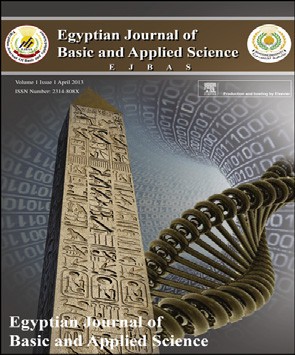
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Analytical approximations of two and three dimensional time-fractional telegraphic equation by reduced differential transform method

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## a b s t r a c t

In this article, an analytical solution based on the series expansion method is proposed to solve the time-fractional telegraph equation (TFTE) in two and three dimensions using a recent and reliable semi-approximate method, namely the reduced differential trans- formation method (RDTM) subjected to the appropriate initial condition. Using RDTM, it is possible to find exact solution or a closed approximate solution of a differential equation. The accuracy, efficiency, and convergence of the method are demonstrated through the four numerical examples.

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# Introduction

Several real phenomena emerging in engineering and science fields can be demonstrated successfully by developing models using the fractional calculus theory. Fractional differential theory has gained much more attention as the fractional order system response ultimately converges to the integer order equations. The applications of the fractional differentiation

for the mathematical modeling of real world physical prob- lems such as the earthquake modeling, the traffic flow model with fractional derivatives, measurement of viscoelastic ma- terial properties, etc., have been widespread in this modern era. Before the nineteenth century, no analytical solution method was available for such type of equations even for the linear fractional differential equations. Recently, Keskin and Oturanc [[1]](#_bookmark30) developed the reduced differential transform

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method (RDTM) for the fractional differential equations and showed that RDTM is the easily useable semi analytical method and gives the exact solution for both the linear and nonlinear differential equations.

Let us assume that *u*(*x*,*y*,*t*) and *i*(*x*,*y*,*t*) denote the electric voltage and the current in a double conductor, then the time- fractional telegraphic equations (TFTEs) in the two dimension (2D) aregiven as

v2a *u* + 2*p* va *u* + *q*2*u* = v2 *u* + v2 *u* + *f* (*x*; *y*; *t*); 9=

v*t*2a

v*t*a

v*x*2

v*y*2

1

; (*x*; *y*; *t*)˛U; *p* > 0; *q* > 0

equation directly without using linearization, transformation, discretization or restrictive assumptions. Also, the RDTM scheme is very easy to implement for the multidimensional time-fractional order physical problems emerging in various fields of engineering and science.

# Fractional calculus

v2a *i* + 2*p*  va *i* + *q*2*i* =  v2 *i* +  v2 *i* + *f* (*x*; *y*; *t*); ;

v*t*2a

v*t*a

v*x*2

v*y*2

2

(1)

In this section, we demonstrate some notations and defini-

tions that will be used further in the study. Fractional calculus

where U = [*a*,*b*] × [*c*,*d*] × [*t*>0]. The initial conditions are

assumed to be

*u*(*x*; *y*; 0)= f1(*x*; *y*); 9>=

*ut*(*x*; *y*; 0)= f2(*x*; *y*);

*i*(*x*; *y*; 0)= c (*x*; *y*);

; (*x*; *y*)˛U (2)

*i x y* 0 1 *x y* >;

*t*( ;

;

)= c2( ;

);

theory is almost more than two decades’ old in the literature.

Several definitions of fractional integrals and derivatives have been proposed but the first major contribution to give proper definition is due to Liouville as follows.

Definition 2.1. *A real function f*(*x*),*x* > 0 *is said to be in the space*

*C*m,m ˛ *ℝ if there exists a real number q*(>m), *such that f*(*x*) = *xqg*(*x*),

Similarly, the three dimensional (3D) time-fractional order

*where g*(*x*) ˛ *C*[0,N) , *and it is said to be in the space Cm if*

telegraphic equation (TFTE) can be given as

v2a *u* + 2*p* va *u* + *q*2*u* = v2 *u* + v2 *u* + v2 *u* + *f* (*x*; *y*; *z*; *t*); 9>=

m

v*t*2a

v*t*a

v*x*2

v*y*2

v*z*2

1

; (*x*; *y*; *z*; *t*)˛U; *p* > 0; *q* > 0 (3)

v*t*2a

v*t*a

v*x*2

v*y*2

v*z*2

2

*f*(*m*) ˛ *C*m,*m*˛*ℕ*.

