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Backbone Coloring of Graphs with Galaxy Backbones [1](#_bookmark0)

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**Abstract**

A (proper) *k*-coloring of a graph *G* = (*V, E*) is a function *c* : *V* (*G*) *→ {*1*,..., k}* such that *c*(*u*) */*= *c*(*v*) for every *uv ∈ E*(*G*). Givena graph *G* and a subgraph *H* of *G*,a *q*-backbone *k*-coloring of (*G, H*) isa *k*-coloring *c* of *G* such that *q ≤ |c*(*u*) *− c*(*v*)*|* for every edge *uv ∈ E*(*H*). The *q*-backbone chromatic number of (*G, H*), denoted by BBC*q* (*G, H*), is the minimum integer *k* for which there exists a *q*-backbone *k*-coloring of (*G, H*). Similarly, a circular *q*-backbone *k*-coloring of (*G, H*) is a function *c* : *V* (*G*) *→ {*1*,..., k}* such that, for every edge *uv ∈ E*(*G*), we have *|c*(*u*) *− c*(*v*)*| ≥* 1 and, for every edge *uv ∈ E*(*H*), we have *k − q ≥ |c*(*u*) *− c*(*v*)*| ≥ q*. The circular *q*-backbone chromatic number of (*G, H*), denoted by CBC*q* (*G, H*), is the smallest integer *k* such that there exists such coloring *c*.

In this work, we first prove that if *G* is a 3-chromatic graph and *F* is a galaxy, then CBC*q* (*G, F* ) *≤* 2*q* + 2.

Then, we prove that CBC3(*G, M* ) *≤* 7 and CBC*q*(*G, M* ) *≤* 2*q*, for every *q ≥* 4, whenever *M* is a matching of a planar graph *G*. Moreover, we argue that both bounds are tight. Such bounds partially answer open questions in the literature. We also prove that one can compute BBC2(*G, M* ) in polynomial time, whenever *G* is an outerplanar graph with a matching backbone *M* . Finally, we show a mistake in a proof that BBC2(*G, M* ) *≤* Δ(*G*) + 1, for any matching *M* of an arbitrary graph *G* [Miˇskuf *et al.*, 2010] and we present how to fix it.

*Keywords:* Graph Coloring; Circular Backbone Coloring; Planar Graphs; Brooks’ Type Theorem.

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# Introduction

For basic notions and undefined terminology we refer to [[3](#_bookmark13)]. Let *G* = (*V, E*) be a simple graph. Given a positive integer *k*, we denote the set *{*1*, ··· , k}* by [*k*]. A *(proper) k-coloring of G* is a function *c* : *V* (*G*) *→* [*k*] such that *c*(*u*) */*= *c*(*v*) for every edge *uv ∈ E*(*G*). We say *G* is *k-colorable* if there exists a *k*-coloring of *G*. The *chromatic number* of *G*, denoted by *χ*(*G*), is the smallest *k* for which *G* is *k*-colorable. We say *G* is *k-chromatic* if *χ*(*G*) = *k* and that it is *k*-colorable if it admits a *k*-coloring.

Let *G* be a graph and *H* a subgraph of *G*. We say that (*G, H*) isa *pair*, where *H* is called *backbone* of *G*. Given two positive integers *q* and *k*,a *q-backbone k-coloring of* (*G, H*) isa *k*-coloring *c* of *G* for which *|c*(*u*) *−c*(*v*)*|≥ q* for every *uv ∈ E*(*H*). The *q-backbone chromatic number of* (*G, H*), denoted by BBC*q*(*G, H*), is the minimum *k* for which there exists a *q*-backbone *k*-coloring of (*G, H*). Problems regarding backbone colorings were first introduced by Broersma *et al.* [[5](#_bookmark15)], based on coloring problems related to frequency assignment.

Observe that if *c* is a proper *k*-coloring of *G*, then *g* defined by *g*(*v*)= *q · c*(*v*) *−* (*q −* 1) is a *q*-backbone (*q · k − q* + 1)-coloring of (*G, H*) for any spanning subgraph *H* of *G*. Hence

BBC*q*(*G, H*) *≤ q · χ*(*G*) *− q* + 1*.*

We now consider special backbone *k*-colorings where the color space is circular,

i.e. it behaves as Z*/k*. To be precise, given a graph *G*, a subgraph *H* of *G*, and a positive integer *q*, a *circular q-backbone k-coloring of* (*G, H*) is a *k*-coloring *c* : *V* (*G*) *→ {*1*,..., k}* such that *q ≤ |c*(*u*) *− c*(*v*)*| ≤ k − q*, for every *uv ∈ E*(*H*). The *circular q-backbone chromatic number* of (*G, H*), denoted by CBC*q*(*G, H*), is the smallest *k* for which there exists a circular *q*-backbone *k*-coloring of (*G, H*).

