Electronic Notes in Theoretical Computer Science 120 (2005) 201–215 

[www.elsevier.com/locate/entcs](http://www.elsevier.com/locate/entcs)

An Algorithm for Computing Fundamental Solutions

# Klaus Weihrauch[1](#_bookmark0)

*Department of Computer Science Fernuniversitat*

*58084 Hagen, Germany*

Ning Zhong[2](#_bookmark0) ,[3](#_bookmark0)

*Department of Mathematical Sciences University of Cincinnati*

*Ohio 45221-0025, USA*

Abstract

For a partial differential operator *P* = P *cαDα* with constant coefficients *cα*, a generalized function *u* is a fundamental solution, if *Pu* = *δ*, where *δ* is the Dirac distribution. In this article we provide an algorithm which computes a fundamental solution for every such differential operator

|*α*|≤*m*

*P* on a Turing machine, if the input- and output-data are represented canonically.

*Keywords:* computable analysis, partial differential equations, fundamental solution

# Introduction

In the theory of differential operators with constant coefficients, fundamental solutions have a central place. Let

Σ

*P* = *P* (*D*)= *cαDα* (1)

|*α*|≤*m*

1 Email: [Klaus.Weihrauch@fernuni-hagen.de](mailto:Klaus.Weihrauch@fernuni-hagen.de)

2 Email: [Ning.Zhong@uc.edu](mailto:Ning.Zhong@uc.edu)

3 The author has been supported in part by the University of Cincinnati’s Summer Faculty

Research Fellowship.

1571-0661 © 2005 Elsevier B.V. Open access under [CC BY-NC-ND license](http://creativecommons.org/licenses/by-nc-nd/3.0/).

doi:10.1016/j.entcs.2004.06.045

be a partial differential operator with constant coefficients *cα*, where *α* =

(*α*1*,... , αn*) is a multi-i√ndex of order |*α*| = *α*1 + ··· + *αn*, *Dj* = (1*/i*)*∂/∂xj*

*α α*1 *α*2 *αn*

with 1 ≤ *j* ≤ *n* and *i* = −1, *D* = (*D*1*,... , Dn*) and *D* = *D D* ··· *D* . A

1

2

*n*

distribution is called a fundamental solution of the partial differential operator

*P* if it is a solution of the point source problem

Σ

*P* (*D*)*u* =

|*α*|≤*m*

*cαDαu* = *δ* (2)

where *δ* is the Dirac measure at 0. In 1954 Malgrange and Ehrenpreis proved that every partial differential operator with constant coefficients has a funda- mental solution. Fundamental solutions are very useful tools in the theory of partial differential equations, for instance, in solving inhomogeneous equations and in providing information about the regularity and growth of solutions. In the case of solving inhomogeneous equations, if *E* is a fundamental solution of the partial differential operator *P* = *P* (*D*) and if *f* is a distribution, then, *E f* is a solution of the equation *P* (*D*)*u* = *f* whenever the convolution is defined.

∗

Many classical differential operators are known to have computable func- tions as fundamental solutions. For example, the Schwartz function

*E* (*t, x*)= (4*πνt*)−*n/*2*e*−|*x*|2*/*4*νt*

*H*

which is a computable real function, is a fundamental solution of the heat equation

*n* 2

*∂u* = *ν* Σ *∂ u,* (*t, x*) ∈ R × R*n.*

*∂t*

*i*

*i*=1

*∂x*2

In general, however, a fundamental solution is a distribution. Has every partial

differential operator *P* (*D*) with computable coefficients a computable funda- mental solution? In fact, this follows from the even more general result we prove in this note: There is a computable operator mapping every differential operator *P* (*D*) (as in ([1](#_bookmark1))) to a fundamental solution. We present an algorithm which in a well-defined realistic model of computation computes a fundamen- tal solution of any given differential operator *P* from its coefficients, where abstract data are encoded canonically by, generally infinite, sequences of sym- bols and computations are (can be) performed by Turing machines.

# Preliminaries

In this note, we consider the representation approach as a model of compu- tation for analysis [[5](#_bookmark11)]. Computable functions on Σ∗ and Σ*ω* (the set of finite

and infinite sequences of symbols, respectively, from the finite alphabet Σ) are defined by Turing machines which can read and write finite and infinite sequences. A multifunction *f* : ⊆ *X* ⇒ *Y* is a function which assigns to every *x* ∈ *X* a set *f* (*x*)⊆*Y* , the set of “acceptable” values (*f* (*x*)= ∅, if *x* /∈ dom(*f* )).

Computability on other sets is defined by using Σ∗ and Σ*ω* as codes or names. A notation (representation) is a multifunction *ν* : Σ∗ ⇒ *M* (*δ* : Σ*ω* ⇒ *M*). For the natural numbers and the rational numbers we use canonical notations *ν*N : ⊆ Σ∗ → N and *ν*Q : ⊆ Σ∗ → Q, respectively. For the real numbers we use the representation *ρ* : ⊆Σ*ω* → R, where *ρ*(*p*)= *x* if *p* encodes a sequence (*ai*)*i* of rational numbers such that |*x* − *ai*| ≤ 2−*i*. For a naming system *γ* : ⊆ *Y* → *M* (*Y* ∈ {Σ∗*,* Σ*ω*}) the representation *γk* : ⊆ *Y* → *Mk* is defined by *γ y*1*,... , yk* := (*γ*(*y*1)*,... γ*(*yk*)) where is a tupling function. For the complex numbers we use the representation *ρ*2 of R2.

