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Anti-Ramsey Threshold of Cycles for Sparse Graphs [1](#_bookmark0)*,*[2](#_bookmark0)

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Abstract

For graphs *t* and *H*, let *t −*r*→*b *H* denote the property that for every *proper* edge colouring of *t* there is a *rainbow* copy of *H* in *t*. Extending a result of Nenadov, Person, Sˇkori´c and Steger (2017), we prove

p

that *n−*1*/m*2 (*&l* ) is the threshold for *t*(*n, p*) *−*r*→*b *C*

p

*Æ*

when *l ≥* 5. Thus our result together with a result of

the third author which says that the threshold for *t −*r*→*b *C*4 is *n−*3*/*4 settles the problem of determining the

p

threshold for *t −*r*→*b *C*

p

*Æ*

for all values of *l*.

*Keywords:* anti-Ramsey, proper colouring, random graphs, threshold.

# Introduction

For a positive integer *r* and graphs *G* and *H*, let *G →* (*H*)*r* denote the following Ramsey property: for every edge colouring of *G* with at most *r* colours, there is a

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monochromatic copy of *H* in *G*. In random graph theory, we are interested in a function *p*ˆ: N *→* [0*,* 1] such that with *p p*ˆ the probability that *G*(*n, p*) satisfies some graph property tends to 1 as *n* tends to infinity. We then say that *G*(*n, p*) satisfies this property asymptotically almost surely (a.a.s.) and if additionally we have that for *p p*ˆ we a.a.s. do not have this property, then we call *p*ˆ the *threshold* for this property. Note that often there are more precise statements known, but for simplicity we will only discuss this kind of threshold.

A classical result in extremal combinatorics by R¨odl and Rucin´ski [[9](#_bookmark22)] determines the threshold for *G*(*n, p*) *→* (*H*)*r*, where *G*(*n, p*) is the binomial random graph. In particular, when *H* contains a cycle, the threshold is expressed in terms of the *maximum 2-density m*2(*H*)= max *{*(*e*(*J* ) *−* 1)*/*(*v*(*J* ) *−* 2) : *J ⊆ H, v*(*J* ) *≥* 3*}*.

**Theorem 1.1 ([**[**9**](#_bookmark22)**])** *Let H be a graph which contains a cycle. Then the thresh- old function pH* = *pH* (*n*) *for the property G*(*n, p*) *→* (*H*)*r is given by pH* (*n*) = *n−*1*/m*2(*H*)*.*

In this paper we investigate an *anti-Ramsey* property of sparse random graphs.

Given graphs *G* and *H*, we denote by *G −*r*→*b

p

*H* the following anti-Ramsey property:

every *proper* edge colouring of *G* contains a *rainbow* copy of *H* in *G*, i.e. a subgraph

of *G* isomorphic to *H* in which all edges have distinct colours. Since *G −*r*→*b

p

*H* is

an increasing property, there exists a threshold *p*rb = *p*rb(*n*) for any fixed graph *H*

*H H*

(see [[1](#_bookmark14)]).

Rodl and Tuza [[10](#_bookmark23)] already studied anti-Ramsey properties of random graphs

in the early 90’s, obtaining an upper bound for the threshold *p*rb for *G*(*n, p*) *−*r*→*b *C*,

*C* p

where *C* is a fixed cycle. In [[4](#_bookmark16)], an upper bound for the threshold *p*rb for any fixed

*H*

graph *H* was obtained.

**Theorem 1.2 ([**[**4**](#_bookmark16)**])** *Let H be a ﬁxed graph. Then there exists a constant C >* 0

*such that a.a.s. G*(*n, p*) *−*r*→*b *H whenever p* = *p*(*n*) *≥ Cn−*1*/m*2(*H*)*. In particu-*

p

*lar, prb ≤ n−*1*/m*2(*H*)*.*

*H*

The threshold for a graph *H* is not always given by *n−*1*/m*2(*H*), as in [[5](#_bookmark18)] it was proved that there are infinitely many graphs *H* for which the threshold *p*rb is asymptotically smaller than *n−*1*/m*2(*H*). However, Nenadov, Person, Sˇkori´c and Steger [[8](#_bookmark21)] proved that at least for sufficiently large cycles and complete graphs *H* the lower bound for *p*rb matches the upper bound *n−*1*/m*2(*H*) of Theorem [1.2](#_bookmark1).

*H*

*H*

**Theorem 1.3 ([**[**8**](#_bookmark21)**])** *If H is a cycle on at least* 7 *vertices or a complete graph on at least* 19 *vertices, then prb* = *n−*1*/m*2(*H*)*.*

*H*

In [[6](#_bookmark19)], extending Theorem [1.3](#_bookmark2), it was proved that for complete graphs *Kk*

with *k ≥* 5 the threshold *p*rb is in fact *n−*1*/m*2(*Kk* ) and that for *K*4 we have *p*rb =

*Kk*

*K*4

*n−*7*/*15 *n−*1*/m*2(*K*4). Our result determines the threshold *p*rb

*C*

*Æ*

for every cycle *Cl*

on *l ≥* 5 vertices.