v2a *i* + 2*p* va *i* + *q*2*i* =  v2 *i* +  v2 *i* +  v2 *i* + *f* (*x*; *y*; *z*; *t*); >;

where U = [*a*,*b*] × [*c*,*d*] × [*e*,*f*] × [*t* > 0],with initial conditions

*u*(*x*; *y*; *z*; 0)= x1(*x*; *y*; *z*);

9>=

*u* (*x*; *y*; *z*; 0)= x (*x*; *y*; *z*);

*integral operator* [[23]](#_bookmark31) *of order* a ≥ 0, *is defined as* Definition 2.2. *For a function f*, *the Riemann*e*Liouville fractional*

8 *x*

a

1

0

*t* 2 ; (*x*; *y*; *z*)˛U (4)

*i*(*x*; *y*; *z*; 0)= j (*x*; *y*; *z*);

>

>;

Z a—1

*i x y z* 0

*t*( ;

;

;

1

*x y z*

)= j2( ;

;

)

< *J f* (*x*)= G(a)

(*x* — *t*)

*f* (*t*)*dt*; a > 0; *x* > 0;

(5)

In Eqs. [(1) and (3)](#_bookmark4) *p* and *q* denote constants. For *p* > 0, *q* = 0,

>:

[(1) and (3)](#_bookmark4) represent time-fractional order damped wave

equations in two and three dimensions respectively.

It has been observed that telegraph equation is more suitable than ordinary diffusion equation in modeling reac- tion diffusion. The hyperbolic partial differential equations model the vibrations of structures (e.g. machines, buildings and beams) and they are the basis for fundamental equations of atomic physics. The telegraph equation is an important equation for modeling several relevant problems in engi- neering and science such as wave propagation [[2]](#_bookmark25), random walk theory [[3]](#_bookmark26), signal analysis [[4]](#_bookmark27) etc. In recent years, from the literature it can be seen that much attention has been given to

> *J*0*f* (*x*)= *f* (*x*)

*The Riemann*e*Liouville derivative has certain disadvantages when*

*trying to model real world problems with fractional differential equations*. *To overcome this discrepancy*, *Caputo and Mainardi* [[24]](#_bookmark32) *proposed a modified fractional differentiation operator D*a *in his work on the theory of viscoelasticity*. *The Caputo fractional derivative al- lows the utilization of initial and boundary conditions involving integer order derivatives*, *which havte clear physical interpretations*.

Definition 2.3. *The fractional derivative of f in the Caputo sense*

[[25]](#_bookmark33) *can be defined as*

1 Z*x*

the development of analytical and numerical schemes for the one dimensional and two dimensional hyperbolic fractional

*D*a*f* (*x*)= *Jm*—a*Dmf* (*x*)=

G(*m* — a)

0

(*x* — *t*)*m*—a—1*f* (*m*)(*t*)*dt*; (6)

and non-fractional TFTE [[5](#_bookmark28)e[22]](#_bookmark28). To the best of our knowledge

till now no one has applied the RDTM to solve the time-frac- tional order telegraphic equations in two and three dimensions.

In this paper, we propose an analytical scheme namely the reduced differential transformation method based on series solution method to find analytical solutions of the time- fractional telegraph equation (TFTE) in two and three di- mensions. The accuracy and efficiency of the proposed

*for m* — 1 < a*s* ≤ *m*, *m*˛*ℕ*, *x* > 0, *f* ˛*Cm* .

The fundamental basic properties of the Caputo fractional

—1

derivative are given as.

