_bookmark18) we discuss the results and future works.

# Preliminaries

In what follows, assume that the number of slots *K* is a constant, that *L* =1 and 0 *< si ≤* 1 for each *Ai ∈ A*.

We partition the ads into two groups: *large ads* and *small ads*. For a con-

stant *ε >* 0, we say that *Ai* is a large ad if *si*

*≥ ε/*(22*K* 2*KK*), otherwise we say it

is a small ad. Then, let *G* = *{Ai*

*∈A| si*

*≥ ε/*(22*K* 2*KK*)*}* be the set of large ads

and *P* = *A\ G* be the set of small ads.

Let *S* denote a feasible solution *Aj ⊆A* scheduled into slots *B*1*, B*2*,..., BK*.

Then the *fullness* of a slot *Bj* is defined as *f* (*Bj*)= Σ*Ai∈Bj si*. Also, the fullness

of solution *S* is *f* (*S*)= Σ*K f* (*Bj*).

*j*=1

The *type t* of an ad *Ai ∈ Aj* with respect to a solution *S* is the subset of slots to which *Ai* is assigned, that is, *Ai ∈ Bj* if and only if *j ∈ t*. Let *T* be a set that contains all the subsets of slots, then *T* contains every possible type and *|T |* = 2*K*. Observe that two ads with the same type have the same frequency, and thus one can think of all ads with the same type as a single ad. For each *t ∈T* , the *occupation ot* of type *t* is the space the ads with this type occupy in each of its slots, i.e., let *Aj*

*t*

be subset of *Aj* composed of ads of type *t*, then *ot* = Σ*A ∈A′ si*.

*i*

*t*

A *conﬁguration* for a subset of ads *Aj* is a feasible solution which schedules

every ad in *Aj*. Lemma [2.1](#_bookmark2) states that if *K* is constant, then the number of possible configurations containing only large ads is polynomial in the number of large ads,

and can be enumerated by a brute-force algorithm.

**Lemma 2.1** *If K is constant, then the set of conﬁgurations for all subsets of G*

*can be listed in polynomial time.*

**Proof.** Since each container has height 1, the number of large ads which can be scheduled into a single slot is at most 1*/*(*ε/*(22*K* 2*KK*)) = (22*K* 2*KK*)*/ε*. Then, a feasible solution can contain at most *R* = *K*(22*K* 2*KK*)*/ε* large ads. Select each subset of *Gj ⊆ G* with *|Gj|≤ R*. Observe that the number of such subsets is at

most *O* , which is polynomial in *|G|* since *R* is constant.

*|G|*

*R*

For each *Gj*, create a multiset *M* in which each ad *Ai ∈ Gj* appears *wi* times, and list every partition of *M* into sets *B*1*, B*2*,..., BK*. Since *|M|≤ KR* is bounded by a constant, the number of such partitions for each *Gj* is bounded by a constant. Now note that any feasible solution of *Gj*, if any, is a partition of *M* . This completes the lemma. *2*

The so-called *ﬁrst-ﬁt heuristic* is the algorithm which iteratively schedules each copy of an ad in the first compatible slot (respecting release date and deadline restrictions and not exceeding the slot size) and stops as soon as an ad cannot be scheduled. Lemma [2.2](#_bookmark3) states that first-fit heuristic is optimal if the optimal value is less then 1*/*2. This result is used in Section [4.](#_bookmark14)

**Lemma 2.2** *Let Opt be an optimal solution to A with f* (*Opt* ) *<* 1*/*2*. The ﬁrst-ﬁt heuristic schedules the whole set A.*

**Proof.** First, note that for each *Ai ∈ A*, *si <* 1*/*2, since otherwise we could sched- ule only *Ai* to obtain a solution *Sj* such that *f* (*Sj*) *≥* 1*/*2 *> f* (*Opt* ), which is a contradiction.

