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Characterizing Consistent Smyth Powerdomains by *FS*-*∧↑*-domains

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**Abstract**

In this paper, we introduce *FS*-*∧↑*-domains, and show that the category with *FS*-*∧↑*-domains as objects and Scott continuous functions as morphisms is a Cartesian closed category. Moreover, we characterize the consistent Smyth powerdomain over a Lawson compact domain by means of *FS*-*∧↑*-domain.

*Keywords:* Domain; Consistency; *FS*-*∧↑*-domain; Consistent Smyth powerdomain

# Introduction

In Domain theory, powerdomains are very important structures, which play an im- portant role in modeling the semantics of nondeterministic programming languages ([[4,5,](#_bookmark7)[6,](#_bookmark8)[7,](#_bookmark9)[9,](#_bookmark10)[11,](#_bookmark13)[12,](#_bookmark14)[13,](#_bookmark15)[14,](#_bookmark16)[15](#_bookmark17)]). For example, the Smyth powerdomain is the free defla- tionary semilattice over a continuous dcpo, where the deflationary binary operator is exactly the Scott continuous meet operator [[14](#_bookmark16)]. However, in many interesting domains, such as *L*-domains, the meet operator is not total but a partial one: two elements have a meet (or a greatest lower bound) if they are consistent, i.e., they have an upper bound. In this case, the partial meet operator is called a consistent meet, denoted by *∧↑*. So a question arises: can we construct a new free algebra over

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a continuous dcpo on which the binary operator is exactly the Scott continuous con- sistent meet? In [[16](#_bookmark18)], we show by methods of topology and order theory that the consistent Smyth powerdomain over a continuous dcpo exists and is a continuous dcpo-*∧↑*-semilattice. Moreover, if a continuous dcpo is Lawson compact, then its consistent Smyth powerdomain is a Lawson compact *L*-domain. This is a difference between the consistent Smyth powerdomain and the classical one, because the clas- sical Smyth powerdomain over a Lawson compact domain is a bounded complete domain.

Note that classical powerdomains, such as the Smyth powerdomain and the Hoare powerdomain, can be characterized by means of some *FS*-domains given of the basic functions in the structure of powerdomains respectively. In [[8](#_bookmark11)], Huth, Jung and Keimel introduced a new concept: linear *FS*-lattice, which is a complete lattice and there exists a directed family of finitely separated linear functions which can approximate *id*, where a function is linear if it preserves all suprema. They proved that the Hoare powerdomain *H*(*L*) over a pointed domain *L* is characterized by a distributive linear *FS*-lattice. In [[10](#_bookmark12)], Meng and Kou introduced *F S∧*-domains and proved that the Smyth powerdomain *S*(*L*) over a Lawson compact domain *L* is characterized by *F S∧*-domains. So for purely mathematical purposes, we have reasons to believe that there exists a kind of *FS*-domains to characterize the consistent Smyth powerdomain over a Lawson compact continuous domain.

In this paper we first use the partial Scott continuous binary operator *∧↑* to con- struct a finitely separated domain: *FS*-*∧↑*-domain, which is a dcpo-*∧↑*-semilattice and there exists a directed family of finitely separated *∧↑*-semilattice homomor- phisms which can approximate *idL*. We have obtained the following conclusions:

* 1. The category with *FS*-*∧↑*-domains as objects and Scott continuous functions as morphisms is a cartesian closed category.
  2. The consistent Smyth powerdomain *SC*(*L*) over a Lawson compact continuous domain *L* is an *FS*-*∧↑*-domain.

Moreover, we characterize consistent Smyth powerdomains by means of *FS*-*∧↑*- domains.