**Theorem 1.4** *If H is a cycle on at least* 5 *vertices, then prb* = *n−*1*/m*2(*H*)*.*

*H*

In Section [2](#_bookmark4) we prove Theorem [1.4](#_bookmark3). Similarly to what happens for complete graphs, the situation for *C*4 is different. In a short abstract of the third author [[7](#_bookmark20)], it

is proved that *p*rb

*C*

4

= *n—*3*/*4 *n—*1*/m*2(*C*4). For completeness, we exhibit in Section [3](#_bookmark13)

the proof of threshold *p*rb obtained in [[7](#_bookmark20)]. We use standard notation and terminology

*C*

4

(see e.g. [[2](#_bookmark15)] and [[3](#_bookmark17)]). In particular, *Cl* denotes a cycle on *l* vertices, *v*(*G*) and *e*(*G*)

denote the number of vertices and edges of *G*, respectively, and we write *G − H*

for *G − V* (*H*).

# Cycles on at least five vertices

The *maximum density* of a graph *G* is denoted by *m*(*G*) =

max *{e*(*J* )*/v*(*J* ): *J ⊆ G, v*(*J* ) *≥* 1*}*. The results of Nenadov, Person,

Sˇkori´c,

and Steger [[8](#_bookmark21)] provide a general framework that allows for a transference of random Ramsey problems into questions for deterministic graphs with bounded density. The proof of Theorem [1.3](#_bookmark2) for cycles relies on the following lemma.

**Lemma 2.1 ([**[**8**](#_bookmark21)**])** *Let l ≥* 7 *be an integer and G be a graph such that m*(*G*) *<*

*m* (*C* )*. Then G —z*r*→*b *C .*

2 *l* p *l*

Note that in fact they prove a slightly stronger statement for which they need a non-strict inequality for the density [[8](#_bookmark21), Corollary 13]. The condition *l ≥* 7 above is simply a consequence of the proof of the lemma, as observed by the authors [[8](#_bookmark21)]. We extend Lemma [2.1](#_bookmark5), proving the following result, which we state in words for clarity. It is easy to see that *m*2(*Cl*)= (*l −* 1)*/*(*l −* 2).

**Lemma 2.2** *Let l ≥* 5 *be an integer and G be a graph such that m*(*G*) *<* (*l −* 1)*/*(*l −* 2)*. Then there exists a proper edge colouring of G such that every l-cycle has two non-adjacent edges with the same colour.*

Theorem [1.4](#_bookmark3) thus follows by replacing Lemma [2.1](#_bookmark5) with our Lemma [2.2](#_bookmark6) in the proof of Nenadov et al. [[8](#_bookmark21)]. We remark that the proof of Lemma [2.2](#_bookmark6) considers all the cycle lengths in the range *l ≥* 5; it is not a proof only for the cases *l* = 5 and *l* = 6. Throughout this section let *l ≥* 5 be an integer and *G* be a graph with *m*(*G*) *<*

(*l −* 1)*/*(*l −* 2). For the proof of our lemma, we shall define a *partial* proper edge colouring of *G* such that every *l*-cycle has two non-adjacent edges with the same colour. Clearly, having defined such a partial edge colouring, we can extend it to a proper edge colouring (for instance, the uncoloured edges may be assigned distinct colours).

Let *Cl*(*G*) be the set of all *l*-cycles of *G*. The *edge intersection graph* of *Cl*(*G*) is the graph whose vertex set is *Cl*(*G*) and whose edges correspond to pairs *{C, Cj}*, *C /*= *Cj*, such that *E*(*C*) *∩ E*(*Cj*) */*= *∅*. A subgraph *H ⊆ G* is a *Cl- component* of *G* if it is the union of all the *l*-cycles corresponding to the vertices of some component of the edge intersection graph of *Cl*(*G*). Clearly *H* can be con- structed from an *l*-cycle *H*1 in *G* as follows. Suppose we have defined *H*1 *⊆ · · ·⊆ Hi*, *i ≥* 1. If there is an *l*-cycle *C* in *G* such that *C /⊆ Hi* and *E*(*C*) *∩ E*(*Cj*) */*= *∅* for some *l*-cycle *Cj ⊆ Hi* then we put *Hi*+1 = *Hi ∪ C*; otherwise we terminate the

construction and set *H* = *Hi*. Note that *H* can be reconstructed by this proce- dure starting from any *l*-cycle *H*1 *⊆ H*. Also note that two *l*-cycles belonging to distinct *Cl*-components may share vertices (obviously they do not share edges).