⊆ ⊆

⟨ ⟩ ⟨ ⟩

For naming systems *γi* : ⊆ *Yi* ⇒ *Mi* a function *h* : ⊆ *Y*1 → *Y*2 (on sequences of symbols) realizes a multifunction *f* : ⊆ *M*1 ⇒ *M*2, if

*γ*2 ◦ *h*(*y*) ∩ *f* (*x*) /= ∅ if *γ*1(*y*) ∈ dom(*f* ) *,*

that is, *h*(*y*) is a name of some *z* ∈ *f* (*x*) if *y* is a name of *x* ∈ dom(*f* ). The multifunction *f* is (*γ*1*, γ*2)-computable, if it has a computable realization, and (*γ*1*, γ*2)-continuous, if it has a continuous realization.

Multifunctions occur naturally in Computable Analysis. As an example, there is an algorithm which maps every *ρ*-name *p* of *x* R (i.e., every se- quence of rational numbers rapidly converging to *x*) to some *n* N such that *x < n*. This algorithm, however, might give another upper bound *n*' of *x*, if fed with another *ρ*-name *p*' of *x*. The algorithm is not “(*ρ, ν*N)-extensional”. Nevertheless, the algorithm realizes the multifuntion *f* : R ⇒ N, defined by *n f* (*x*) *x < n*. There is no (*ρ, ν*N)-computable function *g* : R N such that *x < f* (*x*).

∈

∈

∈ ⇐⇒ →

As generalizations of the “acceptable Go¨del numbering *ϕ* : N → *P* (1)” of the partial recursive functions, for any *a, b , ω* there is a canonical representation *ηab* : Σ*ω F ab* of continuous functions *f* : Σ*a* Σ*b* sat- isfying the “utm-theorem” and the “smn-theorem” [[5](#_bookmark11)]. For naming systems *γ*1 : ⊆ Σ*a* → *M*1 and *γ*2 : ⊆ Σ*b* → *M*2 a *representation* [*γ*1 → *γ*2] of the *total* (*γ*1*, γ*2)-continuous functions is defined by

→ ⊆ →

∈ {∗ }

[*γ*1 → *γ*2](*p*)= *f* : ⇐⇒ *ηab*

*p*

realizes *f* w.r.t. (*γ*1*, γ*2) *,*

and a *multirepresentation* [*γ*1 →*p γ*2] of *all partial* (*γ*1*, γ*2)-continuous functions is defined by

*f* ∈ [*γ*1 →*p γ*2](*p*): ⇐⇒ *ηab*

*p*

realizes *f* w.r.t. (*γ*1*, γ*2) *.*

Notice that [*γ*1 → *γ*2] is the “weakest” representation *δ* of the set of total (*γ*1*, γ*2)-continuous functions such that the evaluation (*f, x*) '→ *f* (*x*) becomes (*δ, γ*1*, γ*2)-computable (Lemma 3.3.14 in [[5](#_bookmark11)]).

For definitions and mathematical properties of distributions see [[1](#_bookmark8),[2](#_bookmark9)]. In [[6](#_bookmark13)] computability on distributions over the real line are studied. The definitions and theorems can be generalized straightforwardly to distributions over R*n* (*n* 1). For the space of continuous functions *f* : R*n* C we use the representation [*ρn ρ*2]: Σ*ω C*(R*n*). For the space *C*∞(R*n*) of infinitely differentiable functions *f* : R*n* C we use the representation *δ*∞ : Σ*ω C*∞(R*n*) defined by

→ ⊆ →

→ ⊆ →

≥ →

*δ*∞(⟨*pα*⟩*α*∈N*n* )= *f* : ⇐⇒ (∀*α* ∈ N ) *D f* = [*ρ* → *ρ* ](*pα*)

*n α n* 2

*ω*

(where ⟨*pα*⟩*α*∈N*n* ∈ Σ is a canonical merging of the infinitely many infinite

sequences *pα* ∈ Σ*ω*, *α* ∈ N*n*). The topology of *C*∞(R*n*) can be defined by the semi-norms *f* → |*Dmf* |*k* = sup|*α*|≤*m*(sup*x*∈*K* |*Dαf* (*x*)|), as *m* varies over the set of non-negative integers and *K* varies over the family of compact subsets of R*n*.

A test function is a function *f* ∈ *C*∞(R*n*) with compact support supp(*f* ) := cls{*x* ∈ R*n* | *f* (*x*) /= 0}. The set of test functions is denoted by D(R*n*) and its topology is induced by *C*∞(R*n*). We use the representation *δD* : ⊆ Σ*ω* → D(R*n*) defined by

*δD*(0*k*1*p*)= *f* : ⇐⇒ *δ*∞(*p*)= *f* and supp(*f* ) ⊆ [−*k*; *k*]*n .*

A Schwartz function is a function *f* ∈ *C*∞(R*n*) such that sup*x*∈R*n* (|*x*|*j* |*Dαf* |) ≤

∞ for all *j* ∈ N and *α* ∈ N . The set of Schwartz functions is denoted by

*n*

S(R*n*). We use the representation *δS* : ⊆Σ*ω* → S(R*n*) defined by

*δD*(⟨*p, q*⟩)= *f* : ⇐⇒ *δ*∞(*p*)= *f* and *q* encodes a function *h* : N*n*+1 → N

such that (∀ *j, α*) sup*x*∈R*n* (|*x*|*j* |*Dαf* |) ≤ *h*(*j, α*)*.*

Of course, the above representations can be replaced by equivalent ones. Ex- amples for the case *n* = 1 are discussed in [[6](#_bookmark13)].

A continuous linear map *T* : D(R*n*) → C (*T* : S(R*n*) → C) is called a distribution (tempered distribution), (where “continuous” means sequentially continuous w.r.t. the canonical convergence relations on D(R*n*) and S(R*n*), resp.). The set of all distributions (tempered distributions) is denoted as '(R*n*) ( '(R*n*)). For the set '(R*n*) we use the representation [*δD ρ*2]

D S D →

(restricted to the linear functions).