Next, we collect some basic notions needed in this paper. The reader can also consult [[1,](#_bookmark3)[3](#_bookmark6)]. A poset *L* is called a directed complete poset (a dcpo, for short) if any directed set of *L* has a sup in *L*. For *x, y ∈ L*, *x* is called to be way below *y* (denoted by *x y*) if for any directed set *D*, *y ≤ ∨D* implies that there is some *d ∈ D* with *x ≤ d*. A poset *L* is called continuous if for all *x ∈ L*, *x* = *∨↑****↓****x*, i.e., the set ***↓****x* = *{a ∈ L* : *a x}* is directed and *x* = *∨{a ∈ L* : *a x}*, where the arrow in the symbol *∨↑* is to emphasize the directedness of ***↓****x*. Specially, a dcpo which is continuous as a poset will be called a (continuous) domain. For a subset *A* of *L*, let *↑A* = *{x ∈ L* : *∃a ∈ A, a ≤ x}*, *↓A* = *{x ∈ L* : *∃a ∈ A, x ≤ a}*. We use *↑a* (resp. *↓a*) instead of *↑{a}* (resp. *↓{a}*) when *A* = *{a}*. *A* is called an upper (resp. a lower) set if *A* = *↑A* (resp. *A* = *↓A*). An element *k ∈ L* is called compact if *k k*. The subset of all compact elements is denoted by *K*(*L*). A dcpo *L* is called algebraic if for all *x ∈ L*, *x* = *∨↑*(*↓x ∩ K*(*L*)).

**Definition 1.1** *Let L be a poset.*

* + 1. *A subset A of L is called consistent if A has an upper bound in L, i.e.,*

*A ⊆ ↓x for some x ∈ L.*

* + 1. *L is called a consistent meet-semilattice (or ∧↑-semilattice) if x ∧ y exists for all consistent x, y ∈ L. To emphasize the fact that x and y are consistent, we will write x ∧↑ y instead of x ∧ y. Moreover, if L is a (continuous) dcpo, then L is called a (continuous) dcpo-∧↑-semilattice.*
    2. *L is called an L-domain if L is a domain and every consistent subset of L*

*has an inf, i.e., ↓x is a complete lattice for all x ∈ L.*

Let *L* be a poset. We call the topology generated by the complements *L\↑x* of principal filters as subbasic open sets the lower topology and denote it by *ω*(*L*). If (*L, ≤*) is a dcpo, we define the Scott topology (denoted by *σ*(*L*)), which has as its topology of closed sets all directed complete lower sets, i.e., lower sets closed under directed sups. The Lawson topology *λ*(*L*) is generated by taking the join of *σ*(*L*) and *ω*(*L*) as subbasic. If *L* is a *∧↑*-semilattice, then the partial operator

*∧↑* : *L × L→L* is Scott continuous.

A *∧↑*-semilattice homomorphism *f* between dcpo-*∧↑*-semilattices (*P, ∧↑*) and (*E, ∧↑*) is a Scott continuous function from *P* to *E* such that *f* (*a∧↑ b*)= *f* (*a*) *∧↑ f* (*b*) whenever *a, b* are consistent in *P* . Note that the function is Scott continuous and conditionally multiplicative (or *cm* for short in [[2](#_bookmark4)]), that is, each *∧↑*-semilattice homomorphism is a Scott continuous *cm* function.

# Categories of *FS*-*∧↑*-domains

For dcpos *L* and *P* , let [*L−→P* ] denote all Scott continuous functions from *L* to *P* with the pointwise order. For dcpo-*∧↑*-semilattices *D* and *E*, let [*D−→∧↑ E*] denote the function space of *∧↑*-semilattice homomorphisms from *D* to *E* with the pointwise order.