Note that any circular *q*-backbone *k*-coloring is a *q*-backbone *k*-coloring. Con- versely, a *q*-backbone *k*-coloring yields a circular *q*-backbone (*k* + *q −* 1)-coloring. Therefore we get

BBC*q*(*G, H*) *≤* CBC*q*(*G, H*) *≤* BBC*q*(*G, H*)+ *q −* 1*,* and also

*q · χ*(*H*) *≤* CBC*q*(*G, H*) *≤ q · χ*(*G*)*.*

Havet *et al.* [[8](#_bookmark18)] conjectured the following problem:

**Conjecture 1.1** *If G is a planar graph and F is a galaxy in G, then* CBC*q*(*G, F* ) *≤*

2*q* + 2*.*

Recall that a *star* on *n* vertices is a tree with *n −* 1 leaves and the remaining vertex is called the *central vertex* of the star (in the case which a star is only an edge, we choose one of its ends as central vertex.). A *galaxy* is a graph whose connected components are stars.

We prove that Conjecture [1.1](#_bookmark1) holds for 3-chromatic graphs, even for those which are not planar. Recall that for planar graphs, by Gr¨otzsch’s theorem, this includes triangle-free planar graphs. It is also known that there are linear-time algorithms to find 3-colorings of triangle-free planar graphs [[9](#_bookmark19)]. By combining such algorithms

with our result, we deduce that a *q*-backbone coloring with 2*q* + 2 colors can be obtained in linear time. We do not know whether our bound is tight when *G* is triangle-free. It is trivially tight when *G* is not triangle-free as it suffices to take a *K*4 with *K*1*,*3 as backbone. Therefore, we pose the question:

**Problem 1.2** *If G is a triangle-free 3-chromatic graph and F a galaxy in G, then*

CBC*q*(*G, F* ) *≤* 2*q* + 1*?*

With respect to matching backbones, Havet *et al.* [[8](#_bookmark18)] asked the following ques- tion:

**Problem 1.3** *If G is a planar graph, M is a matching in G, and q is a positive* *integer, q ≥* 3*, is it true that* CBC*q*(*G, M* ) *≤* 2*q* + 1*?*

In the same article, they prove that it is *NP* -complete to decide whether CBC2(*G, M* ) *≤ k* for *k ∈ {*4*,* 5*}* when *M* is a matching in a planar graph *G*. This is why the problem above does not consider *q* = 2. In fact, they show that if *G* is a pla- nar graph with girth at least 5 and *M* is a matching in *G*, then CBC*q*(*G, M* ) *≤* 2*q*+1. This is why they propose to investigate the following relaxation of Problem [1.3](#_bookmark2).

**Problem 1.4** *If G is a planar graph with girth at least 4 and M is a matching in*

*G, is it true that* CBC*q*(*G, M* ) *≤* 2*q* + 1*?*

We prove a stronger version of Problem [1.3](#_bookmark2), namely we prove that the bound 2*q* +1 holds for *q* = 3, and that for *q ≥* 4, the bound can be improved to 2*q*. Observe that if *M* is non-empty, then 2*q* is the best possible, because, for every planar graph *G*, non-empty matching *M* and positive integer *q*, we have CBC*q*(*G, M* ) *≥* 2*q.*

Therefore, by our result there is always equality if *q ≥* 4, and if *q* = 3 then CBC*q*(*G, M* ) equals either 6 or 7. We also give an example where the upper bound 7 can be attained. So, we pose the following question:

**Problem 1.5** *Given a planar graph G and a non-empty matching M, can one decide in polynomial time whether* CBC3(*G, M* )= 6*?*

In [[4](#_bookmark14)], Broersma *et al.* proved that for *q*+1 *≤ χ*(*G*) *≤* 2*q*, we have BBC*q*(*G, M* ) *≤* 2*χ*(*G*) *−* 2, where *M* is a matching of *G*. This and the fact that CBC*q*(*G, H*) *≤* BBC*q*(*G, H*)+ *q −* 1, gives us that CBC2(*G, M* ) *≤* 5, whenever *G* is a 3-chromatic graph and *M* is a matching of *G*. But since triangle-free planar graphs are 3- colorable by Gr¨otzsch’s Theorem, the case *q* = 2 of Problem [1.4](#_bookmark3) is known. The other cases follow from our result.