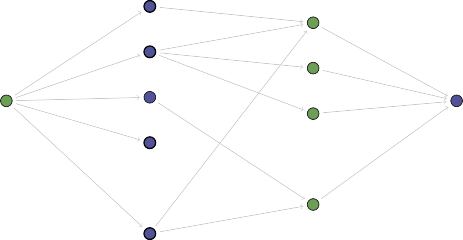
For the sake of contradiction assume that there is an optimal solution *Opt* to *A* with *f* (*Opt* ) *<* 1*/*2 and the first-fit heuristic does not schedule the whole set *A*. Now, let *S* be the solution returned by the first-fit heuristic and let *Aj* be the set of ads scheduled by *S*. We claim that *Aj* = *A*. If not, then the algorithm must have stopped when trying to schedule an ad *Ai*. Since the algorithm respects the release date and the deadline, it must be the case that *Ai* could not be added to some slot *Bj* without exceeding capacity. Since in this case *f* (*Bj*)+ *si >* 1, and *si <* 1*/*2, it follows that *f* (*Bj*) *>* 1*/*2. But then again *f* (*S*) *>* 1*/*2 *> f* (*Opt* ), which is a contradiction. *2*

# An algorithm for small ads

Lemma [2.1](#_bookmark2) states that all configurations of large items can be listed in polynomial time, and thus one can guess the configuration of large ads induced by an optimal solution. This suggests that the hard part of MAXSPACE is obtaining a solution for small ads. In this section, we consider a variant of MAXSPACE which contains only small ads and present an almost optimal algorithm for this problem.

Formally, let *ε >* 0 be a constant such that 1*/ε* is an integer. We

*x y*



*A*1

*t*1

*A*2

*t*2

*A*3

*t*3

*A*4

.

.

*t*2*K*

*Am*

Fig. 1. Example of flow graph to assign ads to types. In blue we have the set *U* and in green *V* .

define SMALL-MAXSPACE-RD as the problem of, given a set of ads *P* ,

where for each *Ai*

*∈ P* , *si*

*< ε/*(22*K* 2*KK*), and for each type *t ∈T* , and inte-

ger *ct ∈ {*0*,* 1*,...,* 1*/ε}*, one wants to find a subset *Aj ⊆ P* for each *t ∈T* such that the occupation Σ*Ai∈A′ si ≤ ctε*, and which maximizes the fullness

*t*

*t*

Σ Σ *|t|si.*

*t∈f Ai∈A′*

*t*

The reasoning behind this problem is that we try to infer the space occupied by each type in any of its slots. Thus, for each type, we guess an integer multiple of *ε* as its capacity. Since a container’s height is *L* = 1, there are exactly 1*/ε* possible guesses for each type capacity. Also, since there are 2*K* types, if *C* is the set of capacity combinations, then the size of *C* is

## 1 2*K*

*|C|* =

*.* (1)

*ε*

For each combination *c* in *C*, we create an assignment of ads to types using a maximum flow algorithm. For this step, we create a graph *H* as follows. The vertices of *H* are a source vertex *x*, a sink vertex *y*, and sets *U* and *V* , where *U* contains a vertex for each ad, and *V* contains a vertex for each type. For each ad *Ai ∈ P* and each type *t ∈ T* we add an edge (*Ai, t*) with flow capacity *∞* if the slots in type *t* correspond to a valid schedule to ad *Ai*, that is, *t* contains exactly *wi* slots, and each slot of *t* respects the release date and the deadline of *Ai*. Furthermore, we add an edge (*x, Ai*) with flow capacity *wisi* for each *Ai ∈ U* , and an edge (*t, y*) with capacity *|t|ct* for each *t ∈T* . Figure [1](#_bookmark5) illustrates this graph.