We start the colouring procedure in some *Cl*-component *H* of *G*. Once we have the partial colouring of *H*, we proceed to assign colours different from those used in *H* to edges of a *Cl*-component of *G−E*(*H*), according to the same steps as those taken for the colouring of *H*. We continue in this manner (taking an uncoloured *Cl*- component, colouring it and removing its edges), until we have considered all the *l*-cycles of *G*. So our aim is to describe the colouring procedure of an arbitrary *Cl*- component *H* of *G*.

Let *t* be such that *H* = *Ht*, i.e. *t* is the number of steps taken for the con- struction of *H* starting from some *l*-cycle *H*1 in *G*. We may assume *t ≥* 2, since the case *t* = 1 is trivial. The following proposition describes the possibilities for a step of construction of *H*. Some configurations are impossible because they imply that *e*(*H*)*/v*(*H*) *≥* (*l −* 1)*/*(*l −* 2), which contradicts *m*(*G*) *<* (*l −* 1)*/*(*l −* 2).

**Proposition 2.3** *Let* 1 *≤ i ≤ t −* 1*. Let C be an l-cycle which is added to Hi to form Hi*+1*. There exists a labelling C* = *u*1*u*2 *··· ulu*1 *such that exactly one of the following conﬁgurations of Hi*+1 *(with respect to C) occurs, where* 2 *≤ k ≤ l and* 3 *≤ j ≤ l:*

(*Ak*) *u*1*u*2 *··· uk is a k-path in Hi and uk*+1*,..., ul ∈/ V* (*Hi*)*;*

(*Bj*) *u*1*u*2 *∈ E*(*Hi*)*, u*2*u*3 *∈/ E*(*Hi*)*, {u*3*,..., ul}\ {uj} ⊆ V* (*Hi*+1) *\ V* (*Hi*)*, uj ∈*

*V* (*Hi*)*.*

With this proposition at hand the proof of Lemma [2.2](#_bookmark6) proceeds as follows: For configurations [(*Ak*)](#_bookmark7) with 2 *≤ k ≤ l −* 2 we assign a new colour to two non-adjacent new edges. All other configurations appear at most twice and in these cases we will colour all previous configurations more carefully so that we are able to proceed.

To prove Proposition [2.3](#_bookmark7) and that bad configurations are rare, we heavily use *m*(*G*) *<* (*l −* 1)*/*(*l −* 2). For each 1 *≤ j ≤ i*, let *rj* be the number of edges in *E*(*Hj*+1) *\ E*(*Hj*), *sj* be the number of vertices in *V* (*Hj*+1) *\ V* (*Hj*), and *cj* be the number of components of *Hj*+1 *− Hj*. We have that *rj ≥ sj* + 1. In fact, if *sj* =0 then *rj ≥* 1, and if *sj ≥* 1 then the components of *Hj*+1 *− Hj* are paths and the number of edges of *Hj*+1 with at least one end in *Hj*+1 *−Hj* is *rj* = *sj* + *cj ≥ sj* + 1. We have that *sj ≤ l −* 2, because an *l*-cycle is added to *Hj* with at least one edge in *E*(*Hj*). By an adequate number of applications of the inequality *a/b >*

(*a* + 1)*/*(*b* + 1) (which is equivalent to *a > b*), we obtain

*l −* 1

*e*(*H* )

*e*(*H* )+ *r*

*l* + *ri* + Σ*i—*1 (*sj* + 1)

*l* + *r*

+ (*i −* 1)(*l −* 1)

*>*  *i*+1 = *i i ≥*  *j*=1 *≥*  *i ,*

*j*=1

*l −* 2

*v*(*Hi*+1)

*v*(*Hi*)+ *si*

*l* + *si* + Σ*i—*1 *sj*

*l* + *si* + (*i −* 1)(*l −* 2)

(1)

which is equivalent to *ri <* ((*l −* 1)*si* + *l*)*/*(*l −* 2).

**Proof of Proposition** [**2.3**](#_bookmark7)Using this last inequality, one can easily check that *si* =

*l −* 3 implies *ri ≤ l −* 1. We get that for *si* = *l −* 3 exactly one of the following three alternatives holds: *Hi*+1 *− Hi* has one component and no edge with both ends in *V* (*Hi*) is added to *Hi*, or *Hi*+1 *− Hi* has one component and one edge with both ends in *V* (*Hi*) is added to *Hi*, or *Hi*+1 *− Hi* has two components and no edge with both ends in *V* (*Hi*) is added to *Hi*. These alternatives correspond to Configurations [(*A*3)](#_bookmark7) and [(*Bj*)](#_bookmark7) with 3 *≤ j ≤ l*, respectively. The inequality also gives us that 0 *≤ si ≤ l −* 4 implies *ri ≤ si* + 1; we have thus Configurations [(*Ak*)](#_bookmark7) with 4 *≤ k ≤ l*. Configuration [(*A*2)](#_bookmark7) is clearly the only possibility for *si* = *l −* 2. *2*