Lemma. *If m*—1 < a ≤ *m*,*m* ˛ *ℕ and f* ˛*Cm*; m ≥ —1, *then*

m

8>< *D*a*J*a*f* (*x*)= *f* (*x*); *x* > 0; 

:

( )= ( )— ( ) ; > ;

*k*=0

*k*!

method are demonstrated by the four test examples. The

main advantage of the method is that it solves the telegraph

> *D*a*J*a*f x f x* P*m f* (*k*) 0+ *xk x* 0

(7)

62 [e gypti an j o ur nal o f b a sic and a pp l i ed sci e n c e s 1 ( 2014) 60](http://dx.doi.org/10.1016/j.ejbas.2014.01.002) e[66](http://dx.doi.org/10.1016/j.ejbas.2014.01.002)

|  |  |
| --- | --- |
| Table 1 e Fundamental operations of the reduced differential transform method. | |
| Original function | Reduced differential transformed function |
| *R*D[*u*(*x*,*y*,*z*,*t*)*v*(*x*,*y*,*z*,*t*)] | *Uk*(*x*; *y*; *z*)5*Vk*(*x*; *y*; *z*)= P*k Ur*(*x*; *y*; *z*)*Vk*—*r*(*x*; *y*; *z*) |
|  |  |
| *R*D[a*u*(*x*,*y*,*z*,*t*) b*v*(*x*,*y*,*z*,*t*)] | a*Uk*(*x*,*y*,*z*) b*Vk*(*x*,*y*,*z*) |
| *R*D[v*N*a/v*tN*a*u*(*x*,*y*,*z*,*t*)] | G(*k*a + *N*a + 1)/G(*k*a + 1)*Uk*+*N*(*x*,*y*,*z*) |
| *R*D[v*m*+*n*+*p*+*s*/v*xm*v*yn*v*zp*v*tsu*(*x*,*y*,*z*,*t*)] | (*k* + *s*)!/*k*!v*m*+*n*+*p*/v*xm*v*yn*v*zpUk*+*s*(*x*,*y*,*z*) |
| *R*D[*xmynzptq*] | {xmynzp, k = q0, otherwise |
| *R*D[*e*l*t*] | l*k*/*k*! |
| *R*D[sin(a*x* + b*y* + g*z* + u*t*)] | *wk*/*k*!sin(p*k*/2! + a*x* + b*y* + g*z*) |
| *R*D[cos(a*x* + b*y* + g*z* + u*t*)] | *wk*/*k*!cos(p*k*/2! + a*x* + b*y* + g*z*) |

*In this study*, *the Caputo fractional derivative is taken since it allows traditional initial and boundary conditions to be included in the derivation of the problem*. *Some other properties of fractional de- rivative can be found in* [[25,26]](#_bookmark33).

N

*w*(*x*; *y*; *z*; *t*)=

X

*k*=0

1

G(*k*a + 1)

v*k* v*tk*

*w*(*x*; *y*; *z*; *t*)#

*t*=*t*0

(*t* — *t*0)*k*a (11)

# Reduced differential transform method

"

*When t* = 0, Eq. [(11)](#_bookmark6) *reduces to*

# (RDTM)

In this section, we introduce the basic definitions of the reduced differential transformations.

N

*w*(*x*; *y*; *z*; *t*)=

X

*k*=0

1

G(*k*a + 1)

"

v*k* v*tk*

*w*(*x*; *y*; *z*; *t*)#

*t*=*t*0

*tk*a (12)

that it can be represented as a product *w*(*x*,*y*,*z*,*t*) = *F*(*x*,*y*,*z*)*G*(*t*). Consider a function of four variables *w*(*x*,*y*,*z*,*t*), and assume On extending the basis of the properties of the one-

dimensional differential transformation [[26,27]](#_bookmark34), the function

*From the* Eq. [(11)](#_bookmark6), *it can be seen that the concept of the reduced*

*differential transform is derived from the power series expansion of the function*.

Definition 2.2. *If u*(*x*; *y*; *z*; *t*)= *R*—1[*Uk*(*x*; *y*; *z*)],*v*(*x*; *y*; *z*; *t*)= *R*—1

*w*(*x*,*y*,*z*,*t*) can be represented as D D

N

X

*w*(*x*; *y*; *z*; *t*)=

XN XN

N

*F*(*i*1; *i*2; *i*3)*xi*1 *yi*2 *zi*3

X

*G*(*j*)*tj*

[*Vk*(*x*; *y*; *z*)]; *and the convolution* 5 *denotes the reduced differential*

*transform version of the multiplication*, *then the fundamental*

*i*1 =0 *i*2 =0 *i*3 =0

*j*=0

*operations of the reduced differential transform are shown in the*

N N N N

X X X X

= *W*(*i*1; *i*2; *i*3)*xi*1 *yi*2 *zi*3 *tj*; (8)

*i*1 =0 *i*2 =0 *i*3 =0 *j*=0

where *W*(*i*1,*i*2,*i*3) = *F*(*i*1,*i*2,*i*3)*G*(*j*) is called the spectrum of *w*(*x*,*y*,*z*,*t*). Let *R*D denotes the reduced differential transform operator

[Table 1](#_bookmark5).