Consider then a maximum *xy*-flow *F* in *H*, which can be obtained in polyno- mial time [[2](#_bookmark22)] and notice that *F* induces an assignment from ads to types. In this assignment, if the maximum flow is such that *FAi,t* units flow from ad *Ai* to type *t*, then we say that ad *Ai* is fractionally assigned to *t* by an amount of *FAi,t/*(*wisi*). Observe that this ratio is at most one. The set of all types *t* for which *FAi,t >* 0 is called the *support* of *Ai* and is denoted by *Sup*(*Ai*).

To eliminate fractional assignments, we group ads with the same support. Let *W* be a subset of types, and *PW* be the set of ads *Ai* with *Sup*(*Ai*)= *W* . In particular, each ad *Ai ∈ PW* is compatible with any type *t ∈ W* . For each type *t ∈ W* , we

define the total flow received by *t* from *PW* as

*zt* =

Σ

*Ai∈PW*

*FAi,t.*

By the fact that each ad in *PW* is fractionally assigned to types in *W* , we know that

Σ *wisi ≥* Σ Σ *FAi,t* = Σ *zt.*

*Ai∈PW*

*Ai∈PW t∈W*

*t∈W*

In other words, the total size of *PW* given by *Ai∈PW wisi* is not smaller than the flow received by types *W* from *PW* . Therefore, we can remove the fractional assignment of all ads in *PW* , and integrally reassign each ad in *PW* to types in *W* , discarding any remaining ad.

Σ

The process of rounding the fractional assignment is summarized in Algorithm [1,](#_bookmark7) which receives as input a fractional assignment of ads to types *W* and returns an integer assignment.

**Algorithm 1** Algorithm for rounding ad assignment.

1: **procedure** Rounding(*F* )

2: **for each** *Ai ∈ P* and *t ∈T* **do**

*Ai,t →* 0

3: *F*

*j*

4: **for each** *W ⊆T* **do**

5: *PW →* all ads *Ai* with *Sup*(*Ai*)= *W*

6: **for each** *t ∈ W* **do**

Σ

7: *zt → Ai∈PW FAi,t*

8: *Zt →∅*

9: **for each** *Ai ∈ PW* **do**

10: **if** *|t|f* (*Zt*)+ *wisi ≤ zt* **then**

11: *Zt → Zt ∪ {Ai}*

12: *Fj*

*A ,t*

*i*

*→ wisi*

13: *PW → PW \ {Ai}*

14: discard remaining ads in *PW*

15: **return** *Fj*

We observe that Algorithm [1](#_bookmark7) is polynomial in the number of ads.

**Lemma 3.1** *Algorithm* [*1*](#_bookmark7) *runs in polynomial time.*

Lemma [3.2](#_bookmark9) bounds the total size of ads discarded by Algorithm [1](#_bookmark7) in each itera- tion.

**Lemma 3.2** *Let W ⊆T and let PW be the set of ads with support W at the begin- ning of the algorithm. Then the total assignment of PW after the execution is*

*Ai,t ≥* Σ Σ *FAi,t*

Σ *F*

Σ

*j*

*— |W|ε/*(22*K* 2*K*)*.*

*Ai∈PW t∈W Ai∈PW t∈W*

**Proof.** Let *PW* and *zt* be as in the algorithm, and recall that

Σ *wisi ≥* Σ Σ *FAi,t* = Σ *zt.*

*Ai∈PW*

*Ai∈PW t∈W*

*t∈W*

If no ad in *PW* is discarded, then the lemma holds trivially. Thus, assume that there exists an ad *Aj* which was discarded in the iteration. Since *Aj* was discarded in every iteration of the Line [6](#_bookmark7), we know that for any type *t*,

*|t|f* (*Zt*)+ *wjsj > zt.*

Therefore,

## Σ

*Ai,t* = Σ

Σ *wisi* = Σ *|t|f* (*Zt*)

*Ai∈PW t∈W*

Σ *F*

*j*

*t∈W Ai∈Zt*

*t∈W*

*>* Σ(*zt − wjsj*)= Σ Σ *FAi,t − |W|wjsj.*

*t∈W*

*Ai∈PW t∈W*

Since *A* is small, *s < ε/*(22*K* 2*KK*), and since *w ≤ K*, the lemma follows. *2*

*j j j*

Corollary [3.3](#_bookmark10) is obtained from Lemma [3.2](#_bookmark9).