For arguments involving similar calculations to ([1](#_bookmark8)) we will refer to this as the *density argument*. For example, when *Hi*+1 has Configuration [(*Al*)](#_bookmark7), then *si* = 0 and *ri* = 1, which with ([1](#_bookmark8)) immediately implies that there cannot be another occurrence of [(*Al*)](#_bookmark7). Similarly, we can show that Configuration [(*Al—*1)](#_bookmark7), where *si* = 1 and *ri* = 2, appears at most twice and any [(*Bj*)](#_bookmark7), where *si* = *l −* 3 and *ri* = *l −* 1, at most once. Furthermore, when one of the configurations appears, the occurrence of the other [(*Ak*)](#_bookmark7) with 3 *≤ k ≤ l −* 2 is restricted, while only [(*A*2)](#_bookmark7) can appear arbitrarily often.

**Proof of Lemma** [**2.2**](#_bookmark6)Choose an arbitrary *l*-cycle *H*1 and assign a colour *c*1 to some pair of non-adjacent edges of *H*1. Let *H* = *Ht*, *t ≥* 2, be the *Cl*-component of *G* constructed from *H*1.

Next we consider the cases according to which configurations given by Proposi- tion [2.3](#_bookmark7) occur during the construction of *H*. For each 1 *≤ i ≤ t−* 1 and each *l*-cycle which is added to *Hi* to get *Hi*+1, we shall choose two non-adjacent edges of the cycle and a colour *c*, and assign *c* to the chosen edges.

The *l*-cycles added to *Hi* are the ones of *Hi*+1 which pass through edges of components of *Hi*+1 *−Hi* (recall that these components are paths) or through edges in *E*(*Hi*+1) *\ E*(*Hi*) with both ends in *V* (*Hi*). Thus, if all the edges in *E*(*Hi*+1) *\ E*(*Hi*) belong to components of *Hi*+1 *−Hi* and each of these components has at least two vertices, then the task is easy: for each component we choose two non-adjacent edges of it and assign a new colour to them, and this guarantees that *Hi*+1 contains no rainbow copy of *Cl*. If nothing else is explicitly stated we will always do this when *Hi*+1 has Configuration [(*Ak*)](#_bookmark7) with 2 *≤ k ≤ l−* 2. Hence the effort in the proof consists in dealing with the other configurations. These will receive colours first. By the density argument Configuration [(*Al*)](#_bookmark7) appears at most once, [(*Al—*1)](#_bookmark7) at most twice, and any [(*Bj*)](#_bookmark7) at most once.

**Case 1** *For all* 1 *≤ i ≤ t −* 1*, Hi*+1 *has Conﬁguration* [(*Ak*)](#_bookmark7) *with* 2 *≤ k ≤ l −* 2*.*

For 1 *≤ i ≤ t−*1, assign a new colour *ci*+1 to two non-adjacent edges in *E*(*Hi*+1)*\ E*(*Hi*).

**Case 2** *There is* 1 *≤ i*1 *≤ t−* 1 *such that Hi*1+1 *has Conﬁguration* [(*Al*)](#_bookmark7) *with respect to some C.*

In this case, for all *i /*= *i*1, *Hi*+1 has Configuration [(*Ak*)](#_bookmark7) with 2 *≤ k ≤ l −* 2, by the density argument. Moreover, for at most one 1 *≤ i*2 *≤ t −* 1, *Hi*2+1 has Configuration [(*A*3)](#_bookmark7) with respect to some *Cj*.

Let *C* = *u*1*u*2 *··· ulu*1, where *P* = *u*1*u*2 *··· ul* is an *l*-path in *Hi*1 and *ulu*1 *∈/ E*(*Hi*1 ). The number of *l*-cycles in *Hi*1+1 which are not in *Hi*1 equals that of *l*- paths in *Hi*1 with ends *u*1 and *ul*. First, we consider the case in which the number of such paths is greater than one, and, for this case, we define the colouring of *H* in such a way that each of these paths contains two non-adjacent edges with the same colour.

Suppose that *Hi*1

contains some *Pj* = *u*1*x*2 *··· xl—*1*ul*, *Pj /*= *P* . We have that *P ∪*

*Pj* contains an even cycle with length at most 2*l−* 2, and in *Hi*

1

the only such cycles

are *l*-cycles (when *l* is even), or (2*l −* 4)-cycles, or (2*l −* 2)-cycles. A (2*l −* 2)-cycle appears during the construction only if, for some *i*, *Hi* has two *l*-cycles *C* and *Cj* such that *C ∩ Cj* is a 2-path. Similarly a (2*l −* 4)-cycle appears during the construction only if, for some *i*, *Hi* has two *l*-cycles *C* and *Cj* such that *C ∩ Cj* is a 3-path.