**Definition 2.1** *[*[*3*](#_bookmark6)*]A dcpo L is called an FS-domain if idL is approximated directly by a family of ﬁnitely separating functions, where a Scott continuous function f* : *L−→L is called ﬁnitely separated if there exists a ﬁnite set Mf such that for each x ∈ L, there exists m ∈ Mf such that f* (*x*) *≤ m ≤ x.*

**Definition 2.2** *A dcpo L is called an FS-∧↑-domain if it is a ∧↑-semilattice and there exists a directed family of ﬁnitely separated ∧↑-semilattice homomorphisms which can approximate idL.*

In other words, an *FS*-*∧↑*-domain is a continuous dcpo-*∧↑*-semilattice which *id* is approximated by a directed family of finitely separated Scott continuous functions preserving existing finite infs. Obviously, every *FS*-*∧↑*-domain is an *FS*-domain. Then we have

**Proposition 2.3** *Each FS-∧↑-domain is a Lawson compact L-domain.*

Next, we show that the category with *FS*-*∧↑*-domains as objects and Scott continuous functions as morphisms is a cartesian closed category.

**Theorem 2.4** *Let D and E be dcpo-∧↑-semilattices, then* [*D−→∧↑ E*] *is a dcpo-∧↑-*

*semilattice.*

**Proof.** Firstly, [*D−→∧↑ E*] is a dcpo. For any directed set *{fj ∈* [*D−→∧↑ E*]: *j ∈ J}* and *x ∈ D*, set *f* (*x*) = *j∈J fj*(*x*). It is obvious that *f* is Scott continuous. If *x, y ∈ D* are consistent, then *f* (*x*)*,f* (*y*) are also consistent in *E*. Then

W

*f* (*x ∧↑ y*)= \_ *fj*(*x ∧↑ y*)= \_(*fj*(*x*) *∧↑ fj*(*y*))

*j∈J j∈J*

= ( \_ *fj*(*x*)) *∧↑* ( \_ *fj*(*y*)) = *f* (*x*) *∧↑ f* (*y*)*.*

*j∈J j∈J*

So *f* is also Scott continuous and a *∧↑*-semilattice homomorphism. Hence [*D−→∧↑ E*] is a dcpo.

Secondly, [*D−→∧↑ E*] isa *∧↑*-semilattice. If *f, g ∈* [*D−→∧↑ E*] are consistent, then *f* (*x*)*, g*(*x*) are consistent for any *x ∈ D*. Then *f* (*x*) *∧↑ g*(*x*) exists. Let (*f ∧↑ g*)(*x*)= *f* (*x*) *∧↑ g*(*x*). For a directed set *{xk ∈ D* : *k ∈ K}*, we have

(*f ∧↑ g*)( \_ (*xk*)) = *f* ( \_ (*xk*)) *∧↑ g*( \_ (*xk*))

*k∈K*

*k∈K*

*k∈K*

= \_ *f* (*xk*) *∧↑* \_ *g*(*xk*)= \_ [*f* (*xk*) *∧↑ g*(*xk*)]

*k∈K*

*k∈K*

*k∈K*

= \_ [(*f ∧↑ g*)(*xk*)]*.*

*k∈K*

Then *f ∧↑ g* is Scott continuous.

For a pair of consistent points *x, y* in *D*, *f* (*x ∧↑ y*) and *g*(*x ∧↑ y*) are consistent in *E*. Then

(*f ∧↑ g*)(*x ∧↑ y*)

= *f* (*x ∧↑ y*) *∧↑ g*(*x ∧↑ y*)

= (*f* (*x*) *∧↑ f* (*y*)) *∧↑* (*g*(*x*) *∧↑ g*(*y*))

= (*f* (*x*) *∧↑ g*(*x*)) *∧↑* (*f* (*y*) *∧↑ g*(*y*))

= (*f ∧↑ g*)(*x*) *∧↑* (*f ∧↑ g*)(*y*)*.*

That is, *f ∧↑ g* is a *∧↑*-semilattice homomorphism. So [*D−→∧↑ E*] is a *∧↑*- semilattices.