*In* [Table 1](#_bookmark5), G *represents the Gama function*, *which is defined as*

N

Z

G(g) := *e*—*tt*g—1d*t*; g˛ℂ (13)

and *R*—1 the inverse reduced differential transform operator.

D

The basic definition and operation of the RDTM method is

described below.

Definition 2.1. *If w*(*x*,*y*,*z*,*t*) *is analytic and continuously differen- tiable with respect to space variables x*,*y and time variable t in the domain of interest*, *then the spectrum function* [[28,29]](#_bookmark35)

0

# RDTM for two dimensional TFTE

Applying the RDTM to the two dimensional TFTE (1), we have the following relation

1 " v*k* #

*R*D[*w*(*x*; *y*; *z*; *t*)]z*Wk*(*x*; *y*; *z*)=

G(*k*a + 1)

v*tk w*(*x*; *y*; *z*; *t*)

*t*=*t*0

(9)

Now applying the method to the initial conditions (2), we get

G(*k*a+2a+1) *U*

G(*k*a+1)

*k*+2

. (14)

*k*+2

G(*k*a+1)

*k*

v*x*2 *k*

v*y*2 *k*

D

2

(x; y)+ 2*p* G(*k*a+a+1) *U*

(x; y)+ *q*2*U* (x; y)= v2 *U* (x; y)+ v2 *U* (x; y)+ *R* *f* (*x*; *y*; *t*) ; 9=

G(*k*a+2a+1) *I*

G(*k*a+1)

*k*+1

*k*

v*x*2

*k*

v*y*2

*k*

D

1

G(*k*a+1)

(x; y)+ 2*p* G(*k*a+a+1) *I*

(x; y)+ *q*2*I* (x; y)= v2 *I* (x; y)+ v2 *I* (x; y)+ *R* *f* (*x*; *y*; *t*) ; ;

*is the reduced transformed function of w*(*x*,*y*,*z*,*t*).

*k*+1

*In this article*, (*lowercase*) *w*(*x*,*y*,*z*,*t*) *represents the original*

*function while* (*uppercase*) *Wk*(*x*,*y*,*z*) *stands for the reduced trans-*

*U*0(*x*; *y*) = f1(*x*; *y*); 9>=

*formed function*. *The differential inverse reduced transform of* 0

*U*1(*x*; *y*)= f2(*x*; *y*);

*I* (*x*; *y*)= c (*x*; *y*);

; (*x*; *y*)˛U. (15)

1 >;

*Wk*(*x*,*y*,*z*) *is defined as*

N

X *k*

*R*—1[*Wk*(*x*; *y*; *z*)]z*w*(*x*; *y*; *z*; *t*)= *Wk*(*x*; *y*; *z*)(*t* — *t*0) a (10)

D

*k*=0

*I*1(*x*; *y*)= c2(*x*; *y*);

From above two equations we get the values of

*Uk*(*x*,*y*),*Ik*(*x*,*y*),*k* = 2,3,4,... etc. Using the differential inverse reduced transform of *Uk*(x,y);*Ik*(x,y),*k* = 0,1,2,3,. , we get the

*Combining* Eqs. [(9) and (10)](#_bookmark8), *we get* approximate solution for *u*(*x*,*y*,*t*) and *i*(*x*,*y*,*t*) as

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*u x y t*

PN *U*

x y *tk*a

*U* x y

*U* x y *t*a

*U* x y *t*2a

*U* x y *t*3a 9>

( , , )=

*k*=0

, )=

*k*=0

*k*

( ,

)

=

0

( ,

)+

1

( ,

)

+

2

( ,

)

+

3

( ,

)

+ ...