Remember that the evaluation function is computable, i.e., there is Type-2 Turing machine which computes a *ρ*2-name of *T* (*φ*) whenever given a [*δD*

→

*ρ*2]-name of *T* and a *δD*-name of *φ* as input. Usually *T* (*φ*) is written as

*<T, φ>* .

We conclude this section with recalling some further definitions and facts [[2](#_bookmark9)]. The Dirac measure *δ* on R*n* at 0 is a tempered distribution defined by *δ*(*φ*) = *<δ, φ>* = *φ*(0) for any Schwartz function *φ* (R*n*). The Dirac measure can be viewed as a point source. The Fourier transform of a Schwartz

∈ S

function *φ* ∈ S(R*n*), denoted as *φ*ˆ or F(*φ*), is defined as

F ∫

(*φ*)(*ξ*)= *φ*ˆ(*ξ*) := (2*π*)−*n/*2

R*n*

*e*−*ixξφ*(*x*)*dx*

where *x* = (*x*1*, x*2*,... , xn*), *ξ* = (*ξ*1*, ξ*2*,... , ξn*) and *xξ* = *x*1*ξ*1+*x*2*ξ*2+·· ·+*xnξn*. The Fourier transform is a linear bijection of S(R) to itself. For every partial derivative,

F(*Djφ*)= *ξj*F(*φ*) and *Dj*F(*φ*)= F(−*ξjφ*) (3) Therefore, for any partial differential operator *P* (*D*) with constant coefficients

*<*F(*P* (*D*))*, φ>* = *P* (*ξ*)*φ*ˆ*,* ∀*φ* ∈ S(R)*.* (4)

*D* = −*i∂*.

Since S(R) is sequentially dense in S'(R), by a duality argument, F and *Dj* can be uniquely extended from S(R) to S'(R) defined by the following formulae:

*<*F *T, φ>* := *<T,* F *φ> ,* (5)

*<DjT, φ>* := *<T,* −*Djφ>* (6)

for any *T* '(R) and *φ* (R). The Fourier transform measure *δ* is a constant:

∈ S ∈ S

−*n/*2

*δ*ˆ of the Dirac

F(*δ*)= (2*π*) *.* (7)

Let R be the reflexion operator on S(R*n*) defined by R(*u*)(*x*) := *u*(−*x*). Lemma 2.1 *For any E* ∈ D'(R*n*) *let* ⟨*E*˜*, φ>* := *<E,* R*φ> . Then E is a fundamental solution of P* (*D*)*, that is, P* (*D*)*E* = *δ, iff*

*<E*˜*, P* (*D*)*φ>* = *φ*(0) *for all φ* ∈ D(R*n*) *.*

Proof By ([6](#_bookmark2)), *<E,* R*Djφ>* = *<E,* −*Dj*R*φ>* = *<DjE,* R*φ>* , and there- fore,

*<E*˜*, P* (*D*)*φ>* = *<E,* R*P* (*D*)*φ>* = *<P* (*D*)*E,* R*φ> .*

We obtain *P* (*D*)*E* = *δ*, iff (∀*φ*) *<P* (*D*)*E,* R*φ>* = R*φ*(0) = *φ*(0), iff

(∀*φ*) *<E*˜*, P* (*D*)*φ>* = *φ*(0).

# Computing a Fundamental Solution

Formally, the problem of constructing a fundamental solution of a partial differential operator *P* (*D*) is very easy. Indeed, suppose that we have

*P* (*D*)*E* = *δ.*

Since F *DjT* = *ξj*F *T* (from ([3](#_bookmark2)), ([5](#_bookmark2)) and ([6](#_bookmark2))), by taking the Fourier transform we obtain *P* (*ξ*)F *E* = (2*π*) , hence

−*n/*2

(2*π*)−*n/*2

F *E* =

*,*

*P* (*ξ*)

and *E* should be defined as the inverse Fourier transform of (2*π*)−*n/*2*/P* (*ξ*). This is meaningful if *P* (*ξ*) has no real zeros. In the general case, we will overcome the difficulty by selecting domains of integration which avoids the zeros of *P* (*ξ*).

For *m* ≥ 1 let P(*m*) be the linear space of all polynomials *P* (*ξ*) =

Σ

*cαξα*, *cα* ∈ C, in *n* variables of degree ≤ *m*, where *ξα* = *ξα*1 · *...* · *ξαn*

|*α*|≤*m*

1

*n*

for *α* = (*α*1*,..., αn*) and |*α*| = *α*1 + *...* + *αn*. This space has dimension

*N*(*m, n*) := (*m* + *n*)!*/m*!*n*!, the monomials *ξα*, |*α*|≤ *m*, can b e considered as

a basis. On P(*m*) we consider the norm || Σ|*α*|≤*m cαξα*||*m* :=

Σ|*α*|≤*m* |*cα*|2.

Lemma 3.1 *[*[*3*](#_bookmark10)*] From m a ﬁnite set Am* = {*νi* | 1 ≤ *i* ≤ *L*(*m, n*)}⊆ Q*n of vectors, L*(*m, n*) := (*m* + 1)(*n*+1)*, can be computed such that for all P* ∈ P(*m*)*, P* /≡ 0*,*

(∃ *ν* ∈ *Am*)(∀*z* ∈ C*,* |*z*| = 1)*P* (*zν*) /= 0 *.* (8)

*Am* can be chosen as follows: *A* = *k ν* : *ν A*'*,k* = 0*,* 1*,..., m* , where *A*' = *ξ* R*n* : *ξi* 0*,* 1*,* 2*,..., m ,* 1 *i n* . Observe that *Am* = (*m* + 1)(*n*+1) =: *L*(*m, n*).