Finally, by the Scott continuity of the operation *∧↑*, we obtain the following conclusion. If the sup of the directed set *{fj ∈* [*D−→∧↑ E*] : *j ∈ J}* and *g ∈* [*D−→∧↑ E*] are consistent, then for *x ∈ D*,

[*g ∧↑* ( \_ *ƒj*)](*x*)= *g*(*x*) *∧↑* ( \_(*ƒj*(*x*))

*j∈J j∈J*

= \_[*g*(*x*) *∧↑ ƒj*(*x*)] = \_[(*g ∧↑ ƒj*)(*x*)]

*j∈J*

= [ \_(*g ∧↑ ƒj*)](*x*)*.*

*j∈J*

*j∈J*

So *∧↑* : [*D−→∧↑ E*] *×* [*D−→∧↑ E*]*−→*[*D−→∧↑ E*] is Scott continuous.

We have obtained that [*D−→∧↑ E*] is a dcpo-*∧↑*-semilattice. *2*

**Theorem 2.5** *Let D and E be FS-∧↑-domains, then* [*D−→∧↑ E*] *and* [*D−→E*] *are*

*FS-∧↑-domains.*

**Proof.** Suppose that *D* and *E* are approximate identities for *D* and *E* respectively. Then we claim that the family

*D⊗ E* = *{δ ⊗ ϵ* : *δ ∈ D,ϵ ∈ E}*

defined by

*ƒ '→ ϵ*2*ƒδ*2

for *ƒ ∈* [*D−→∧↑ E*] is an approximate identity for [*D−→∧↑ E*] and *δ ⊗ ϵ* is finitely separating. The proof is similar with the case of *FS*-domains.

It suffices to show that *δ ⊗ϵ ∈* [*D−→∧↑ E*]*−→∧↑* [*D−→∧↑ E*]. Firstly, it is obvious that *δ ⊗ ϵ* is Scott continuous. Secondly, for a pair of consistent points *ƒ, g ∈* [*D−→∧↑ E*], we have (*δ ⊗ ϵ*)(*ƒ* )*,* (*δ ⊗ ϵ*)(*g*) are consistent and for any *x ∈ D*

[(*δ ⊗ ϵ*)(*ƒ ∧↑ g*)](*x*)= [*ϵ*2(*ƒ ∧↑ g*)*δ*2](*x*)

= *ϵ*2[*ƒδ*2(*x*) *∧↑ gδ*2(*x*)] = *ϵ*2*ƒδ*2(*x*) *∧↑ ϵ*2*gδ*2(*x*)

= [*ϵ*2*ƒδ*2 *∧↑ ϵ*2*gδ*2](*x*)= [(*δ ⊗ ϵ*)(*ƒ* ) *∧↑* (*δ ⊗ ϵ*)(*g*)](*x*)*.*

So we conclude that *δ ⊗ ϵ* is a *∧↑*-semilattice homomorphism. Then [*D−→∧↑ E*] is an *FS*-*∧↑*-domain. Similarly, [*D−→E*] is also an *FS*-*∧↑*-domain. *2*

Note that usually the category with *FS*-*∧↑*-domains as objects and *∧↑*- semilattice homomorphisms as morphisms is not a cartesian closed category. How- ever, if the category considers Scott continuous functions as morphisms, then from the preceding paragraph we have the following conclusion:

**Theorem 2.6** *The category with FS-∧↑-domains as objects and Scott continuous functions as morphisms is a cartesian closed category.*

# Characterize consistent Smyth powerdomains by *FS*-

*∧↑***-domains**

In the following paragraph, we will relate *FS*-*∧↑*-domain and consistent Smyth powerdomain with the functions: *∧↑*-semilattice homomorphisms. We characterize consistent Smyth powerdomains *SC*(*L*) over a Lawson compact continuous domain *L* by means of *FS*-*∧↑*-domains.