First consider that *P ∪ Pj* is a (2*l −* 2)-cycle in *Hi* (*P j* and *P* are internally disjoint). So we may assume without loss of generality that *i*1 = 2, *H*2 has Configu- ration [(*A*2)](#_bookmark7) and *P ∪ Pj ⊆ H*2. Assign a colour *c*1 to the edges of *P ∪ Pj* alternately. Note that, if *l* is even then there may be a third *l*-path in *H*2 between *u*1 and *ul*, and one can easily check that such a path will have two edges with the same colour. Consider that *P ∪ Pj* contains a (2*l −* 4)-cycle. We may assume that there is

1

no *l*-path *P jj* in *Hi* between *u*1 and *ul* such that *P jj ∪P* or *P jj ∪Pj* is a (2*l−*2)-cycle.

1

Without loss of generality *x*2 = *u*2, *H*2 has Configuration [(*A*3)](#_bookmark7) and (*P ∪ Pj*) *− u*1 *⊆ H*2. Alternately colour the edges of (*P ∪ Pj*) *− u*1 with two colours *c*1 and *c*2. If *l* is even then there may be a third (*l −* 1)-path in *H*2 between *u*2 and *u*5, and one can easily check that such a path will have two edges with the same colour.

Now consider that *P ∪Pj* contains an *l*-cycle, *l* even. We may assume that there is no *l*-path *P jj* in *Hi*1 between *u*1 and *ul* such that *P jj ∪ P* or *P jj ∪ Pj* contains a cycle with length at least 2*l −* 4. Without loss of generality *H*1 is an *l*-cycle contained in *P ∪ Pj*. and we alternately colour the edges of *H*1 with two colours *c*1 and *c*2.

Finally, assume that *Hi*1 contains no *l*-path linking *u*1 and *ul* other than *P* . Suppose that *P* has two consecutive edges in some *l*-cycle *Cj*. Obviously, *Cj* can- not contain both *u*1*u*2 and *ul—*1*ul*. Assign a colour *c*1 to two non-adjacent edges in *E*(*Cj*) in such a way that, if *{u*1*u*2*, u*2*u*3*} ⊆ E*(*Cj*), then *c*1 is not assigned to *u*2*u*3, and, if *{ul—*2*ul—*1*, ul—*1*ul} ⊆ E*(*Cj*), then *c*1 is not assigned to *ul—*2*ul—*1. Without loss of generality *H*1 = *Cj*. For 1 *≤ i ≤ t −* 1, *i /*= *i*1, assign a new colour *ci*+1 to two non-adjacent edges in *E*(*Hi*+1) *\ E*(*Hi*). We have that some edge in *E*(*P* ) *\ {u*2*u*3*, ul—*2*ul—*1*}* is uncoloured. Assign a new colour *ci*1+1 to *ulu*1 and to an uncoloured edge in *E*(*P* ) *\ {u*2*u*3*, ul—*2*ul—*1*}*.

Suppose now that no two consecutive edges of *P* lie in the same *l*-cycle in *Hi*1 . We have that *P* cannot have two non-consecutive edges in same *l*-cycle, otherwise *P* would have length greater than *l −* 1. So any *l*-cycle in *Hi*1 has at most one edge of *P* . For 1 *≤ i ≤ t−* 1, *i /*= *i*1*, i*2, assign a new colour *ci*+1 to two non-adjacent edges in *E*(*Hi*+1) *\* (*E*(*Hi*) *∪ E*(*P* )). Colour two non-adjacent edges in *E*(*Hi*2+1) *\ E*(*Hi*2 ) (when *l* = 5 there are exactly two such edges) with a new colour *ci*2+1. Again some edge in *E*(*P* ) *\ {u*2*u*3*, ul—*2*ul—*1*}* is uncoloured. Assign a new colour *ci*1+1 to *ulu*1

and to an uncoloured edge in *E*(*P* ) *\ {u*2*u*3*, ul—*2*ul—*1*}*.

**Case 3** *There are* 1 *≤ i*1 *< i*2 *≤ t —* 1 *such that Hi*1+1 *and Hi*2+1 *have Conﬁgura- tion* [(*Al—*1)](#_bookmark7) *with respect to some C and to some Cj, respectively.*

By the density argument, this case occurs only if *l* = 5 and we have that *Hi*+1

has Configuration [(*A*2)](#_bookmark7) for all *i /*= *i*1*, i*2. Let *C* = *u*1*u*2*u*3*u*4*u*5*u*1, and let *Cj* =

*v*1*v*2*v*3*v*4*v*5*v*1, where *P* = *u*1*u*2*u*3*u*4 and *Pj* = *v*1*v*2*v*3*v*4 are 4-paths in *Hi* , *u*5 *∈/*

1

*V* (*Hi*1 ) and *v*5 *∈/ V* (*Hi*2 ). We see that *P* is the only 4-path between *u*1 and *u*4

in *Hi*1

and thus *C* is the only 5-cycle added to *Hi*1

to form *Hi* +1. As for *Pj*, there

may be a 4-path *P jj* in *Hi* between *v*1 and *v*4 other than *Pj*. If such is the case

1

2

then we have that *Pj ∪ P jj* contains a 4-cycle or a 6-cycle.