*m*

{ ∈ ∈ { } ≤ ≤ } | |

{ ∈ }

Lemma 3.2 *There is a Type-2 Turing machine which, for every polynomial*

*P* ∈ P(*m*) *of degree m >* 0*, computes a number l* ∈ N *such that*

2−*l <* sup

*n*

*ν*∈*Am*

inf

|*z*|=1

|*P* (*ξ* + *zν*)| *for all ξ* ∈ R *.* (9)

*(*|*α*| ≤ *m) of a polynomial P* (*ξ*) =

*More precisely, from the degree m an*Σ*d ρ*2*-names of the coeﬃcients cα* ∈ C

*computes a number l such that (*[*9*](#_bookmark5)*).*

Proof

|*α*|≤*m cαξα of degree m the machine*

By Lemma [3.1](#_bookmark4) there is some *ν* ∈ *Am* such that for all *z* ∈ C, *P* (*zν*) /= 0 if

|*z*| = 1. Since *z* '→ |*P* (*zν*)| is continuous,

0 *<* sup

*ν*∈*Am*

inf

|*z*|=1

|*P* (*zν*)| *.* (10)

For every *ν* ∈ *Am* the function

(*c*˜*, z*) '→ |*P* (*zν*)| is (*ρ*2*N*(*m,n*)*, ρ*2*, ρ*)-computable

(*c*˜ := (*cα*)|*α*|≤*m* C*N*(*m,n*)). By Theorem 3.3.15 in [[5](#_bookmark11)] on type conversion, the function

∈

*c*˜ '→ (*z* '→ |*P* (*zν*)|) is (*ρ*2*N*(*m,n*)*,* [*ρ*2 → *ρ*])-computable.

Since *z* C *z* = 1 is a *κ*2-computable compact set (Def. 5.1.2 in [[5](#_bookmark11)]) the function

{ ∈ | | | }

*f* inf

'→

|*z*|=1

*f* (*z*) is ([*ρ*2*, ρ*]*, ρ*)-computable

by Cor. 6.2.5 in [[5](#_bookmark11)]. Therefore, for each *ν* ∈ *Am*, the function

*c*˜ inf

'→

|*z*|=1

|*P* (*zν*)| is (*ρ*2*N*(*m,n*)*, ρ*)-computable. (11)

By Thm. 6.1.2 in [[5](#_bookmark11)], the function

*c*˜ '→ sup inf |*P* (*zν*)| is (*ρ , ρ*)-computable. (12)

2*N* (*m,n*)

*ν*∈*Am* |*z*|=1

Σ

Since *Bm* := {*c*˜ | |*α*|≤*m* |*cα*|2 = 1} is compact and *c*˜ '→ sup*ν*∈*A*

is continuous by ([12](#_bookmark5)), by ([10](#_bookmark5))

*m*

inf|*z*|=1 |*P* (*zν*)|

inf sup inf |*P* (*zν*)| *>* 0 *.*

||*c*˜||*m*=1 *ν*∈*Am* |*z*|=1

Since *Bm* is *κ*2*N*(*m,n*)-computable, by ([12](#_bookmark5)) and Cor. 6.2.5 in [[5](#_bookmark11)],

inf sup inf |*P* (*zν*)| *>* 0 is *ρ*-computable*.*

||*c*˜||*m*=1 *ν*∈*Am* |*z*|=1

Since all the above computations are uniform in the degree *m*, a rational number *Cm >* 0 can be computed from *m* such that

*Cm <* sup inf |*P* (*zν*)| if ||*P* ||*m* = 1 *.* (13)

*ν*∈*Am* |*z*|=1

Σ

Now consider arbitrary polynomials *P* (*ξ*)= |*α*|≤*m cαξα* of degree *m*. There is a Type-2 machine which on input *m* and the coefficients (*cα*)|*α*|≤*m* computes some rational number *aP* such that 0 *< aP <* |*cβ*| for some index *β* with

*β* = *m*.

| |

For fixed *ξ* ∈ R*n*, *P* (*ξ* + *ξ*')= Σ

*cα*(*ξ* + *ξ*')*α* =

Σ *dα*(*ξ*) *ξ*'*α* where the coeffi-

cients of the polynomials *dα*(*ξ*) can be determined by algebraic computation. By an easy observation, *dβ*(*ξ*) = *cβ* since |*β*| = *m*. Let *Q*(*ξ*') := *P* (*ξ* + *ξ*').

Then *aP <* |*cβ*|≤ |*dα*(*ξ*)|2 = ||*Q*||*m*. Since *Q/*||*Q*||*m* has norm 1, by ([13](#_bookmark5)),

*Cm <* sup*ν*∈*Am* inf|*z*|=1 |*Q*(*zν*)|*/*||*Q*||*m*, therefore

√Σ

*aP* · *Cm <* ||*Q*||*m* · *Cm* ≤ sup

*ν*∈*Am*

inf

|*z*|=1

*Q*(*zν*) = sup

*ν*∈*Am*

| |

inf

|*z*|=1

|*P* (*ξ* + *zν*)| *.*

Finally some *l* ∈ N can be computed such that 2−*l* ≤ *aP Cm*.

In the following we denote, for any (*ξ*1*,... , ξn*) ∈ R*n*, |(*ξ*1*,..., ξn*)| := sup*j ξj* . *X* will denote the closure and *X*◦ will denote the interior of a set

| |

*X*. For vectors in R*n* let (*xn,... , xn*) *<* (*yn,... , yn*), iff *xi < yi* for all *i*. For

*a, b* ∈ R*n*, *a < b*, let (*a*; *b*] be the semi-open interval (box) {*x* ∈ R*n* | *a < x* ≤ *b*}. Let B1 := {(*a*; *b*] | *a, b* ∈ Q*n* and *a < b*} be the set of all semi-open rational intervals with canonical notation *νB*1. The set {*I*◦ | *I* ∈ B1} is a basis of the topology on R*n*. Let B be the set of all finite unions of elements from B1 with standard notation *νB*. Notice that ∅∈ B and B is closed under union, intersection and difference (*I, J*) '→ *I* \ *J* and that every *J* ∈ B is a finite union of pairwise disjoint semi-open intervals from B1. Union and difference are (*νB, νB, νB*)-computable.