**Definition 3.1** *[*[*16*](#_bookmark18)*] A consistent deflationary semilattice is a continuous dcpo L with a Scott continuous binary partial operator ∧↑ deﬁned only for consistent pairs of points that satisfy three equations for commutativity x∧↑ y* = *y ∧↑ x, associativity x∧↑* (*y ∧↑ z*)= (*x∧↑ y*) *∧↑ z, and idempotency x∧↑ x* = *x together with the inequality x ≥ x ∧↑ y for any x, y, z ∈ L. The free consistent deflationary semilattice over a domain L is called the consistent Smyth powerdomain over L.*

**Definition 3.2** *[*[*16*](#_bookmark18)*] Let L be a poset and F a nonempty subset of L. Two elements x and y in F are called linearly connected in F provided there exists a consistent path in ↑F from x to y, i.e. ﬁnitely many x*0*, x*1*,..., xn in ↑F such that x* = *x*0 *† x*1 *† ... † xn* = *y, denoted by x ∼F y. F is called linearly connected if any two elements of F are linearly connected in F.*

Let *L* be a continuous domain and let

*BC*(*L*)= *{F ⊆fin L* : *F /*= *∅* and *F* is linearly connected*}.*

Let *SC*(*L*) be the family generated by *↑BC*(*L*) = *{↑F* : *F ∈ BC*(*L*)*}* as a basis, i.e., for all *A ∈ SC*(*L*), *A* = *↑{↑F* : *F ∈ BC*(*L*) & *↑F A}*, then *SC*(*L*) is a continuous dcpo-*∧↑*-semilattice in [[16](#_bookmark18)].

**Theorem 3.3** *[*[*16*](#_bookmark18)*] Let L be a continuous domain. The embedding j of L into SC*(*L*) *is given by j*(*x*)= *↑x for x ∈ L. If P is a dcpo-∧↑-semilattice and ƒ* : *L−→P a Scott continuous function, then there exists uniquely a ∧↑-homomorphism ƒ*¯ *such that ƒ*¯*j* = *ƒ. Thus, SC*(*L*) *is isomorphic to the consistent Smyth powerdomain over L.*

**Definition 3.4** *The relation in a continuous dcpo-∧↑-semilattice D is called consistent -multiplication if for consistent elements a, b ∈ D, x a, b implies x a ∧↑ b.*

For a continuous *L*-domain *L*, *L* is distributive with *∨* and *∧↑* if it holds the following statements: for any consistent points *x, y, z*, *x∧↑* (*y∨z*)= (*x∧↑ y*)*∨*(*x∧↑ z*).

**Definition 3.5** *Let L be a poset. An element m ∈ L is a minimal upper bound (or mub for short) for a subset A if m is an upper bound for A that is minimal in the set of all upper bounds of A.*

**Lemma 3.6** *Let D be an algebraic Lawson compact L-domain with the consistent*

*-multiplicative property. If D is distributive with ∨ and ∧↑, then D is an FS-∧↑- domain.*

**Proof.** Let *D* = *{*(*x ⇒ x*) *∈ D→D* : *x x}*, where (*x ⇒ x*) is the one-step function defined by

(*x ⇒ x*)(*z*)= ⎧⎨ *x, z ∈* ***↑****x,*

⎩ 0*↓z,* otherwise*.*

For any finite compact element subset *K* = *{x*1*, ..., xn} ⊆ K*(*D*) for *n ∈ N* and

*∅ /*= *G ⊆ K*, define *MG ⊆fin mub*(*G*): for any *m ∈ MG*, there is *x ∈ ∩xi∈G↑xi* such that *m x*. By compactness of *D*, there is a finite set *MG ⊆ mub{xi* : *xi ∈ G}* such that *mub{xi* : *xi ∈ G}⊆ m∈MG* ***↑****m*.