One of the following three alternatives holds for *P* in *Hi*1 : the three edges of *P* lie in the same 5-cycle, or two consecutive edges of *P* lie in the same 5-cycle but the third one does not, or any 5-cycle in *Hi*1 contains at most one edge of *P* .

Consider that the first alternative holds for *P* . Without loss of generality all the edges of *P* lie in *H*1 and *i*1 = 1. Hence *H*1 is of the form *H*1 = *u*1*u*2*u*3*u*4*x*5*u*1 for some *x*5. Note that *Cjj* = *u*1*x*5*u*4*u*5*u*1 is a 4-cycle in *H*2. Suppose that all the edges of *Pj* lie in *H*2. Therefore, without loss of generality *i*2 = 2. If the ends of *Pj* are two adjacent vertices in *V* (*Cjj*) then we colour *u*1*u*2 and *u*3*u*4 with *c*1 and we colour two non-adjacent edges of *Cj* with a new colour *c*2. If the ends of *Pj* are *u*1 and *u*4 then we colour *u*1*u*2 and *u*3*u*4 with *c*1. If the ends of *Pj* are *x*5 and a vertex in *{u*2*, u*3*}* then we assign *c*1 to *u*1*u*2 and *u*3*u*4 and we assign a new colour *c*2 to *v*5*x*5 and *u*2*u*3. The case in which the ends of *Pj* are *u*5 and a vertex in *{u*2*, u*3*}* is symmetric. Suppose that the ends of *Pj* are *u*1 and *u*3. Note that there are two 4- paths between *u*1 and *u*3 (and thus two possibilities for *Pj*): *u*1*u*5*u*4*u*3 or *u*1*x*5*u*4*u*3. We assign *c*1 to *u*4*x*5, *u*1*u*2 and *u*3*v*5, and assign *c*2 to *u*2*u*3, *u*4*u*5 and *v*5*u*1. The case in which the ends of *Pj* are *u*4 and *u*2 is symmetric.

Now suppose that *Pj* has at most two edges in *H*2. If *Pj* has two edges in *H*2 then these must be consecutive. Hence we may assume without loss of generality that *v*3*v*4 *∈/ E*(*H*2). Since there is no 6-cycle in *Hi*2 and the unique 4-cycle in *Hi*2 has its edges in *H*2, a 4-path *P jj /*= *Pj* between *v*1 and *v*4 must pass through *v*3*v*4. Colour *u*1*u*2 and *u*3*u*4 with *c*1, and *v*3*v*4 and *v*5*v*1 with a new colour *ci*2 .

Let us consider the second alternative for *P* . Without loss of generality *H*1 contains the edges *u*1*u*2 and *u*2*u*3 but does not contain *u*3*u*4. Thus *H*1 is of the form *H*1 = *u*1*u*2*u*3*x*4*x*5*u*1 for some *x*4 and *x*5. Note that *Cjj* = *u*1*x*5*x*4*u*3*u*4*u*5*u*1 is a 6-cycle in *Hi*1+1. We see that *Cjj* is the unique 6-cycle in *Hi*2 , and that *Hi*2 contains no 4-cycle. Hence, the number of 4-paths linking *v*1 and *v*4 is at most two. If there are two such paths, these correspond to two internally disjoint paths along *Cjj*. Suppose that *E*(*Pj*) *⊆ E*(*Cjj*). Alternately colour the edges of *Cjj* with two colours *c*1 and *c*2 and, for 1 *≤ i ≤ t—* 1, *i /*= *i*1*, i*2, assign a new colour *ci*+2 to two non-adjacent edges in *E*(*Hi*+1) *\* (*E*(*Hi*) *∪ {u*3*u*4*}*). Assume that *E*(*Pj*) */⊆ E*(*Cjj*). Thus *Pj* is the unique 4-path between *v*1 and *v*4. If *E*(*Pj*) *⊆ E*(*H*1) then *E*(*Pj*) *∩ {u*1*u*2*, u*2*u*3*} /*= *∅* (by assumption *Pj* cannot be *u*1*x*5*x*4*u*3), and we colour *u*4*u*5 and the two non- adjacent edges in *E*(*Pj*) with *c*1. Assign a new colour *ci*+1 to two non-adjacent edges

in *E*(*Hi*+1) *\ E*(*Hi*), for 1 *≤ i ≤ t —* 1, *i /*= *i*1*, i*2. Thus assume that *E*(*Pj*) */⊆ E*(*H*1) (possibly *Pj* = *P* ). Therefore *Pj* has an edge *vjvj*+1 which does not belong to *E*(*H*1). Colour *u*2*u*3, *x*4*x*5 and an edge in *{u*5*u*1*, u*5*u*4*}\ {vjvj*+1*}* with *c*1, and give a new colour *ci*2+1 to *vjvj*+1 and to some edge in *{v*5*v*1*, v*5*v*4*}* not incident with *vj* nor with *vj*+1.