Using the same result and the fact that an outerplanar graph is 3-chromatic, we have that BBC2(*G, M* ) *≤* 4, whenever *G* is an outerplanar graph and *M* is a matching of *G*. In this work, we prove that one can compute BBC2(*G, M* ) in polynomial time, whenever *G* is an outerplanar graph with a matching backbone *M* .

Apart from Problem [1.5](#_bookmark4), the only remaining questions concerning *q*-backbone chromatic number of (*G, M* ) are for *q* = 2. Broersma *et al.* [[5](#_bookmark15)] proved that BBC2(*G, M* ) *≤* 6 and ask: 1) can this result be proved without using the Four

Color Theorem? and 2) Can this be improved to 5? Both questions are still open, although in [[1](#_bookmark11)] some partial answers are given. They hint for positive answers by proving that if *G* has no induced cycles of length 4 or 5, then CBC2(*G, M* ) *≤* 5, and that if *G* is diamond-free, then CBC2(*G, M* ) *≤* 6 (none of their proofs use the Four Color Theorem). Given that the original questions posed by Broersma *et al.*

[[5](#_bookmark15)] seem to be very hard, we ask the following simpler question that could be a good intermediate step for a definite answer for their questions.

**Problem 1.6** *Let G be a planar C*4*-free graph, and M be a matching in G. Is it true that* CBC2(*G, M* ) *≤* 5*?*

Finally, with respect to general graphs, Miˇskuf *et al.* [[10](#_bookmark20)] presented a proof for a Brooks’ type theorem for BBC, i.e. for any graph *G* and any matching *M* in *G*, we have BBC2(*G, M* ) *≤* Δ(*G*) + 1. We found a mistake in their proof and we present here how to fix it.

# Galaxy backbones

In this section, we want to prove that CBC*q*(*G, F* ) *≤* 2*q* +2 when *F* is a galaxy of a 3-chromatic graph.

**Theorem 2.1** *If G is a 3-chromatic graph and F is a galaxy, then*

CBC*q*(*G, F* ) *≤* 2*q* + 2*.*

**Proof.** Let *c* : *V* (*G*) *→* [3] be a 3-coloring of *G*. Define *Li* = *{v ∈ V* (*G*) *| c*(*v*) = *i* and *dF* (*v*)= 1*}* and for each *v ∈ Li*, consider *v* the vertex such that *vv ∈ E*(*F* ). We now define a circular *q*-backbone coloring *cj* : *V* (*G*) *→* [2*q* + 2] as follows:

* 1. If *v ∈ c—*1(1), then *cj*(*v*)= 1.
  2. If *v ∈ c—*1(2), then
  3. If *v ∈ c—*1(3), then

⎧ *q* + 1, if *v ∈ L*2 and *c*(*v*)= 1;

*cj*(*v*)= 2*q* + 2, if *v ∈ L*2 and *c*(*v*)= 3;

⎪⎪⎨

⎪⎪⎩ *q* + 3, otherwise.

*cj*(*v*)= ⎧⎨ 2, if *v ∈ L*3 and *c*(*v*)= 2;

⎩ *q* + 2, otherwise.

First, we prove that *cj* is a proper coloring. In fact, *C*1*, C*2*, C*3 *⊂ V* (*G*) be partitions of the 3-partition induced by *c* and consider *Cj* = *{v ∈ V* (*G*): *cj*(*v*)=

*i*

*, C*

*j*

*i}*, for each *i ∈ {*1*,* 2*,q* +1*,q* +2*,q* +3*,* 2*q* +2*}*. Observe that *Cj*

*q*+1

*j*

*q*+3

*, C*

2*q*+2 *∈*

*C*2, *Cj , Cj ∈ C*3 and *C*1 = *Cj* . Then, *Cj*

is a independent set, for all

*q*+2 2 1 *i*

*i ∈ {*1*,* 2*,q* + 1*,q* + 2*,q* + 3*,* 2*q* + 2*}*. This give us that *cj* must be a proper

coloring of *G*. Now, we prove that it is a circular backbone coloring. For this, we prove that, given a central vertex *v*, all of its neighbors in the backbone are colored with an appropriate color. First observe that *v* receives color 1, *q* +3 or *q* + 2. In the first case, the colors allowed for its neighbors in *F* are *q* +1 or *q* + 2. In the second case, all the its neighbors in *F* are colored with colors 1 or 2. Finally, in the last case, all the its neighbors in *F* are colored with colors 1 or 2*q* + 2. Hence *cj* is a circular q-backbone coloring of (*G, F* ).

*2*

# Matching backbones

In this section our goal is to prove the upper bounds for planar graphs *G* with matching backbones *M* .