;

*k*( , )

= 0( ,

)+ 1( , )

+ 2( , )

+ 3( , )

+ ...,

=. (16)

*i x y t*

( ,

PN *I*

x y *tk*a

*I* x y

*I* x y *t*a

*I* x y *t*2a

*I* x y *t*3a >

# RDTM for three dimensional TFTE

Applying the RDTM to the three dimensional TFTE (4), we have the following relation

*Using the RDTM to the initial conditions* [(21)](#_bookmark12), *we have*

*U*0(*x*, *y*)= *ex*+*y*; *U*1(*x*, *y*)= —3*ex*+*y*. (23)

G(*k*a+2a+1) *U*

G(*k*a+1)

*k*+2

. (17)

*k*+2

G(*k*a+1)

*k*+1

*k*

v*x*2 *k*

v*y*2 *k*

v*z*2 *k*

D

2

(x, y, z)+ 2*p* G(*k*a+a+1) *U*

(x, y, z)+ *q*2*U* (x, y, z)= v2 *U* (x, y, z)+ v2 *U* (x, y, z)+ v2 *U* (x, y, z)+ *R* *f* (*x*, *y*, *z*, *t*) ; 9=

G(*k*a+2a+1) *I*

G(*k*a+1)

*k*+1

*k*

v*x*2

*k*

v*y*2

*k*

v*z*2

*k*

D

1

G(*k*a+1)

(x, y, z)+ 2*p* G(*k*a+a+1) *I*

(x, y, z)+ *q*2*I* (x, y, z)= v2 *I* (x, y, z)+ v2 *I* (x, y, z)+ v2 *I* (x, y, z)+ *R* *f* (*x*, *y*, *z*, *t*) , ;

Now applying the method to the initial conditions (4), we get

*From* Eq. [(23)](#_bookmark9) *into* Eq. [(22)](#_bookmark13), *we get the following Uk*(x, y) *values successively*

*I*0(*x*, *y*, *z*)= j1(*x*, *y*, *z*),

>

*U*1(*x*, *y*, *z*)= x2(*x*, *y*, *z*),

, (*x*, *y*, *z*)˛U. (18)

(—3)*k*

l + 1

*U*0(*x*, *y*, *z*) = x1(*x*, *y*, *z*), 9>=

*I*1(*x*, *y*, *z*)= j2(*x*, *y*, *z*), ;

l

*U*

*x*, *y*)=

G

*ex*+*y*; *k* ≥ 2. (24)

*k*( G *k* + 1 l

Applying the same procedure as in the case of 2D TFTE, we

get the approximate solution for *u*(*x*,*y*,*z*,*t*) and *i*(*x*,*y*,*z*,*t*) as

*where* a = 1/l,l > 0. *Using the differential inverse reduced transform*

*of Uk*(*x*,*y*), *we get*

*u x y z t*

PN *U*

x y z *tk*a

*U* x y z

*U* x y z *t*a

*U* x y z *t*2a 9>

( , ,

, )=

*k*=0

, )=

*k*=0

*k*

( ,

,

)

=

0

( ,

,

)+

1

( ,

,

)

+

2

( ,

,

)

+ ...

;

*k*( , , )

= 0( ,

, )+

1( , , )

+ 2( , , )

+ ...,

=. (19)

*i x y z t*

( ,

,

PN *I*

x y z *tk*a

*I* x y z

*I* x y z *t*a

*I* x y z *t*2a >

# Numerical examples

*u x y t*

( ,

PN *U*

*x y tk*a

*k*( ,

)

PN *U*

=

*k*( ,

)

*k*=0

*x y tk*.l

In this section, we describe the method explained in the

, )=

*k*=0

Sections [4 and 5](#_bookmark7) by taking four examples of both linear and

*t*

= *U* (*x*, *y*) + *U* (*x*, *y*)*t*1.l + *U* (*x*, ) 2.l + *U* (*x*, *y*)*t*3.l + ...

(25)

nonlinear 2D and 3D TFTEs to validate the efficiency and

3

0

1

2

*y t*

3

l

G(2+1)

l

l

)

reliability of the RDTM scheme.

= *ex*+*y* 1 + (—3) 1.l + G l+1 (—3)2 *t*2.l + (—3)3

3.l + ... .

Example 6.1. *Consider the* 2*D linear TFTE*

*t*

G( +1

Eq. [(25)](#_bookmark10) *represents the solution of the TFTE* [(20)](#_bookmark11). *When* l = 1, *i*.*e*.

v2a *u*

va*u*

v2*u*

v2*u*

a = 1, *we get*

v*t*2a + 2 v*t*a + *u* = v*x*2 + v*y*2 (20)

*subject to the initial conditions*

2!