Integration of total continuous functions over intervals is computable [[5](#_bookmark11)]. We need a generalization to partial functions. In [[4](#_bookmark12)] such multirepresentations have been used for partial continuous functions on computable metric spaces. The following lemma generalizes Thm. 6.4.1 in [[5](#_bookmark11)]. It can be proved by using standard techniques.

Lemma 3.3 (i) *The function f* '→ ∫|*z*|=1 *f* (*z*) *dz for continuous f* : ⊆C → C

*which is deﬁned if* {*z* | |*z*| = 1}⊆dom(*f* ) *is* ([*ρ*2 →*p ρ*2]*, ρ*2)*-computable.*

(ii) *The function* (*f, I*) '→ ∫*I f* (*ξ*) *dξ for continuous f* : ⊆R → C *and I* ∈B

*n*

*which is deﬁned if I*⊆dom(*f* )*, is* ([*ρn* →*p ρ*2]*, νB, ρ*2)*-computable.*

Since the boundary of every *I* ∈ B has measure 0, for every continuous function *f* ,

∫

∫

∫

*f* (*ξ*) *dξ* = *f* (*ξ*) *dξ* = *f* (*ξ*) *dξ ,*

*I I I*◦

if *I*⊆dom(*f* ). After these preparations we can prove our main theorem.

Theorem 3.4 *There is a Type-2 Turing machine which for every differential operator P* (*D*) = |*α*|≤*m cαDα* /≡ 0 *computes a fundamental solution E* ∈

Σ

D (R*n*)*. More precisely, from m and ρ*2*-names of the cα* ∈ C *(*|*α*| ≤ *m) it*

'

*computes a* [*δD* → *ρ*2]*-name of a fundamental solution E.*

Proof

Σ

First, we assume that the degree *m* of the polynomial *P* (*ξ*)= |*α*|≤*m cαξα* is fixed. Let *c*˜ := (*cα*)|*α*|≤*m* ∈ C*N*(*m,n*) be the vector of coefficients of *P* (*ξ*). Let *Am* = {*νi* | 1 ≤ *i* ≤ *L*(*m, n*)} be any set satisfying Lemma [3.1](#_bookmark4). Let *l* ∈ N be some constant such that ([9](#_bookmark5)). For 1 ≤ *j* ≤ *L*(*m, n*) Define

Ω*j* := *ξ* R*n* 2−*l <* inf

{ ∈ |

|*z*|=1

|*P* (*ξ* + *zνj*)|} *.* (14)

Let *M*−1 := *M*0 := ∅∈ У and *Mk* := ((—*k,...,* —*k*); (*k,..., k*)] ∈У for *k* ≥ 1. For *k, j* ∈ N, 1 ≤ *j* ≤ *L*(*m, n*), let *T k* ∈У be sets such that

*j*

*Tk*⊆Ω*j, Tk* ∩ *Tk* = ∅ for *i* /= *j* and *Tk* = *Mk* \ *Mk*−1 *.* (15)

*j i j j*

*j*

Since for each *k* the *Tk* (1 ≤ *j* ≤ *L*(*m, n*)) are a partition of *Mk* \ *Mk*−1 and

*j*

'

the *M* \ *M* are partition of R*n*, *Tk* ∩ *Tk* = ∅ if *k* /= *k*' or *j* /= *j*'. Define

*k*

*Tj* := *k*

*k*−1

*Tk*. Then

*j*

*j j*'

*Tj*⊆Ω*j, Ti* ∩ *Tj* = ∅ for *i* /= *j* and *Tj* = R*n .* (16)

*j*

For any *u* ∈ Ð(R*n*) and *k* ∈ N define

*E*˜*k* (*u*) := (2*π*)−*n/*2

*L*Σ(*m,n*) ∫

*dξ*  1 ∫

*u*ˆ(*ξ* + *zνj*) 1

*dz* (17)

and

*k*

*j*=1 *j*

*T*

*L*Σ(*m,n*) ∫

2*πi*

1 ∫

|*z*|=1 *P* (*ξ* + *zνj*) *z*

*u*ˆ(*ξ* + *zν* ) 1

*E*˜(*u*) := (2*π*)−*n/*2

*j*=1

*dξ*

*Tj* 2*πi*

*j*

|*z*|=1 *P* (*ξ* + *zνj*) *z*

*dz .* (18)

If *P* (*ξ* + *zνj*) becomes 0 in the domain of integration, the integrals possibly do not exist. By ([14](#_bookmark7)), however, for any *j*, any *ξ* ∈ Ω*j* and for |*z*| = 1,

*u*ˆ(*ξ* + *zνj*) 1 ≤ |*u*ˆ(*ξ* + *zνj*)| 1

*l*

*P* (*ξ* + *zνj*) *z*

|*P* (*ξ* + *zνj*)| |*z*|

*j*

≤ 2 · |*u*ˆ(*ξ* + *zν* )| *.* (19)

The integrals exist, since *z* = 1 is considered and the function *u*ˆ (R*n*) is rapidly decaying.

| | ∈ £

Define *E* by *<E, φ>* := *<E*˜*,* Y*φ>* for all *φ* ∈ Ð(R*n*). Then *<E*˜*, φ>* =