The proof of the upper bounds follow by contradiction as we suppose the ex- istence a minimal counter-example. Let us formally define this notion. Given a pair (*G, H*), a *subpair* (*Gj,Hj*) of (*G, H*) is a pair such that *Hj ⊆ H* and *Gj ⊆ G*. We say that (*Gj,Hj*) is a *proper subpair* of (*G, H*) if it is a subpair of (*G, H*) and *Hj ⊂ H* and *Gj ⊂ G*. A pair (*G, H*) is called (*k, q*)-minimal if CBC*q*(*G, H*) *> k*, but CBC*q*(*Gj,Hj*) *≤ k* for every proper subpair (*Gj,Hj*) of (*G, H*). Note that if CBC*q*(*G, H*) *> k*, then there exists a subpair (*Gj,Hj*) of (*G, H*) that is (*k, q*)- minimal.

Let Z*k* be a color space. A subset *S ⊆* Z*k*, a positive integer *q* and *i ∈* Z*k*, we say that the color *i* is *q-bad for S* if *S ⊆ {i − q* + 1*, ··· ,i* + *q −* 1*}*.

Given a plane graph *G*, we denote by *F*(*G*) the set of faces of *G* and by *d*(*f* ) the degree of a face *f* in *G*. Now, given a pair (*G, H*) and a vertex *u ∈ V* (*G*), the (*k, q*)*-total degree of u in* (*G, H*) is given by:

*t*

*d*

(*G,H*)*,k,q*

(*u*) = *dG*(*u*)+ (2*q −* 2)*dH* (*u*)*.*

If *G, H, k, q* are clear from context, we omit them in the notation.

**Lemma 3.1** *Let* (*G, H*) *be a* (*k, q*)*-minimal pair, with k ≥* 2*q. If uv ∈ E*(*H*)*, then*

*dt*(*u*)+ *dt*(*v*) *≥* 2*k* + 2*q −* 2*.*

## Sketch of the proof.

First, write *dt*(*u*)= *k* + *l* and *dt*(*v*)= *k* + *lj*. Then, we proved that *dt*(*u*) *≥ k*, for every *u ∈ V* (*G*), so that *l, lj ≥* 0.

Let *f* be a circular *q*-backbone *k*-coloring of (*G − u − v, H − u − v*). Note that *af* (*u*)= *k −* (*k* + *l −* 2*q* + 1) = 2*q −* (*l* + 1), and analogously *af* (*v*)= 2*q −* (*lj* + 1). By contradiction, suppose that *dt*(*u*)+ *dt*(*v*) *≤* 2*k* + 2*q −* 3. Then, we get:

*k* + *l* + *k* + *lj ≤* 2*k* + 2*q −* 3 *⇔ l* + *lj ≤* 2*q −* 3*.*

Therefore, *af* (*v*) *≥* 2*q −* 1 *−* (2*q −* 3 *−l*)= *l* + 2. Then, we also proved that if *S ⊆* Z*k*

has cardinality 2*q −p*, where *p ≥* 0 and *k ≥* 2*q*, there are at most *p* colors in Z*k* that

are *q*-bad for *S*. So we have that at most *l*+1 colors are *q*-bad for *Af* (*u*). Therefore, there exists a color *c ∈ Af* (*v*) that is not *q*-bad for *Af* (*u*), a contradiction. *2*

The lemma below follows directly from Euler’s Formula.

**Lemma 3.2** *Let G be a plane graph, M be a matching of G, and q be a positive integer. Then,*

Σ (*dt*(*v*) *−* 2*q −* 2) + Σ

(*d*(*f* ) *−* 4) *≤ −*8*.*

*v∈V* (*G*) *f∈7* (*G*)

**Theorem 3.3** *Let G be a plane graph, M be a matching in G, and q be a positive integer. Then:*

CBC3(*G, M* ) *≤* 7*, and*

CBC*q*(*G, M* ) *≤* 2*q, if q ≥* 4*.*

## Sketch of the proof.

First, we prove that it holds when *M* is a perfect matching and *q ≥* 4. For this, suppose otherwise and let (*G, H*) be a minimal counter-example. We apply the discharging method. Start by giving charge *dt*(*u*) to every *u ∈ V* (*G*) and *d*(*f* ) *−* 4 to every *f ∈ F*(*G*). Let *α* = 2*q* + 2. By Lemma [3.1](#_bookmark5), for each *uv ∈ M* we have:

*dt*(*u*)+ *dt*(*v*) *≥* 2*k* + 2*q −* 2= 6*q −* 2= 2*α* + 2*q −* 6*.*

This tells us that the sum of the charges on each edge of the matching is enough to ensure that every vertex can end up with non-negative charge, with surplus of 2*q −* 6 *≥* 2 on each edge of the matching. Because *M* is a perfect matching, we get a surplus of at least *n* which is clearly bigger than the number of triangles, thus contradicting Lemma [3.2](#_bookmark6). One can verify that when *q* = 3 and *k* = 2*q* +1 we can apply a similar argument.