**Corollary 3.3** *The difference between the maximum fractional flow and modiﬁed flow is not larger than ε. That is,*

*j*

Σ Σ *F*

*Ai,t*

*Ai∈P t∈f*

*≥ FAi,t − ε.*

*Ai∈P t∈f*

Σ Σ

**Proof.** Consider the value of variables *W* and *PW* of Algorithm [1](#_bookmark7). Using Lemma [3.2](#_bookmark9), we have that

*Ai,t* = Σ Σ

Σ Σ *F*

*j*

*j*

Σ *F*

*Ai,t*

*Ai∈P t∈f W ⊆f Ai∈PW t∈W*

*≥* , Σ

Σ Σ

*Ai∈PW t∈W*

Σ

⎝

*W ⊆f*

Σ *FAi,t*

*— |W|ε/*(22*K* 2*K*)⎞

⎠

*≥ F*

*Ai∈P t∈f*

Σ Σ

*Ai,t*

*—*

*W ⊆f*

2*K*(*ε/*(22*K* 2*K*))

= *FAi,t − ε,*

Σ

*Ai∈P t∈f*

where the last inequality holds because *|W|≤* 2*K*, and the last equality holds be- cause there are 2*|f |* = 22*K* distinct choices for *W* . *2*

The complete algorithm for small ads is presented in Algorithm [2](#_bookmark11). Given pa- rameter *ε >* 0, this algorithm receives as input a set of small ads *P* and a vector *c* which contains the capacity of each type. The algorithm returns a feasible schedule for *Pj ⊆ P* in *K* slots. Based on vector *c*, the algorithm creates a flow graph and ex- ecutes a maximum flow algorithm to assign ads to type. The Algorithm Rounding

transforms a fractional assignment into an integral assignment. Note that this as- signment can be easily converted into a solution for SMALL-MAXSPACE-RD.

**Algorithm 2** Algorithm for small ads.

1: **procedure** AlgP*ε*(*P* , *c*)

2: *H →* create flow graph using values of *c*

3: *F →* MaxFlow(*H*)

4: *Fj →* Rouding(*F* )

5: Create a solution *Sj* according to integral assignment *Fj*

6: **return** *Sj*

In Lemma [3.4](#_bookmark12) and Lemma [3.5](#_bookmark13) we prove that Algorithm [2](#_bookmark11) is polynomial in the instance size, and that it discards at most *ε* of the space of an optimal schedule.

**Lemma 3.4** *Algorithm* [*2*](#_bookmark11) *executes in polynomial time.*

**Lemma 3.5** *Let SP be the solution Algorithm* [*2*](#_bookmark11) *returns and let Opt P be an optimal solution for SMALL-MAXSPACE-RD for a set P of ads. Then, f* (*SP* ) *≥ f* (*Opt P* ) *− ε.*

**Proof.** Let *F* be a maximum flow in *H* and *Fj* be the output of Rounding. Define

*f* (*F* )= Σ Σ *FA ,t* and *f* (*F j*)= Σ Σ *Fj .*

*i*

*Ai∈P t∈f*

*Ai∈P t∈f*

*Ai,t*

Observe that *Opt P* induces a feasible flow for *H* with value *f* (*Opt P* ). This implies that *f* (*F* ) *≥ f* (*Opt* *P* ), as *F* is a maximum flow. Also, note that *f* (*SP* )= *f* (*F j*), then using Corollary [3.3](#_bookmark10) we have