Finally, consider the third alternative for *P* . In *Hi*2 there are neither 4-cycles nor 6-cycles, and therefore *Pj* is the unique 4-path between *v*1 and *v*4. We may assume without loss of generality that *H*1 contains *u*2*u*3 and that *u*2*u*3 is uncoloured. Suppose that *Pj* = *P* . Colour *u*2*u*3, *u*5*u*1 and *v*5*u*4 with a new colour *ci* +1, and assign a new colour *ci*+1 to two non-adjacent edges in *E*(*Hi*+1) *\ E*(*Hi*), for 1 *≤ i ≤ t —* 1, *i /*= *i*1*, i*2. Now suppose that *Pj /*= *P* . Since *Pj* is the unique 4-path in *Hi*

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linking *v*1 and *v*4, *Pj* and *P* cannot have both ends in common. Without loss of generality *v*1 */*= *u*1. Colour *u*2*u*3 and *u*5*u*1 with a new colour *ci*1+1. If *v*2*v*3 = *u*2*u*3 then colour *v*5*v*1 with *ci*1+1; otherwise colour *v*2*v*3 and *v*5*v*1 with a new colour *ci*2+1.

**Case 4** *There is exactly one* 1 *≤ i*1 *≤ t—*1 *such that Hi*1+1 *has Conﬁguration* [(*Al—*1)](#_bookmark7)

*with respect to some C.*

By the density argument, *Hi*+1 has Configuration [(*Ak*)](#_bookmark7) with 2 *≤ k ≤* 4 for all *i /*= *i*1. Let *C* = *u*1*u*2 *··· ulu*1, where *P* = *u*1 *··· ul—*1 is an (*l —* 1)-path in *Hi*1 and *ul ∈/ V* (*Hi*1 ). The number of *l*-cycles in *Hi*1+1 which are not in *Hi*1 equals that of (*l —* 1)-paths with ends *u*1 and *ul—*1. Next we consider the case in which the number of such paths is greater than one, and, for this case, we define the colouring of *H* in such a way that each of these paths contains two non-adjacent edges with the same colour.

Suppose that *Hi*1

contains some *Pj* = *u*1*x*2 *··· xl—*2*ul—*1, *Pj /*= *P* . We see that *P ∪*

*Pj* contains an even cycle with length at most 2*l—* 4, and in *Hi*

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the only such cycles

are *l*-cycles (when *l* is even), or (2*l —* 6)-cycles, or (2*l —* 4)-cycles. A (2*l —* 4)-cycle appears during the construction only if, for some *i*, *Hi* has two *l*-cycles *C* and *Cj* such that *C ∩ Cj* is a 3-path. Similarly a (2*l —* 6)-cycle appears during the construction only if, for some *i*, *Hi* has two *l*-cycles *C* and *Cj* such that *C ∩ Cj* is a 4-path.

First, consider that *P ∪ Pj* is a (2*l —* 4)-cycle in *Hi*1 (*P j* and *P* are internally disjoint). So we may assume without loss of generality that *i*1 = 2, *H*2 has Con- figuration [(*A*3)](#_bookmark7) and *P ∪ Pj ⊆ H*2. Alternately colour the edges of *P ∪ Pj* with two colours *c*1 and *c*2. Note that if *l* is even then there may be a third (*l —* 1)-path in *H*2 between *u*1 and *ul—*1, and one can easily check that such a path will have two edges with the same colour.

Consider that *P ∪ Pj* contains a (2*l —* 6)-cycle. We may assume that there is no (*l —* 1)-path *P jj* in *Hi*1 between *u*1 and *ul—*1 such that *P jj ∪ P* or *P jj ∪ Pj* is a (2*l —* 4)-cycle. Without loss of generality *x*2 = *u*2, *H*2 has Configuration [(*A*4)](#_bookmark7) and (*P ∪ Pj*) *— u*1 *⊆ H*2. Alternately colour the edges of (*P ∪ Pj*) *— u*1 with two colours *c*1 and *c*2, and colour the two non-adjacent edges of *Cj ∩ H*1 with a new colour *c*3. If *l* is even then there may be a third (*l —* 2)-path in *H*2 between *u*2 and *ul—*1. Such a path passes through the edges of *Cj ∩ H*1, and therefore will have two edges with the same colour.

Now consider that *P ∪ Pj* contains an *l*-cycle, *l* even. We may assume that there is no (*l —* 1)-path *P jj* in *Hi* between *u*1 and *ul* such that *P jj ∪ P* or *P jj ∪ Pj* contains a cycle with length at least 2*l —* 6. Without loss of generality *H*1 is an *l*- cycle contained in *P ∪ Pj*. Alternately colour the edges of *H*1 with two colours *c*1 and *c*2, and assign a new colour *ci*+2 to two non-adjacent edges in *E*(*Hi*+1) *\ E*(*Hi*) for 1 *≤ i ≤ t —* 1, *i /*= *i*1.