*<E*˜*,* Y2*φ>* = *<E,* Y*φ>* for all *φ* ∈ Ð(R*n*). We show in the following

that *E* is a fundamental solution of *P* (*D*). By Lemma [2.1](#_bookmark3), *E* is a fundamental solution of *P* (*D*) iff *E*˜(*P* (*D*)*v*)= *v*(0) for any *v* ∈ Ð(R*n*). For any *u* = *P* (*D*)*v*, *v* ∈ Ð(R),

*E*˜(*u*)= *E*˜(*P* (*D*)*v*)

Σ ∫

*L*(*m,n*)

= (2*π*)−*n/*2

*dξ*  1 ∫

F(*P* (*D*)*v*)(*ξ* + *zνj*) 1 *dz*

*j*=1 *Tj*

2*πi*

|*z*|=1

*P* (*ξ* + *zνj*) *z*

*L*Σ(*m,n*) ∫

1 ∫

*P* (*ξ* + *zν* )*v*ˆ(*ξ* + *zν* ) 1

= (2*π*)−*n/*2

*j*=1

*dξ*

*Tj* 2*πi*

|*z*|=1

*j*

*P* (*ξ* + *zνj*)

*j dz*

*z*

= (2*π*)−*n/*2

*L*Σ(*m,n*) ∫

*dξ*  1 ∫

1

*v*ˆ(*ξ* + *zνj*)

*dz .*

We observe that

*j*=1 *Tj*

2*πi*

|*z*|=1 *z*

1 ∫

1

*v*ˆ(*ξ* + *zν* ) *dz*

∫

2*πi*

∫

= 1

2*πi*

|*z*|=1

|*z*|=1

*j z*

(2*π*)−*n/*2

R*n*

∫

*e*−*ix*(*ξ*+*zνj* )*v*(*x*)*dx*1 *dz*

*z*

∫

= (2*π*)

−*n/*2

R*n*

∫

*e*−*ixξ*

*v*(*x*)*dx*

1 2*πi*

|*z*|=1

*e*−*ixzνj*

*dz*

*z*

= (2*π*)−*n/*2

R*n*

∫

*e*−*ixξv*(*x*)*e*−*ix*0*νj dx* (Cauchy integral)

= (2*π*)−*n/*2

R

= *v*ˆ(*ξ*) *.*

Thus we obtain

*e*−*ixξv*(*x*)*dx*

*L*Σ(*m,n*) ∫

*E*˜(*P* (*D*)*v*)= (2*π*)−*n/*2

*j*=1

∫

*v*ˆ(*ξ*)*dξ*

*Tj*

= (2*π*)−*n/*2

R*n*

∫

= (2*π*)−*n/*2

R*n*

*v*ˆ(*ξ*)*dξ*

*ei*0*ξv*ˆ(*ξ*)*dξ*

= *v*(0) *.*

Next we determine an upper bound of |*E*˜*k* (*u*)|. Since *u*ˆ ∈ £(R*n*),

(∃ *k* ∈ N) (|*u*ˆ(*ξ*)| |*ξ*| ≤ 1 if |*ξ*|≥ *k* ) *.* (20) Define *ν*¯ := max*ν*∈*Am* |*ν*|.

' *n*

' *n*+2 '

Let *k* ≥ *k* +*ν*¯+1 and *k* ≥ 2*ν*¯+2. Then for *ξ* ∈ R , *z* ∈ C and 1 ≤ *j* ≤ *L*(*m, n*),

'

|*ξ* + *zνj*|≥ |*ξ*|— *ν*¯ ≥ *k* if |*ξ*|≥ *k* — 1 and |*z*| = 1 *.* (21)

We obtain

|*E*˜*k*(*u*)|≤ (2*π*)−*n/*2

*L*(*m,n*) 1

*dξ*

Σ ∫

*T k* 2*π*

*j*=1

*j*

|*u*ˆ(*ξ* + *zνj*)| 1

|*z*|=1 |*P* (*ξ* + *zνj*)| |*z*|

∫

∫

|*dz*|

≤ 2 (2*π*)

*l*

−*n/*2

*L*(*m,n*) 1

*dξ*

Σ ∫

*T k* 2*π*

|*z*|=1

|*u*ˆ(*ξ* + *zνj*)||*dz*| (by ([14](#_bookmark7)), ([15](#_bookmark7)))

*j*=1 *j*

*L*Σ(*m,n*) ∫

1 ∫

|*u*ˆ(*ξ* + *zν* )|· |*ξ* + *zν* |*n*+2

*j*

≤ 2 (2*π*)

*l*

−*n/*2

*j*=1

*dξ*

*k* 2*π*

*T*

*j*

|*z*|=1

*j*

|*ξ* + *zνj*|

*n*+2 |*dz*|

≤ 2 (2*π*)

*l*

−*n/*2

*L*Σ(*m,n*) ∫

*dξ*  1 ∫

1

*n*+2 |*dz*|

*j*=1

*T*

*k* 2*π*

*j*

|*z*|=1 |*ξ* + *zνj* |

(by ([20](#_bookmark7))and ([21](#_bookmark7)), since |*ξ*|≥ *k* — 1 for *ξ* ∈ *T k*)

*j*

≤ 2 (2*π*)

*l*

−*n/*2

*L*Σ(*m,n*) ∫

*dξ*  1 ∫

1

*n*+2 |*dz*|

*j*=1

*T*

*k* 2*π*

*j*

|*z*|=1 (*k* — 1 — *ν*¯)

≤ 2 (2*π*)

*l*

−*n/*2

*L*(*m,n*)

*T k*

Σ ∫

*j*=1

*j*

1

*dξ*

(*k* — 1 — *ν*¯)*n*+2

≤ 2 (2*π*)