Now, suppose (*G, M* ) is a pair, and *M* is not a perfect matching. Then, we can add, for each vertex *u* that is not saturated by *M* , a pendant vertex *uj* to *G* and *M* in order to obtain a pair (*Gj,Mj*) containing (*G, M* ) and such that *Mj* is a perfect matching of *Gj*. The lemma follows by the previous paragraph and the fact that (*G, M* ) *⊆* (*Gj,Mj*). *2*

Both upper bounds provided by Theorem [3.3](#_bookmark7) are tight, under the hypothesis that *M /*= *∅*. For *q ≥* 4, if *M /*= *∅*, then we have CBC*q*(*G, M* ) *≥* 2*q*, for any graph

*G*. In [[8](#_bookmark18)], they present an example to show that there exists a planar graph *G*3 and a perfect matching *M*3 of *G*3 such that BBC2(*G*3*, M*3) = 5. The same example also satisfies CBC3(*G*3*, M*3)= 7.

**Proposition 3.4** CBC3(*G*3*, M*3)= 7*.*

**Proof.** The upper bound is provided by Theorem [3.3](#_bookmark7). To prove the lower bound, suppose, by contradiction, that there exists a circular 3-backbone 6-coloring *c* of (*G*3*, M*3). Observe that, if uv is an edge of *M*3, then *{c*(*u*)*, c*(*v*)*}* is either *{*1*,* 4*}* or



a’

d

c’

b’

a

d’

b

c

Fig. 1. CBC3(*G*3*, M*3)=7

*{*2*,* 5*}* or *{*3*,* 6*}*. If the first case (resp. second, third) occurs, we say that *uv* is an 14-edge (resp. a 25-edge, a 36-edge).

One may assume, without loss of generality, that *abj* is a 14-edge. Since *d* is a neighbor of both *a* and *bj*, *d* has a different color, and without loss of generality, we may assume that *ajd* is a 25-edge. Then, since the only non-neighbor of *c* is *cj*, we deduce that *cjd* is a 36-edge. Consequently, since *b* is adjacent to *aj, d, c, dj*, then *bcj* must be a 14-edge. This is a contradiction, because *a* is adjacent to *b* and *cj* and *c*(*a*) *∈ {*1*,* 4*}*. *2*

# Polynomial-time algorithm for outerplanar graphs

In this section, we give a polynomial algorithm that, given an outerplanar graph *G* and a matching *M* of *G*, decides whether *BBC*(*G, M* ) *≤* 3. Since *BBC*(*G, M* ) *≤* 2 if and only if *G* is bipartite and *M* is empty, and because *BBC*(*G, M* ) is always at most 4, this implies that one can compute *BBC*(*G, M* ) in polynomial time.

**Theorem 4.1** *Let G be a connected outerplanar graph on n vertices and m edges, and let M ⊆ E*(*G*) *be a matching in G. Then, deciding if BBC*(*G, M* ) *≤* 3 *can be done in time O*(*m* + *n*)*.*

**Proof.** A *tree decomposition* of a graph *G* is a pair *D* = (*T, {Xt}t∈V* (*T* )) such that: *T* is a rooted tree; for every vertex *v ∈ V* (*G*), there exists *t ∈ V* (*T* ) such that *v ∈ Xt*; for every edge *uv ∈ E*(*G*), there exists *t ∈ V* (*T* ) such that *{u, v} ⊆ Xt*; for every *v ∈ V* (*G*), the subset *{t ∈ V* (*T* ) *| v ∈ Xt}* induces a subtree of *T* . A tree decomposition is *nice* if the vertices of *T* can be classified as one of the following types.

* 1. *leaf* : *t* is a leaf in *T* ;
  2. *forget* : *t* has exactly one child *tj* and there exists *u ∈ V* (*G*) such that *Xt* =

*Xt′ \ {u}*;

* 1. *introduce*: *t* has exactly one child *tj* and there exists *u ∈ V* (*G*) such that

*Xt* = *Xt′ ∪ {u}*; and

* 1. *join*: *t* has exactly two children, *t*1 and *t*2, and *Xt* = *Xt*1 = *Xt*2 .