Set

*Lc*(*K*)= *{∧↑ {xi}* : *∅ /*= *G ⊆ K, ∩x ∈G↑xi /*= *∅},*

*xi∈G*

*i*

*MLc*(*K*) = *{m ∈ MG* : *∅ /*= *G ⊆ Lc*(*K*)*, ∩xi∈G↑xi /*= *∅},*

*G*

*K∗* = *Lc*(*K*) *∪ ML* (*K*)*,*

*c*

*K*1 = *K∗, K*2 = *K∗, ..., Kn*+1 = *K∗,*

1 *n*

*F*(*K*)= *Kn.*

*n∈N*

By the distributive property, the set *F*(*K*) is finite. Let us define the mapping

*ƒK* : *D→D* as follows: for *x ∈ D*,

*ƒ* (*x*)= ⎧⎨ *m, x ∈* ***↑****m \* ***↑***(*↑m ∩ F*(*K*))*, m ∈ F*(*K*)*,*

*K*

⎩ 0*↓x,* otherwise*.*

If *G*1 = *G*2, *m*1*, m*2 *∈ MG*1 and *m*1 */*= *m*2, then ***↑****m*1 *\* ***↑***(*↑m*1 *∩ F*(*K*)) *∩* ***↑****m*2 *\*

***↑***(*↑m*2 *∩ F*(*K*)) = *∅.* Otherwise, there is *x ∈* ***↑****m*1 *\* ***↑***(*↑m*1 *∩ F*(*K*)) *∩* ***↑****m*2 *\* ***↑***(*↑m*2 *∩ F*(*K*)). But *m*1 */*= *m*2. This is a contradiction with which *D* is an *L*-domain. If *G*1 */*= *G*2 and *mi ∈ MGi* for *i* = 1*,* 2, then ***↑****m*1 *\****↑***(*↑m*1 *∩F*(*K*)) *∩* ***↑****m*2 *\****↑***(*↑m*2 *∩F*(*K*)) = *∅.* Otherwise, there is *x ∈* ***↑****m*1 *\* ***↑***(*↑m*1 *∩ F*(*K*)) *∩* ***↑****m*2 *\* ***↑***(*↑m*2 *∩ F*(*K*)). Then *x ∈*

***↑***(*m*1 *∨↓x m*2), a contradiction. On the other hand, suppose *m*1 *∈ MG*1 *, m*2 *∈ MG*2 and *G*1 */*= *G*2. If *a ∈* ***↑****m*1 *\* ***↑***(*↑m*1 *∩ F*(*K*)) and *b ∈* ***↑****m*2 *\* ***↑***(*↑m*2 *∩ F*(*K*)) and *ƒK*(*a*) */*= *ƒK*(*b*). We must have *a /*= *b*. Otherwise, *a ∈* ***↑***(*m*1*∨↓am*2), but *ƒK*(*a*)= *m*1. This is a contradiction with the definition of *ƒK*. If *G*1 = *G*2, *m*1*, m*2 *∈ MG*1 and *m*1 */*= *m*2, then *a /*= *b* because *D* is a *L*-domain. Then *ƒK* is well defined.

It’s obvious that *ƒK* is monotone with finite range and *ƒK ≤ idD*. For any

*d ∈ D*, if *d* = 0*↓d* and *d /∈* ***↓****m* for any *m ∈ F*(*K*), then *ƒ−*1(***↑****d*) = ***↑****d*; if there is

*K*

*−*1 *↑ ↑*

some *m ∈ F*(*K*) such that *d m*, then *ƒK* (*↑d*)=

*{↑m* : *d m}* and otherwise,

*ƒ−*1(***↑****d*) = *∅*. All cases show that *ƒ−*1(***↑****d*) is a Scott open set of *D*. Then *ƒK* is

*K K*

Scott continuous.