*l*

−*n/*2 (2*k*)*n*

(*k* — 1 — *ν*¯)*n*+2 (by ([15](#_bookmark7)))

≤ 2 · (2*π*)

*l*

−*n/*2

22*n*+2 1 (since *k* 2*ν*¯ + 2)

*k*2

≥

·

Since Σ

≤

*k>K k*2

1 1

*K*

we obtain for all *K* ≥ *k*' + 2*ν*¯ + 2,

Σ |*E*˜ (*u*)| ≤ 2*l*+2*n*+2 · (2*π*)−*n/*2 1 *,*

∞

*k K*

*k*=*K*+1

and by ([15](#_bookmark7)) - ([18](#_bookmark7)),

∞

Σ

*E*˜(*u*) =

*E*˜*k* (*u*) (22)

*k*=0

and for all *K* ≥ *k*' + 2*ν*¯ + 2,

|*E*˜(*u*) — Σ *E*˜ (*u*)| ≤ 2*l*+2*n*+2 · (2*π*)−*n/*2 1

*K*

*.* (23)

*k K*

*k*=0

It remains to prove that the fundamental solution *E* can be *computed* from the polynomial *P* . Concretely, a polynomial *P* of degree *m* is given by the list

*c*˜ = (*cα*)|*α*|≤*m*, *cα* C, of its coefficients represented by *ρ*2*N*(*m,n*). Let *Am* be a set of vectors according to Lemma [3.1](#_bookmark4).

∈

1. By Lemma [3.2](#_bookmark5), the multifunction *c*˜|⇒ *l* such that ([9](#_bookmark5)) is (*ρ*2*N*(*m,n*)*, ν*N)- computable.
2. Next we compute the sets Ω*j* defined in ([14](#_bookmark7)). For each number *j* the func- tion (*l, c*˜*, ξ*) inf|*z*|=1 *P* (*ξ*+*zνj*) 2−*l* is (*ν*N*, ρ*2*N*(*m,n*)*, ρn, ρ*)-computable. The proof is essentially that of ([11](#_bookmark5)). By Thm. 3.3.12 in [[5](#_bookmark11)] on type con- version, for each *j* the function

'→ | |—

−*l*

(*l, c*˜) (*ξ* inf

'→ '→

|*z*|=1

|*P* (*ξ* + *zνj*)|— 2 )

is (*ν*N*, ρ*2*N*(*m,n*)*,* [*ρn* → *ρ*])-computable. For the open subsets of R*n* we use the representation *θ<* defined by *θ<*(*p*) = *U* , iff *p* ∈ Σ*ω* is (encodes) a list of all *I* ∈У1 such that *I*⊆*U* (Def. 5.1.15 in [[5](#_bookmark11)]). Notice that

*U* =

*I*⊆*U I*◦ =

*I*⊆*U I*. Since the set {*y* ∈ R | *y >* 0} is r.e. open, by

Thm. 6.2.4.1 in [[5](#_bookmark11)] for each *j* the function

(*l, c*˜) '→ Ω*j* is (*ν*N*, ρ*2*N*(*m,n*)*, θ<*)-computable.

1. We describe how to compute sets *T k* from sequences *pj* ∈ Σ*ω* such that *θ<*(*pj*) = Ω*j*. Consider *k* ∈ N. Simultaneously for all *j* = 1*,... , L*(*m, n*) produce the lists of intervals *I* ∈ У1 encoded by the *pj*. Halt as soon asa finite set of intervals *C*⊆У1 has been found such that *Mk* \ *Mk*−1⊆ *I*∈*C I* (finite covering of a compact set). For each *j* let *Cj*⊆*C* be the set of

◦

*j*

intervals from *C* so far listed by *pj*. Let *Tk* := ∅ and determine the

0

*Tk*⊆У successively by the rule

*j*

*Tk* := ( *I* ∩ (*Mk* \ *Mk*−1)) \ *Tk .*

*j*

*I*∈*Cj*

*j*'

*j*'*<j*

If ([9](#_bookmark5)) holds for the number *l*, then *j* Ω*j* = R*n*, a set *C* will be found by compactness of *Mk* \ *Mk*−1, and ([15](#_bookmark7)) and ([16](#_bookmark7)) are true. As a summary, we have a multifunction

(*l, c*˜) |⇒ ((*k, j*) '→ *Tk*) which is (*ν*N*, ρ*2*N*(*m,n*)*,* [*ν*2 → *νB*])-computable.

*j*

N

1. Next we prove that (*c*˜*, k, u,* ((*k, j*) '→ *T k*)) '→ *E*˜*k* (*u*) defined in ([17](#_bookmark7)) is (*ρ*2*N*(*m,n*)*, ν*N*, δD,* [*ν*2 → *νB*]*, ρ*2)-computable.

*j*

N

First let *j* be fixed. Since the identity from Ð(R*n*) to £(R*n*) is (*δD, δS*)- computable, the Fourier transform is (*δS, δS*)-computable and the identity

from £(R*n*) to *C*(R*n*) is (*δS,* [*ρn* → *ρ*2])-computable, the function

*u*ˆ(*ξ* + *zνj*) 1 (*u, c*˜*, ξ, z*) '→ *P* (*ξ* + *zν* ) *z*

*j*

is (*δD*

*, ρ*2*N*(*m,n*)*, ρn, ρ*2*, ρ*2)-computable *.*

Therefore by type conversion,

*u*ˆ(*ξ* + *zνj*) 1

(*u, c*˜*, ξ*) '→ (*z* '→ ) is (*δ*

*D*

*, ρ*2*N* (*m,n*) *, ρn,* [*ρ*2 →

*ρ*2])-computable *.*

*P* (*ξ* + *zνj*) *z*

By Lemma [3.3](#_bookmark6) the function

*p*

(*u, c*˜*, ξ*)