The *width* of a tree decomposition *D* is the maximum size of a subset *Xt* minus one. The *treewidth of G* is the minimum width of a tree decomposition of *G* and it is denoted by tw(*G*). It is known that if *G* is outerplanar, then tw(*G*) *≤* 2, and that a (nice) tree decomposition *D* = (*T, {Xt}t∈V* (*T* )) of width tw(*G*) such that

*|V* (*T* )*|* = *O*(*n*) can be computed in time *O*(*n* + *m*) [[2,](#_bookmark12)[7](#_bookmark17)]. Here, we use such a tree decomposition to solve our problem. For this, for each node *t ∈ V* (*T* ), denote by *Tt* the subtree of *T* rooted at *t*; by *Vt* the subset *t′∈V* (*Tt*) *Xt′* ; by (*Gt, Ht*) the pair (*G*[*Vt*]*,H*[*Vt*]); and for each coloring *f* : *Xt → {*1*,* 2*,* 3*}*, define the following:

⎧ 1 *,* if there is a 2-backbone 3-coloring *fj* of (*Gt, Ht*) such that *f ⊆ fj*; *Bt*(*f* )=

⎨

⎩ 0 *,* otherwise.

Now, we present how to compute each *Bt*(*f* ), given that the values are computed in a post-order traversal of *T* . Hence, consider a node *t* and a 3-coloring *f* of *G*[*Xt*]. If *t* is a leaf, then *Bt*(*f* ) = 1 if and only if *f* is a 2-backbone 3-coloring of (*G*[*Xt*]*,H*[*Xt*]), which can be tested in constant time, since *|Xt| ≤* 3. So, suppose otherwise and consider that we know the values of *Bt′* (*f* ) for each child *tj* of a node

*t*. We analyse all the possible cases according to the type of *t*:

1. *t* is forget: let *tj* be the child of *t* and *u ∈ V* (*G*) be such that *Xt* = *Xt′ \ {u}*. Observe that (*Gt, Ht*)= (*Gt′ , Ht′* ). Thus, we get that there exists a 2-backbone 3-coloring *fj* of (*Gt, Ht*) that extends *f* if, and only if, *fj* is a 2-backbone 3- coloring of (*Gt′ , Ht′* ) that contains *f* . Hence, if we define *fi* as *f ∪ {*(*u, i*)*}* for each *i ∈ {*1*,* 2*,* 3*}*, we get that:

*Bt*(*f* ) = 1 if, and only if, *Bt′* (*fi*)=1 for some *i ∈ {*1*,* 2*,* 3*}*.

1. *t* is introduce: let *tj* be the child of *t* and *u ∈ V* (*G*) be such that *Xt* = *Xt′ ∪{u}*. Then, there exists a 2-backbone 3-coloring *fj* of (*Gt, Ht*) that extends *f* if, and only if, *f* is a 2-backbone 3-coloring of (*G*[*Xt*]*,H*[*Xt*]) and there exists a 2- backbone 3-coloring *fjj* of (*Gt′ , Ht′* ) that extends *fj* = *f*T*X ′* (*f* restricted to *Xt′* ). Hence, we get that:

*t*

*Bt*(*f* ) = 1 if, and only if, *Bt′* (*f* )= 1*.*

*j*

1. *t* is join: let *t*1*, t*2 be the children of *t*. Because *Xt* separates *Gt*1 *− Xt* from *Gt*2 *− Xt*, we get that the union of two 2-backbone 3-colorings of *Gt*1 and *Gt*2 that agree on *Xt* is a 2-backbone 3-coloring of (*Gt, Ht*). Thus:

*Bt*(*f* ) = 1 if, and only if, *Bt*1 (*f* )= *Bt*2 (*f* )= 1*.*

Now, observe that each step can be done in constant time, because there are at most 33 possible colorings of each *Xt*. Since there are *&*(*n*) nodes in *T* , we get that, once we find the tree decomposition of *G*, one can compute all the values *Bt*(*f* ) in *&*(*n*) time. Once we arrive at the root *r* of *T* , the answer to whether *BBC*(*G, M* ) *≤* 3 is “yes” if and only if *Br*(*f* ) = 1, for some 2-backbone 3-coloring *f* of *Xr*. *2*

We mention that, after the revision and acceptance of this extended abstract, we found an error in our previous proof. Nevertheless, the result is still true and we found a much simpler and general proof. Observe that the above proof can be generalized for any graph *G* with bounded treewidth, for any fixed *k*, and for any possible backbone *H*. This, combined with the fact that *BBC*(*G, H*) *≤* 2*χ*(*G*) *—* 1, which equals 5 when *G* is an outerplanar graph, means that the backbone color- ing problem on outerplanar graphs is polynomial-time solvable for every possible backbone *H*.