3!

*k*!

*u*(*x*, *y*, *t*)= *ex*+*y* 1 + (—3)*t* + (—3)2 *t*2 + (—3)3 *t*3 + ......... + (—3)*ktk* +

*u x y* 0

( ,

,

*ex*+*y*

= *ex*+*y*—3*t*.

*ut*(*x*, *y*, 0) = —3*ex*+*y*,

)=

,

. (21)

*Applying the RDTM to* Eq. [(20)](#_bookmark11), *we obtain the following recurrence*

*relation*

Example 6.2. *Consider the following 3D linear TFTE*

(26)

G(*k*a + 2a + 1) *U* (x, y)+ 2

G(*k*a + 1) *k*+2

G(*k*a + a + 1) *U* (x, y)

G(*k*a + 1) *k*+1

v2a *u*

va*u*

v2*u*

v2*u*

v2*u*

v2 v2

= *Uk*(x, y)+ *Uk*(x, y)— *Uk*(x, y). (22)

v

v*x*2 *y*2

v*t*2a + 2 v*t*a + *u* = v*x*2 + v*y*2 + v*z*2 (27)

*subject to initial conditions*

64 [e gypti an j o ur nal o f b a sic and a pp l i ed sci e n c e s 1 ( 2014) 60](http://dx.doi.org/10.1016/j.ejbas.2014.01.002) e[66](http://dx.doi.org/10.1016/j.ejbas.2014.01.002)

*u*(*x*, *y*, *z*, 0)= sinh(*x*)sinh(*y*)sinh(*z*),

*ut*(*x*, *y*, *z*, 0)= —sinh(*x*)sinh(*y*)sinh(*z*),

. (28)

*Applying the RDTM to* Eq. [(27)](#_bookmark14), *we obtain the following recurrence*

G(*k*a + 2a + 1) *U*

G(*k*a + a + 1)

(x, y)+ 2 *U*

(x, y)

*relation*

G(*k*a + 1)

v2

= v*x*2 *Uk*(x, y)+ v*y*2 *Uk*(x, y)—

*k*+2

v2

G(*k*a + 1)

X*k*

*r*=0

*Ur*(x, y)*Uk*—*r*(x, y)

*k*+1

G(*k*a + 1) *Uk*+2(x, y, z)+ 2

G(*k*a + 2a + 1)

G(*k*a + a + 1)

G(*k*a + 1) *Uk*+1(x, y, z)

+ *e*2(*x*+*y*)

(—4)*k*!

— *e*(*x*+*y*)

(—2)*k*!

. (36)

v2 v2 v2

*k*! *k*!

= v*x*2 *Uk*(x, y, z)+ *y*2 *Uk*(x, y, z)+ *z*2 *Uk*(x, y, z)— *Uk*(x, y, z).

v v

(29)

*Using the RDTM to the initial conditions* [(28)](#_bookmark15), *we have*

*U*0(*x*, *y*, *z*)= sinh(*x*)sinh(y)sinh(z);

*Using the RDTM to the initial conditions* [(34)](#_bookmark21), *we get*

*U*0(*x*, *y*)= *ex*+*y*; *U*1(*x*, *y*)= —2*ex*+*y*. (37)

*Using* Eq. [(37)](#_bookmark19) *in* Eq. [(36)](#_bookmark16), *we get the following Uk*(x,y) *values*

*successively*

(30)

*U* (*x*, *y*, *z*)= —sinh(*x*)sinh(y)sinh(z).

l

1

*U*

*x*, *y*)=

G

*ex*+*y*; *k* ≥ 2. (38)

*k*(

G *k* + 1

l

(—2)*k*  l + 1

*From* Eq. [(30)](#_bookmark18) *into* Eq. [(29)](#_bookmark17), *we get the following Uk*(x,y,z) *values*

*successively*

*Using the differential inverse reduced transform of Uk*(*x*,*y*), *we get*

.