*f* (*SP* )= *f* (*F j*) *≥ f* (*F* ) *− ε ≥ f* (*Opt P* ) *− ε.* *2*

# A PTAS for the general case

In the following, we derive a PTAS for the general case, which tackles both small and large ads. Consider an optimal solution *Opt* which schedules a subset of ads *A∗* into slots *B∗, B∗,.* *, B∗* . Note that *Opt* induces a feasible configuration of large ads *SG*

1 2 *K*

into slots *B*1*, B*2*,..., BK* such that *Bj* = *B∗ ∩ G*. Since all candidate configurations

*j*

can be listed in polynomial time by Lemma [2.1,](#_bookmark2) we may assume that we guessed the configuration *SG* of large ads induced by *Opt* . We are left with the residual problem of placing small ads.

For each slot *j*, 1 *≤ j ≤ k*, the space which is unused by large ads is

*uj* =1 *−*

Σ

*Ai∈Bj*

*si.*

While these values do not completely specify the capacity *ct* for each type *t*, which is part of the input of SMALL-MAXSPACE-RD, the number of possible capacity possibilities is a constant defined by equation ([1](#_bookmark6)), and thus, again, we can guess the vector *c*. We say that a vector *c* is *compatible* with *SG* if, for each 1 *≤ j ≤ K*,

Σ

*ctε ≤ uj.*

*t∈T* :

*j∈t*

Let *o∗* be the occupation of small ads with type *t* in the solution *Opt* . Observe that the optimal solution induces a vector *c* which is compatible with *SG* and such that *ctε ≥ o∗ − ε* for each type *t*. Since we try each capacity vector, we may assume that we guessed *c* and solved the instance of SMALL-MAXSPACE-RD with *c*.

*t*

*t*

Given parameter *ε >* 0, Algorithm [3](#_bookmark15) receives as input a set of ads *A* and returns a feasible solution. First of all, the algorithm tries to schedule all ads with the first-fit heuristic and, if it is possible, the solution is returned. Otherwise, for each configuration of large ads *SG* and each capacity vector *c* compatible with *SG*, the algorithm calls the algorithm AlgP*ε* and obtains a schedule for small ads *SP* . By combining the solution for large ads in *SG* and the solution for small ads in *SP* , it obtains a feasible solution *S* to the problem. The algorithm returns the best solution *Smax* found among all pair of configuration and capacity vector.

**Algorithm 3** A PTAS for whole set of ads *A*.

1: **procedure** Alg(*A*, *ε*)

2: Create a solution *Smax* using first-fit

3: **if** *Smax* schedules the whole set *A* **then**

4: **return** *Smax*

5: *G* = *{A ∈A| s ≥ ε/*(22*K* 2*KK*)*}*

*i* *i*

6: *P →A \ G*

7: *R →* enumerate the set of configurations for large ads as in Lemma [2.1](#_bookmark2)

8: **for each** *SG ∈ R* **do**

9: *C →* enumerate the set of capacity vectors *c* compatible with *SG*

10: **for each** *c ∈ C* **do**

11: *SP →* AlgP*ε*(*P, c, ε*)

12: *S → SG ∪ SP*

13: **if** *f* (*S*) *> f* (*Smax* ) **then**

14: *Smax → Sj*

15: **return** *Smax*

We show that Algorithm [3](#_bookmark15) is a PTAS. First, Lemma [4.1](#_bookmark16) shows Algorithm [3](#_bookmark15) runs in time polynomial in the size of the instance. Then, Lemma [4.2](#_bookmark17) highlights that the returned solution is feasible.