# Brooks’ Type Theorem

This section is devoted to correcting a proof of a Brook’s Type Theorem demon- strated by Miˇskuf et al. in [[10](#_bookmark20)]:

**Theorem 5.1** *Let M be a matching in a graph G of maximum degree* Δ(*G*)*. Then*

*BBC*2(*G, M* ) *≤* Δ(*G*)+ 1*.*

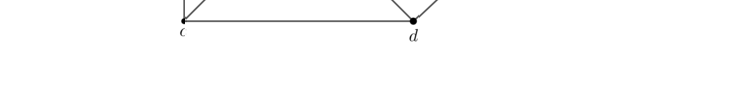
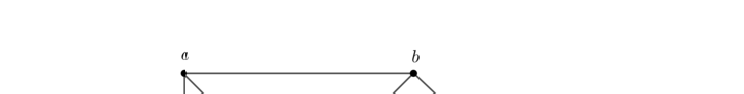
For a better understanding of their proof, it is necessary to define a special structure. Let *x, y* be two non-adjacent neighbors of a vertex *v* in a connected graph *G* such *G — x — y* is connected. Then we say that (*v*; *x, y*) is a *fork*. That being said, the following lemma was required:

**Lemma 5.2** *Let G be a 2-connected graph with all the vertices of the same degree d ≥* 3 *except a particular vertex v which is of degree < d. Then, G has a fork* (*w*; *x, y*) *such that v /*= *x and v /*= *y.*

Assuming that *G* is neither an odd cycle nor a complete graph, they prove the Theorem [5.1](#_bookmark8) considering an order *v*1*,..., vn* of the vertices of *G* such that each *vi* (*i < n*) has a succeeding neighbor. So, they claim that *G* is a regular graph and *M* is a perfect matching. Otherwise, we may choose for *vn* a vertex that is of degree

*<* Δ or that it is not incident with an edge of *M* . In both cases, the procedure will color also the vertex *vn*. Finally, they use the Lemma [5.2](#_bookmark9) to conclude that *G* is 2-connected. So, since *G* is neither an odd cycle nor a complete graph, they use the Theorem [5.3](#_bookmark10) below to ensure the existence of a fork in *G* that give us an order to color the vertices of *G* using at most Δ(*G*) + 1 colors.

In contrast, we found out that Lemma [5.2](#_bookmark9) is not true. A simple counterexample is described in the following figure:



Observe that the above graph satisfies the hypotheses of the Lemma [5.2](#_bookmark9), but the only forks in this graph are (*b*; *a, v*), (*b*; *c, v*), (*d*; *a, v*) and (*d*; *c, v*), which contradicts the Lemma [5.2](#_bookmark9).

On the other hand, the theorem is still true and we find a way to fix the demon- stration made in [[10](#_bookmark20)]. Before presenting it and to better understand our proof, we recall some definitions and results on connectivity.

**Theorem 5.3** *(Bryant [*[*6*](#_bookmark16)*]) For a 2-connected graph the following three statements are equivalent:*

* 1. *G is a complete graph or a cycle;*
  2. *the removal from G of any two non-adjacent vertices disconnects it;*
  3. *the removal from G of any two vertices at distance 2 apart disconnects it.*

Notice that the above theorem claims that each 2-connected graph distinct from a cycle and a complete graph contains a fork.

A *block* of a graph *G* is a maximal connected subgraph of *G* that has no cut- vertex. If *G* itself is connected and has no cut-vertex, then *G* is a block. The *block-cutpoint graph* of a connected graph *G* is the graph whose vertices are the blocks and the cut-vertices of *G*, with an cut-vertex *v* adjacent to an block *B* if and only if *v ∈ V* (*B*). Observe that a block-cutpoint graph of *G* is a tree and all its leaves are blocks of *G*. Each block of *G* which is a leaf of the block-cutpoint graph of *G* is called *leaf block*.

Given a leaf block *B* of *G*, we say that a vertex *u ∈ V* (*B*) is *internal* if it is not a cut-vertex. Also, if *G* is 1-connected and the block-cutpoint graph of *G* is a path, we say that *G* has a *path structure*.

Now, we can prove the following:

**Theorem 5.4** *Let M be a matching in a graph G of maximum degree* Δ*. Then*

BBC2(*G, M* ) *≤* Δ+ 1*.*

**Proof.** The proof follows the same steps as Brooks’ theorem. We just need to be careful in the case *G* has cut-vertices, as we may not be able to combine backbone colorings of the blocks of *G* into a coloring of *G*.