(—1)*k*

*U*

*x*, *y*, *z*)=

l + 1

N

*u*(*x*, *y*, *t*) =

*k*=0

*k*=0

P

N

*Uk*(*x*, *y*)*tk*a =

P

*k* l

*Uk*(*x*, *y*)*t*

*k* ( G *k* + 1 l

G

sinh(*x*)sinh(*y*)sinh(*z*); *k* ≥ 2. (31)

l

. . .

= *e*

1 + (—1)*t*

*t*

G( +1

+ *t*

G +1

+ ...

.

= *U*0(*x*, *y*)+ *U*1(*x*, *y*)*t*

1

l

+ *U*2(*x*, *y*)*t*

2

l

+ *U*3(*x*, *y*)*t*

3

l

+ ...

(39)

*Using the differential inverse reduced transform of Uk*(x,y,z), *we*

3

*get*

*x*+*y*

1.l

l+1 (—2)2

2

+ G

l

2.l

(—2)3

3.l

*u x y z t*

( ,

,

PN *U*

*x y z tk*a

,

)

PN *U*

*k*=0

*x y z tk*.l

l ) (l )

= *U* (*x*, *y*, *z*)+ *U* (*x*, *y*, *z*)*t*1.l + *U* (*x*, *y*, *z*)*t*2.l + *U* (*x*, *y*, *z*)*t*3.l + ...

, )=

*k*=0

*k*( ,

=

*k*( ,

,

)

0

1

2

3

(32)

= sinh(*x*)sinh(*y*)sinh(*z*) 1 + (—1)*t*1.l + G l+1 (—1)2 2.l + (—1)3 *t*3.l + ... .

2

l

*t*

G( +1

G(3+1)

l

)

l

Eq. [(32)](#_bookmark20) *represents the solution of the TFTE* [(27)](#_bookmark14). *When* l = 1, *i*.*e*.

a = 1, *we get*

u(*x*, *y*, *z*, *t*) = *e*—*t* sinh(*x*)sinh(*y*)sinh(*z*), (33)

*which is the closed form solution of the non-fractional form of the*

*TFTE* [(27)](#_bookmark14).

Example 6.3. *Consider the following* 2*D nonlinear TFTE*

*When* l = 1, *we get the exact solution of the non-fractional form of the TFTE* [(34)](#_bookmark21) *as*

u(*x*, *y*, *t*)= *e*(*x*+*y*)—2*t*. (40)

Example 6.4. *Consider the* 3*D nonlinear TFTE given as*

v2*u* v*x*2

v2*u*

+ v*y*2 =

v2a *u* v*t*2a

va*u*

+ 2 v*t*a

+ *u*2 — e2(x+y)—4t+e(x+y)—2t (34)

v2*u* v*x*2

v2*u*

+ v*y*2 +

v*z*2 = v2*u*

v2a *u* v*t*2a

va*u*

+ 2 v*t*a

+ *u*2 — e2(x—y—z)—4t+e(x—y—z)—2t (41)

*subject to the initial conditions*

*under the initial conditions*

*u*(*x*, *y*, 0)= *ex*+*y*,

*ut*(*x*, *y*, 0)= —2*ex*+*y*,

. (35)

*u*(*x*, *y*, *z*, 0)= *ex*—*y*—*z*,

*ut*(*x*, *y*, *z*, 0)= —*ex*—*y*—*z*,

. (42)

*Applying the RDTM technique to* Eq. [(34)](#_bookmark21), *we obtain the following iterative formula*:

*Applying the RDTM technique to* Eq. [(41)](#_bookmark22), *we obtain the following*

*iterative formula*:

[e g ypti an j o ur nal o f b a sic and a pp l i e d sci en c e s 1 ( 2014) 60](http://dx.doi.org/10.1016/j.ejbas.2014.01.002) e[66](http://dx.doi.org/10.1016/j.ejbas.2014.01.002) 65

G(*k*a + 2a + 1)

G(*k*a + 1) *Uk*+2(*x*, *y*, *z*) + 2

G(*k*a + a + 1)

G(*k*a + 1) *Uk*+1(*x*, *y*, *z*)

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v2 v2 v2

= *Uk*(*x*, *y*, *z*)+ *Uk*(*x*, *y*, *z*)+ *Uk*(*x*, *y*, *z*)

v*x*2 v*y*2 v*z*2

X*k*  (—4)*k*!