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Finally, assume that *P* is the only (*l—*1)-path in *Hi* between *u*1 and *ul—*1. Let *Cj* be an *l*-cycle containing the edge *u*2*u*3. Assign a colour *c*1 to two non-adjacent edges in *E*(*Cj*) *\ {u*2*u*3*}*. Without loss of generality *H*1 = *Cj*. For 1 *≤ i ≤ i*1 *—* 1, assign a new colour *ci*+1 to two non-adjacent edges in *E*(*Hi*+1) *\ E*(*Hi*). We have that *u*2*u*3 is uncoloured. Assign a new colour *ci*1+1 to *ulu*1 and *u*2*u*3.

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**Case 5** *There is* 1 *≤ i*1 *≤ t —* 1 *such that Hi*1+1 *has Conﬁguration* [(*Bj*)](#_bookmark7) *with*

1. *≤ j ≤ l with respect to some C.*

By the density argument, *Hi*+1 has Configuration [(*A*2)](#_bookmark7) for all *i /*= *i*1. Let *C* = *u*1*u*2 *··· ulu*1, where *uj ∈ V* (*Hi*) for some 3 *≤ j ≤ l*, *{u*3*,..., ul}\ {uj}⊆ V* (*Hi*+1) *\ V* (*Hi*). If there is a path *P* in *Hi*1 between *u*1 and *uj* such that *V* (*P* )*∪{uj*+1*,..., ul}*

induces an *l*-cycle in *Hi* +1 or there is a path *Pj* in *Hi*

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between *u*2 and *uj* such

that *V* (*P j*) *∪ {u*3*,..., uj—*1*}* induces an *l*-cycle in *Hi* +1, then *Hi* +1 can be con-

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structed in such a way that each of the last two steps has Configuration [(*Al—j*+3)](#_bookmark7)

and [(*Aj*)](#_bookmark7), respectively, and therefore Case [1,](#_bookmark9) [2,](#_bookmark10) [3](#_bookmark11), or [4](#_bookmark12) happens. So we may suppose that *Hi*1 contains none of these paths and colour *u*2*u*3 and *ulu*1 with colour *ci*1+1.

*2*

# 3 Cycle on four vertices

In this section we give a sketch of the proof that *p*rb

*C*4

= *n—*3*/*4. By a classical result

of Bollob´as (see [[3](#_bookmark17)]), we know that if *p n—*3*/*4, then a.a.s. *G*(*n, p*) containsa copy of *K*2*,*4. It is not hard to see that in any proper colouring of the edges of *K*2*,*4 there

is a rainbow copy of *C*4, which implies that *p*rb

*C*4

*≤ n—*3*/*4.

For the lower bound, define a sequence *F* = *C*1*,..., Cl* of copies of *C*4 in *G* as

*i*  4 *i—*1 4 *i*

*—*3*/*4

a *C*4*-chain* if for any 2 *≤ i ≤ l* we have *E*(*C*4) *∩ j*=1 *E*(*C*4)

*/*= *∅*. Let *p n*

and *G* = *G*(*n, p*). We want to show that a.a.s. there exists a proper colouring of *G*

that contains no rainbow copy of *C*4. It is enough to consider *C*4-chains that are maximal with respect to the number of *C*4’s. The first step is to properly colour some edges in all maximal *C*4-chains so that all *C*4’s in *G* will be non-rainbow. Then, since all *C*4’s are coloured we can just give a new colour for each one of the remaining uncoloured edges. For the first step, we use Markov’s inequality and a union bound to obtain that a.a.s. *G* does not contain any graph *H* with *m*(*H*) *≥* 4*/*3 and *|V* (*H*)*|≤* 12.

Let *F* = *C*1*,..., Cl* be an arbitrary *C*4-chain in *G* with *m*(*F* ) *≥* 4*/*3. Let 2 *≤ i ≤*

4 4

*l* be the smallest index such that *Fj* = *C*1*,..., Ci* has density *m*(*Fj*) *≥* 4*/*3. Then,

4 4

since *F jj* = *C*1*,..., Ci—*1 has density *m*(*Fjj*) *<* 4*/*3, we can explore the structure

4 4

of *G* to conclude that *|V* (*F jj*)*| ≤* 10, which implies *|V* (*Fj*)*| ≤* 12, a contradiction.

Therefore, a.a.s. *G* contains no copies of *C*4-chains *F* with *m*(*F* ) *≥* 4*/*3. Since *F* is not too dense, a careful analysis of such chains shows that it is possible to obtain

the desired colouring, which proves that *p*rb

*C*4

*≥ n—*3*/*4.

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