Without loss of generality, we assume that *G* is connected. In case *G* is not regular, then one can create an order *σ* = *v*1*,..., vn*(*G*) over *V* (*G*) such that each *vi*

has at least one neighbor *vj* with *j > i* for every *i ∈ {*1*,..., n—*1*}* and *dG*(*vn*(*G*)) *<* Δ. By applying the greedy algorithm over this ordering of *V* (*G*), the obtained coloring uses at most Δ + 1 colors as each vertex *vj* has at most Δ *—* 1 colored neighbors and at most one edge of the backbone *M* has *vj* as endpoint. Thus, we assume that *G* is regular.

If *G* has a fork (*z*; *x, y*), then we can produce an order *σ* = (*x, y, v*3*,..., vn—*1*, z*) of *V* (*G*) such that each *vi* has at least one neighbor *vj* with *j > i* for every *i ∈*

*{*3*,...,n —* 1*}*, and *z* has two non-adjacent neighbors, namely *x* and *y*, such that when we use the greedy algorithm on *G* using the order *σ* the vertices *x* and *y* have the same color. Then, we can use this order to, given a matching *M* in *G*, construct a 2-backbone coloring of (*G, M* ) that uses at most Δ + 1 colors. Consequently, we also assume that *G* has no fork.

If *G* is a complete graph or a cycle, then the upper bound holds (see [[10](#_bookmark20)] for details). Thus, we consider that *G* is a *k*-regular graph with no forks and *G* is not a complete graph nor a cycle. Consequently, observe that *k ≥* 3. Note also that *G* cannot be 2-connected, due to Theorem [5.3](#_bookmark10).

Let *B* be a leaf-block of *G* and *u* be the only cut-vertex of *G* belonging to *V* (*B*). **Case 1:** *κ*(*B — u*) = 1. In this case, we claim that *G* has a fork, contradicting our hypothesis. In fact, note that *u* has a neighbor in each leaf-block of *B — u* that is not a cut-point of *B — u*. If *dB*(*u*) *≥* 3, let *x* and *y* be neighbors of *u* in distinct leaf-blocks of *B — u*, and that are not cut-vertices of *B — u*. Observe that (*u*; *x, y*) is a fork of *G*. Otherwise, *dB*(*u*) = 2 and the block-cutpoint tree of *B — u* is a path. Since *G* is *k ≥* 3 regular, note that each leaf block in *B — u* has at least 4 vertices. In case *B — u* has only two blocks *B*1 and *B*2, let the unique cutpoint of *B — u* be

*z*. Since those are the only blocks, note that *z* must have two neighbors *y ∈ V* (*B*1) and *x ∈ V* (*B*2) such that neither *y* nor *x* is a neighbor of *u*. Then, (*z*; *x, y*) is a fork of *G*. Finally, if *B — u* has at least three blocks, let *B*1 be a leaf-block, *B*2 be the only block sharing the cut-vertex *z* with *B*1. Once more, one may choose *y ∈ NB*1 (*z*) such that *y* is not a neighbor of *u* and any vertex *x* in *B*2 (even if *B*2 is a single edge) and then (*z*; *x, y*) is a fork of *G*.

**Case 2:** *B — u* is 2-connected. We shall prove that *u* has exactly two neighbors *x* and *y* in *B* and *xy* is the only edge that does not belong to *B — u*. In the sequel, we use such structure to build an ordering over *V* (*G*) such that the greedy algorithm uses at most Δ + 1 colors in a 2-backbone coloring of *G*. We claim that *B — u* has a fork (*z*; *x, y*). If not, by Theorem [5.3](#_bookmark10), *B — u* should be either a complete graph or a cycle. It is not possible as *G* is *k*-regular and *u* has neighbors in *B — u* and *G — V* (*B*). Moreover, Theorem [5.3](#_bookmark10) ensures that every two non-adjacent vertices form a fork. Since *G* has no fork, the only possibility to such fork exist in *B — u* is that *x* and *y* be the only neighbors of *u* in *B — u*. Thus, the leaf-block *B* has the structure we claimed: *u* has exactly two neighbors *x* and *y* in *B* and *xy* is the only edge that does not belong to *B — u*. Finally, one can construct and ordering over *V* (*G*) such that the two first vertices are the neighbors of *u* in *B*, then we have all vertices of *V* (*B*) *\ u, x, y*, in the sequel we place all vertices of *V* (*G*) *\ V* (*B*) in such a way that each vertex has a neighbor that appears latter in the sequence, and the

last vertex is *u*. Observe that each vertex on such sequence either has one neighbor that appears latter in the sequence, or is a neighbor of *x* and *y* which will be colored with the same color. Thus, the greedy algorithm uses at most Δ + 1 colors to build a 2-backbone coloring of (*G, M* ).

*2*

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