[1993;9:789](http://refhub.elsevier.com/S2314-808X(14)00006-2/sref2)e[812](http://refhub.elsevier.com/S2314-808X(14)00006-2/sref2).

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—

*r*=0

*Ur*(*x*, *y*, *z*)*Uk*—*r*(*x*, *y*, *z*) + *e*2(*x*+*y*+*z*)

*k*!

(—2)*k*!

1. [Jordan PM, Puri A. Digital signal propagation in dispersive](http://refhub.elsevier.com/S2314-808X(14)00006-2/sref4) [media. J Appl Phys 1999;85:1273](http://refhub.elsevier.com/S2314-808X(14)00006-2/sref4)e[82](http://refhub.elsevier.com/S2314-808X(14)00006-2/sref4).
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— *e*(*x*+*y*+*z*)

*k*! . (43)

[for the one-space dimensional linear hyperbolic equation.](http://refhub.elsevier.com/S2314-808X(14)00006-2/sref5) [Appl Math Lett 2004;17:101](http://refhub.elsevier.com/S2314-808X(14)00006-2/sref5)e[5](http://refhub.elsevier.com/S2314-808X(14)00006-2/sref5).

*Using the RDTM to the initial conditions* [(42)](#_bookmark23), *we get*

*U*0(*x*, *y*, *z*)= *ex*—*y*—*z*; *U*1(*x*, *y*, *z*)= —*ex*—*y*—*z*. (44)

*Using* Eq. [(44)](#_bookmark29) *in* Eq. [(43)](#_bookmark24), *we get the following Uk*(x,y,z) *values*

*successively*

1. [Mohanty RK. An unconditionally stable difference formula](http://refhub.elsevier.com/S2314-808X(14)00006-2/sref6)

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(—1)*k*  l + 1

*ex*—*y*—*z k* ≥ 2.

[Appl 2010;60:1964](http://refhub.elsevier.com/S2314-808X(14)00006-2/sref8)e[72](http://refhub.elsevier.com/S2314-808X(14)00006-2/sref8).

*k*( ,

l

*U*

*x y*, *z*)=

G

G *k* + 1

(—1)*k*

*U* (*x*, *y*, *z*)=

;

l

l + 1

*ex*—*y*—*z*; *k* ≥ 2. (45)

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*k* G *k* + 1 l

l

G

*Using the differential inverse reduced transform of Uk*(x,y,z), *we get*

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*u x y z t*

( ,

,

PN *U*

*x y z tk*a

*k*( ,

,

)

PN *U*

*k*=0

*x y z*

*k*.l

[direction implicit scheme for the two space dimensional](http://refhub.elsevier.com/S2314-808X(14)00006-2/sref11)

= *U* (*x*, *y*, *z*)+ *U* (*x*, *y*, *z*)*t*1.l + *U* (*x*, *y*, *z*)*t*2.l + *U* (*x*, *y*, *z*)*t*3.l + ...

, )=

*k*=0

=

*k*( ,

,

)*t*

[linear hyperbolic equation. Numer Methods Partial Differ Eq](http://refhub.elsevier.com/S2314-808X(14)00006-2/sref11)

[2001;7:684](http://refhub.elsevier.com/S2314-808X(14)00006-2/sref11)e[8](http://refhub.elsevier.com/S2314-808X(14)00006-2/sref11).

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0

*x*—*y*—*z*

= *e*

1 + (—1)*t*

+ *t*

G +1

1

1.l

2

l+1 (—1)2

+ G

*t*

G( +1

2.l

(—1)3

3

3.l

+ ...

.

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l ) (l )

2

3

l

(46)

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*When* a = 1, *the exact solution of the non-fractional form of the*

*nonlinear TFTE* [(41)](#_bookmark22) *is obtained as*

u(*x*, *y*, *z*, *t*) = *ex*—*y*—*z*—*t*. (47)

# Conclusions

In the present study, we have illustrated the reduced differ- ential transform method for the analytical solution of two and three dimensional second order hyperbolic linear and nonlinear TFTEs. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. The effectiveness of the method is shown from the computational results. These results show that the RDTM technique is highly accurate, rapidly converge and is very easily implementable mathematical tool for the multidimensional physical problems emerging in various fields of engineering and sciences.

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