**Lemma 4.1** *Algorithm* [*3*](#_bookmark15) *executes in polynomial time when K is a constant.*

**Lemma 4.2** *Algorithm* [*3*](#_bookmark15) *returns a feasible solution.*

**Proof.** Since each configuration for large ads is feasible, *SG* respects release date and deadline restrictions. Solution *SP* returned by Algorithm AlgP*ε* also respects the release date and deadline restrictions to small ads. Thus, it only remains to show that for the union of *SG* and *SP* , the capacity of each slot capacity is not exceeded. Indeed, consider 1 *≤ j ≤ K* and let *Bj* be slot of large ads scheduled

by *SG*. Also, let *Aj*

*t*

be the set of small ads scheduled by *SP* with type *t*. The

occupation of slot *j* in the combined solution is then

Σ *si* + Σ

Σ *si ≤* Σ

*si* + Σ *ctε ≤* Σ

*si* + *uj ≤* 1*,*

*Ai∈Bj*

*t∈T* : *Ai∈A′ j∈t t*

*Ai∈Bj*

*t∈T* :

*j∈t*

*Ai∈Bj*

where the first inequality holds because *SP* is a feasible solution for SMALL- MAXSPACE-RD with vector *c*, and the second inequality holds because *c* is com- patible with *SG*. *2*

**Theorem 4.3** *Algorithm* [*3*](#_bookmark15) *is a PTAS to MAXSPACE-RD with constant K.*

**Proof.** By Lemma [4.1](#_bookmark16) and Lemma [4.2](#_bookmark17), the algorithm executes in polynomial time and returns a feasible solution.

Let *Opt* be an optimal schedule and let *Smax* be the schedule returned by Al- gorithm [3](#_bookmark15). If the first-fit heuristic schedules the whole set *A*, *f* (*Smax* )= *f* (*Opt* ) and the proof ends. Thus, assume the algorithm continues and, by Lemma [2.2](#_bookmark3), that *f* (*Opt* ) *≥* 1*/*2.

Denote by *A∗* the set of ads scheduled by *Opt* and let *Sj*

*G*

*P*

and *Sj*

be the config-

urations of large and small ads induced by *A∗ ∩ G* and *A∗ ∩ P* , respectively, such

that *f* (*Opt* ) = *f* (*Sj* )+ *f* (*Sj* ). Also, let *cj* be the capacity vector such that for

*G P*

each type *t*, *cj* is the largest integer with *cj ε ≤ o∗*, where *o∗* is the occupation of ads

*t t t t*

in *A∗ ∩ P* with type *t*. Clearly, *cj* is compatible with *Sj* . Therefore, the algorithm

*t G*

constructs a solution *Sj* for the pair of configuration *Sj* and capacity vector *cj*.

*G*

Let *Opt P* be an optimal solution for SMALL-MAXSPACE-RD with ads *P* and vector *cj*. We claim that *f* (*Opt P* ) *≥ f* (*Sj* ) *−* 2 *·* 2*KKε*. To prove this, for each type *t*, let *Aj* be a maximal subset of ads in *A∗ ∩ P* with type *t* and such that

*P*

*t*

*Ai∈At*

Σ *′*

*t*

*si ≤ cj ε.*

Thus, the sets *Aj*

*t*

form a feasible solution *Sjj*

for SMALL-MAXSPACE-RD with

vector *cj*. Since *Aj* is maximal, either *Aj* contains all ads in *A∗ ∩ P* with type *t*

*P*

*t*

*t*

and Σ *′ si* = *o∗ ≥ cj ε*, or there is a small ad *Aj ∈ A∗ ∩ P* with type *t* which

*Ai∈At*

*t*

*t*

could not be added to *Aj* and Σ *′ si > cj ε − sj*. In either case,

Σ *′*

*t*

*Ai∈At*

*t*

*si*

*t*

*Ai∈At*

*> cj ε − ε/*(22*K* 2*KK*)*.*

This implies that the fullness of *Opt P* , which is at least the fullness of *Sjj* , is

*P*

*f* (*Opt P* ) *≥ f* (*Sjj* )= Σ Σ *|t|si*

*P*

*t∈f Ai∈A′*

*t*

*>* Σ(*|t|cj ε − |t|ε/*(22*K* 2*KK*))

*t*

*t∈f*

*>* Σ(*|t|*(*o − ε*) *− |t|ε/*(2 2 *K*))

*∗* 2*K K*

*t*

*t∈f*

*≥ f* (*Sj* ) *−* 2*KK*(*ε* + *ε/*(22*K* 2*KK*))*.*

*P*

Therefore, indeed *f* (*Opt P* ) *≥ f* (*Sj* ) *−* 2 *·* 2*KKε*.