It is easy to show that *{ƒK* : *K ⊆ K*(*D*)*, |K| ⊆fin N}* is a directed set approxi- mated to *idD*. For any *x ∈ D*, we know *sup{ƒK*(*x*): *K ⊆ K*(*D*)*, |K| ⊆fin N} ≤ x*. If *x /≤ sup{ƒK*(*x*) : *K ⊆ K*(*D*)*, |K| ⊆fin N}*, then there is *u x* but *u /≤ sup{ƒK*(*x*) : *K ⊆ K*(*D*)*, |K| ⊆fin N}*. By *u x*, there is some compact element *v* such that *u v x*. By (*v⇒v*)(*x*) = *v* and (*v⇒v*) *≤ sup{ƒK* : *K ⊆*

*K*(*D*)*, |K| ⊆fin N}*, then *u ≤ v ≤ sup{ƒK*(*x*): *K ⊆ K*(*D*)*, |K| ⊆fin N}*, a contra- diction.

To show that *{ƒK* : *K ⊆ K*(*D*)*, |K| ⊆fin N}* is the approximate identity over *D*, it is sufficient to prove that these functions are also *∧↑*-homomorphisms. Suppose that *a* and *b* are consistent witnessed by *c*. Let *a ∧↑ b* = *x* and let

*a ∈* ***↑****m*1 *\* ***↑***(*↑m*1 *∩ F*(*K*))*,*

and

*b ∈* ***↑****m*2 *\* ***↑***(*↑m*2 *∩ F*(*K*))

*x ∈* ***↑****m*0 *\* ***↑***(*↑m*0 *∩ F*(*K*))*,*

where *mi ∈ MG* for *i* = 0*,* 1*,* 2*.* Then *ƒK*(*a*) *∧↑ ƒK*(*b*) exists. By *m*0 *x ≤ a* and *m*0 *x ≤ b*, if *G*0 = *G*1 = *G*2, then *m*1 = *m*0 and *m*2 = *m*0, and then *m*0 = *m*1 *∧↑ m*2. Otherwise, we obtain *m*0 *∨↓a m*1 *a* and *m*0 *∨↓a m*1 *∈ F*(*K*), but *a ∈* ***↑****m*1 *\* ***↑***(*↑m*1 *∩ F*(*K*)). Then *m*0 *≤ m*1. Similarly, *m*0 *≤ m*2. Thus, we have *m*0 *≤ m*1 *∧↑ m*2. On the other hand, by-multiplicative property, *m*1 *∧↑ m*2 *a ∧↑ b* = *x.* By the definition of *ƒK*, we conclude *m*0 = *m*1 *∧↑ m*2. Hence, *ƒK*(*a ∧↑ b*)= *m*0 = *m*1 *∧↑ m*2 = *ƒK*(*a*) *∧↑ ƒK*(*b*). *2*

*i*

**Theorem 3.7** *Let D be a Lawson compact L-domain with consistent - multiplicative property. If D is distributive with ∨ and ∧↑, then D is an FS-∧↑- domain.*

**Proof.** For any finite subset *X* = *{x*1*, ..., xn} ⊆ D*, *Y* = *{y*1*, ..., yn} ⊆ D* and

*yi xi* for any *n ∈ N* and *I* = *{*1*,* *, n}*, set

*Lc*(*X*)= *{∧↑*

*i∈F*

*{xi}* : *F ∈* Φ(*I*)= Φ(Ψ*X* )*},*

*Uc*(*X*)= *{∨↓x{xi* : *i ∈ F}* : *F ∈* Φ(Ψ*LC* (*X*))*, ∃x ∈ D, s.t., {xi* : *i ∈ F}⊆ ↓x}, X∗* = *Lc*(*X*) *∪ Uc*(*X*)*,*

*X*1 = *X∗, X*2 = *X∗, ..., Xn*+1 = *X∗,*

1 *n*

*F*(*X*)= *Xn,*

*n∈N*

where Φ(*I*)= *{F ⊆ I* : *∩i∈F* ***↑****xi /*= *∅},* Φ*k*(*I*)= *{F ∈* Φ(*I*): *|F |* = *k}, MI* = max*{i ∈ I* : *∃F ∈* Φ(*I*)*, s.t., |F |* = *i},*

and let Ψ*X* = *I,* and

Ψ*L* (*X*) = *{i↓*(*i* + *k*(2))*, ..., i↓*(*i* + *k*(2))*, ..., i↓*(*i* + *k*(*e*+1))*↓...↓*(*i* + *k*(*e*+1)):1 *≤ i ≤ n},*