'→ ∫

|*z*|=1

*u*ˆ(*ξ* + *zνj*) 1

*dz* is (*δD*

*P* (*ξ* + *zνj*) *z*

*, ρ*2*N*(*m,n*)*, ρn, ρ*2)-computable *.*

Again after type conversion by Lemma [3.3](#_bookmark6), the function

(*I, u, c*˜) '→ ∫

*dξ* ∫

*u*ˆ(*ξ* + *zνj*) 1

*dz* is (*ν*

*B D*

*,δ , ρ*2*N*(*m,n*)*, ρ*2)-computable *.*

*I* |*z*|=1

*P* (*ξ* + *zνj*) *z*

Finally, since *π* is computable, we can conclude that

(*c*˜*, k, u,* ((*k, j*) '→ *Tk*)) '→ *E*˜*k* (*u*)

*j*

is (*ρ*2*N*(*m,n*)*, ν*N*, δD,* [*ν*2 → *νB*]*, ρ* )-computable. By Thm. 3.1.7 in [[5](#_bookmark11)] on

N

2

primitive recursion the function

(*c*˜*, K, u,* ((*k, j*) '→ *Tk*)) '→ Σ *E*˜*k*(*u*)

*K*

*j*

*k*=0

is (*ρ*2*N*(*m,n*)*, ν*N*, δD,* [*ν*2 → *νB*]*, ρ*2)-computable.

N

1. As above, from *u* ∈ Ð(R*n*), *u*ˆ ∈ £(R*n*) can be computed. By the defini- tion of *δS*,a number *k*' can be computed from *u*ˆ such that |*v*(*ξ*)||*ξ*|*n*+2 ≤ 1 if |*ξ*| ≥ *k*'. From *l*, *k*' and *L* ∈ N, a number *K* can be computed such that *K* ≥ *k*' + 2*ν*¯ +2 and *K* ≥ 2*l*+2*n*+2 · (2*π*)−*n/*2 · 2*L*+1. Then by ([23](#_bookmark7)),

*K*

Σ

|*E*˜(*u*) — *E*˜*k*(*u*)| ≤ 2−*L*−1 *.*

*k*=0

By (d) above, from *c*˜*, K, u,* ((*k, j*) '→ *T k*) and *L* a rational *bL* ∈ C can be

*j*

computed such that

*K*

Σ

|*bL* — *E*˜*k* (*u*)| ≤ 2−*L*−1 *.*

*k*=0

From the sequence *b*0*, b*1*,...* we can compute a *ρ*2-name of *E*˜(*u*). There- fore, after type conversion, the function

(*c*˜*, l,* ((*k, j*) '→ *Tk*)) '→ (*u* '→ *E*˜(*u*))

*j*

is (*ρ*2*N*(*m,n*)*, ν*N*,* [*ν*2 → *νB*]*,* [*δD* → *ρ*2])-computable.

N

From (a), (c) and (e), the multifunction *c*˜|⇒ *E*˜ is (*ρ*2*N*(*m,n*)*,* [*δD* → *ρ*2])-com- putable. Since *<E, u>* = *<E*˜*,* Y*u>* = *<*Y*E*˜*, u>* and Y is computable on Ð'(R*n*) by Lemma 4.7 in [[6](#_bookmark13)], the operator *u* |⇒ *E* mapping a polynomial of degree *m* to some fundamental solution is (*ρ*2*N*(*m,n*)*,* [*δD* → *ρ*2])-computable.

So far we have assumed a fixed degree *m* of the polynomial. Since the set *Am* can be computed from *m* (Lemma [3.1](#_bookmark4)) and all the other computations are uniform also in *m* the proof is finished.

In the proof multivalued functions occur several times. A critical part is the computation of the sets *T k* which determines a partition of R*n* into (at most) *L*(*m, n*) sets. The resulting distribution depends essentially on this partition. However, the function from C*N*(*m,n*) to these sets cannot be com- putable (hence continuous) and single-valued at the same time, provided it is not trivial (Lemma 4.3.15 in [[5](#_bookmark11)]).

*j*

Problem: Is there a *single-valued* computable function mapping any polyno- mial of degree *m* to a fundamental solution?

As usual in recursion theory, we have not merely proved the *existence* of a computable function (from names of input objects to names of output objects) but the proof is *constructive*, i.e. it explains how a concrete Turing machine or computer program can be constructed for computing this function. Of course the algorithm is not trivial since subroutines for integration, for Fourier transform on Schwartz space etc. have to be included. The presented algorithms can be improved, since, e.g., no derivatives of the Fourier transform

*u*ˆ of the input test function *u* ∈ £(R*n*) are needed for computing the *E*˜*k* (*u*)

by integration.

Problem: Can our informal algorithm be converted to a feasible numerical algorithm for computing fundamental solutions efficiently or is the problem

inherently difficult?

# References

1. Jos´e Barros-Neto. *An introduction to the theory of distributions*, volume 14 of *Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1973.
2. Jeffrey Rauch. *Partial differential equations*, volume 128 of *Graduate Texts in Mathematics*. Springer, New York, 2nd edition, 1997.
3. Fran¸cois Treves. *Linear Partial Differential Equations with Constant Coefficients*. Gordon and Breach Science Publishers, New York, 1966.
4. Klaus Weihrauch. Computability on computable metric spaces. *Theoretical Computer Science*, 113:191–210, 1993. Fundamental Study.
5. Klaus Weihrauch. *Computable Analysis*. Springer, Berlin, 2000.
6. Ning Zhong and Klaus Weihrauch. Computability theory of generalized functions. *Journal of the Association for Computing Machinery*, 50(4):469–505, 2003.