*P*

If *SP* is the solution obtained by Algorithm

AlgP*ε* for vector *cj*, then, by

Lemma [3.5](#_bookmark13), *f* (*SP* ) *≥ f* (*Opt P* ) *− ε*. Let *Sj* be the solution considered by combin-

ing *Sj*

*G*

and *SP* . Since the algorithm returns the best found solution,

*f* (*Smax* ) *≥ f* (*Sj*)= *f* (*Sj* )+ *f* (*SP* )

*G*

*≥ f* (*Sj* )+ *f* (*Opt P* ) *− ε*

*G*

*≥ f* (*Sj* )+ *f* (*Sj* ) *−* 2 *·* 2*KKε − ε*

*G P*

*≥ f* (*Opt* ) *−* 3 *·* 2*KKε*

*≥ f* (*Opt* ) *−* 6 *·* 2*KKε · f* (*Opt* )

= (1 *−* 6 *·* 2*KKε*)*f* (*Opt* )*,*

where the fourth inequality holds because 2 *·* 2*KK >* 1, and the last inequality holds because *f* (*Opt* ) *>* 1*/*2.

For any *εj >* 0, by letting *ε* = *εj/*(6 *·* 2*KK*), the obtained solution has fullness at least (1 *− εj*)*f* (*Opt* ). Therefore, Algorithm [3](#_bookmark15) is a PTAS for MAXSPACE-RD. *2*

# 5 Final remarks

This paper presented a PTAS for MAXSPACE-RD, which is a generalization of MAXSPACE that deals with release dates and deadlines. To our knowledge, this is the first approximation scheme to this MAXSPACE variant. When the number of bins is given in the input, we can show that MAXSPACE-RD is strongly NP- hard, and thus does not admit an FPTAS. We left open the question of whether the problem with a constant number of slots admits an FPTAS. In future works, we will consider the variant in which the value of an ad is given in the input, and may be unrelated to its size. This variant is a generalization of the Multiple Knapsack Problem [[3](#_bookmark23)], which is strongly NP-hard even for *K* = 2.

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**A Omitted proofs**

**Proof.** [of Lemma [3.1](#_bookmark8)] The loops of Lines [4](#_bookmark7) and [6](#_bookmark7) execute a constant number of iterations, since *|T |* = 2*K* and the number of subsets of *T* is 22*K* . The inner loop (Line [9](#_bookmark7)) executes a polynomial number of iterations since *|PW |* is polynomial. Then, the algorithm executes in polynomial time. *2*

**Proof.** [of Lemma [3.4](#_bookmark12)] The maximum flow is solved in polynomial time in the size of graph *H* [[2](#_bookmark22)] and *H* is polynomial in the size of the instance since it has exactly one vertex per small ad and a constant number of vertices for types. The Rouding algorithm is also polynomial, by Lemma [3.1.](#_bookmark8) Then, Algorithm [2](#_bookmark11) is polynomial in the instance size. *2*

**Proof.** [of Lemma [4.1]](#_bookmark16) The number of configurations for large ads is polynomial, by Lemma [2.1](#_bookmark2). Thus, the loop of Line [8](#_bookmark15) executes a polynomial number of iterations. Also, the number of capacity vectors which are compatible with each such configu- ration is at most a constant, by equation ([1](#_bookmark6)). Thus, the loop of Line [10](#_bookmark15) executes a polynomial number of iterations. The call to AlgP*ε* also runs in polynomial time, by Lemma [3.4](#_bookmark12). Therefore, the algorithm runs in polynomial time. *2*