*c* 1 *t s*1 *se*

where

1

*t*

*k*(2) = min*{k* : *↑xi ∩ ↑xi*+*k /*= *∅,* 0 *≤ k ≤ n − i}, k*(2) = max*{k* : *↑xi ∩ ↑xi*+*k /*= *∅,* 0 *≤ k ≤ n − i}*

and *i↓*(*i* + *k*(2)) means that *xi ∧↑ x* exists, and

1

*i*+*k*(2)

1

*e* = max*{|E|* : *↑xi ∩* ( *↑xi*+*k*) */*= *∅,* 0 *≤ k ≤ n − i},*

*k∈E*

*{*(*i* + *k*(*e*+1))*, ...,* (*i* + *k*(*e*+1))*}* = max*{E* : *↑xi ∩* ( *↑xi*+*k*) */*= *∅,* 0 *≤ k ≤ n − i},*

*s*1 *se*

*k∈E*

*i↓*(*i* + *k*(*e*+1))*↓...↓*(*i* + *k*(*e*+1)) denotes that *∧↑{x ,x*

*, ..., x*

*}* exists. From

*s*1 *se*

*i i*+*k*(*e*+1)

*i*+*k*(*e*+1)

the definition of Ψ, we know that *|*Ψ*Lc*(*X*)*| ≤* (*Lc*(*X*))!. Then Ψ*Lc*(*X*) is finite. Similarly, for some *Xk*, the set Ψ*Xk* is finite.

*s*1

*se*

Let us define *ME ⊆fin mub{yi* : *i ∈ E}* for a finite set *E*: for any *m ∈ ME*, there is *x ∈ ∩i∈E↑xi* such that *m x*. By compactness of *D*, there is a finite set *ME ⊆ mub{yi* : *i ∈ E}* such that *mub{xi* : *i ∈ E}⊆ m∈ME* ***↑****m*.

Let *ma* = *∧↑ {yi}*, if *a* = *∧↑ {xi}* for *F ∈* Φ(Ψ*X* ). Let *ma* = *∨↓x{yi* : *i ∈*

*i∈F*

*i∈F*

1

*k*

*F}*, if there is some *x ∈ D* such that *{xi* : *i ∈ F} ⊆ ↓x* for *F ∈* Φ(Ψ*Xk*

2

*a* = *∨↓x{xi* : *i ∈ F}*.

) and

By the distributive property and compactness of *D*, the set *{ma* : *a ∈ F*(*X*)*}* is finite. Let us define a mapping *ƒI* : *D→D* as follows: for *x ∈ D*,

*ƒ* (*x*)= ⎧⎨ *ma, x ∈* (***↑****ma ∩* ***↑****a*) *\* ***↑***(*↑a ∩ F*(*X*))*, a ∈ F*(*X*)*,*

*I*

⎩ 0*↓x,* otherwise*.*

Then *{ƒI* : *I ⊆fin N}* is an approximate identity over *D*. *2*

In [[16](#_bookmark18)], we show that the consistent Smyth powerdomain over a Lawson compact continuous domain is a Lawson compact continuous *L*-domain.

**Theorem 3.8** *[*[*16*](#_bookmark18)*] If L is a Lawson compact continuous domain, then the consis- tent Smyth powerdomain SC*(*L*) *is a Lawson compact continuous L-domain satisﬁed the consistent -multiplicative property and the distributive property with ∨ and*

*∧↑. 2*

By Theorem [3.7](#_bookmark1) and Theorem [3.8](#_bookmark2), we have the following conclusion:

**Theorem 3.9** *If L is a Lawson compact continuous domain, then the consistent* *Smyth powerdomain SC*(*L*) *over L is an FS-∧↑-domain.*

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