Electronic Notes in Theoretical Computer Science 167 (2007) 3–15 

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Computability and the Implicit Function Theorem

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Abstract

We prove computable versions of the Implicit Function Theorem in the single and multivariable cases. We use Type Two Effectivity as our foundation.

*Keywords:* Computable Analysis, Computability.

# Introduction

The Implicit Function Theorem guarantees, under certain conditions, the ex- istence of unique local continuous functional solutions to equations of the form

* 1. *φ*(*x, y*)= 0

given an initial condition of the form

* 1. *y*(*a*)= *b.*

Under a surprisingly weak assumption, namely the differentiability of *φ*, the differentiability of the solution is also guaranteed. A very simple application of this, encountered by most single-variable Calculus students when they learn to use implicit differentiation to calculate tangent lines to curves (although they are not aware of the statement of the Implicit Function Theorem as it requires notation from multivariable calculus), is given by the equation

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doi:10.1016/j.entcs.2006.08.004

* 1. *x*2 + *y*2 =1

and the initial condition

1 1

* 1. *y*( √2 )= √2 *.*

Once granted the assumption that there *is* a differentiable functional solution

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to ([3](#_bookmark1)) that satisfies ([4](#_bookmark1)) on an open interval containing √2 , one can verify

through direct calculation that the derivative of the solution satisfies

*y*' = *x. y*

—

Then, using the initial condition, we can determine that *y*'( 1 )= −1.

√2

The Implicit Function Theorem also has important applications to dif- ferential equations, numerical analysis, and geometric analysis. A thorough discussion of the Implicit Function Theorem, its many variations, and its ap- plications may be found in [[2](#_bookmark10)].

Here we state and prove a computable version of the Implicit Function The- orem in its single variable case, which is what we have just broadly described, and in its multivariable case. We use Type Two Effectivity theory as devel- oped in Weihrauch [[5](#_bookmark11)] as our foundation. The reasoning for the multivariable case builds on that for the single-variable case. Hence, even though the multi- variable case implies the single-variable case, we present both arguments. Our goal is to show that, in general terms, if *φ*, *a*, and *b* are computable, then the unique continuous functional solution to ([1](#_bookmark1)) that satisfies ([2](#_bookmark1)) is computable. In addition, we show that if *φ* has a computable derivative, then this solution has a computable derivative. We also prove uniform versions of these results. Unless otherwise mentioned, all computability notation is as in Soare [[4](#_bookmark12)]. Unless otherwise mentioned, all computable analysis notation is as in

Weihrauch [[5](#_bookmark11)].

We define a few notations and helpful conventions. First, we write *f* : *A B* if *dom*(*f* ) *A* and *ran*(*f* ) *B*. If *f* is a function and *X* is a set, then

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⊆

*f* [*X*] =*df* {*y* | ∃*x* ∈ *dom*(*f* ) ∩ *X f* (*x*)= *y*}*.*

We note that *X* is not required to be a subset of the domain of *f* . Unless otherwise mentioned, the following conventions are followed for sake of brevity.

* + 1. A *computable real number* is a *ρ*-computable real number.
    2. A point *a* ∈ R*n* is *computable* if and only if it is *ρn*-computable.
    3. A function *φ* :⊆ R*n* → R*m* is *computable* if it is (*ρn, ρm*)-computable.
    4. A finite interval is *computable* if its endpoints are computable. An interval of the form (−∞*, a*), (−∞*, a*], (*a,* ∞), or [*a,* ∞) is *computable* if *a* is

computable.

* + 1. *C*(*U* ) is the set of all continuous functions from *U* into *U* .
    2. A function *F* :⊆ *C*(*U* ) → R is *computable* if it is (*δU , ρ*)-computable. That is, if there is a computable function *H* :⊆ Σ*ω* → Σ*ω* such that

*co*

*F* ◦ *δU* = *ρ* ◦ *H*.

*co*

* + 1. A function *F* :⊆ *C*(*U* ) × R → R is *computable* if it is (*δU , ρ, ρ*)- computable. That is, if there is a computable function *H* :⊆ Σ*ω* → Σ*ω* such that *F* ◦ [*δU , ρ*]= *ρ* ◦ *H*.

*co*

*co*

* + 1. A function *F* :⊆ R → *C*(*U* ) is *computable* if it is (*ρ, δU* )-computable.

*co*

# Single-variable case

Theorem 2.1 (Single-Variable Implicit Function Theorem) *Suppose*

*E* ⊆ R2 *is open. Suppose φ* : *E* → R *and ∂φ are continuous. Suppose*

*∂y*

*φ*(*a, b*)= 0 *and ∂φ*(*a, b*) /= 0*. Then, there exist open intervals U, V with a* ∈ *U and b* ∈ *V such that there exists a unique f* : *U* → *V such that φ*(*x, f* (*x*)) = 0 *for all x* ∈ *U and f* (*a*)= *b. Furthermore, f is continuous.*

*∂y*

We prove the following computable version of this theorem.

Theorem 2.2 (Computable Single-Variable Implicit Function Theo- rem) *Suppose E* ⊆ R2 *is open. Suppose φ* : *E* → R *is computable and ∂φ is continuous. Suppose a, b* R *are computable, φ*(*a, b*) = 0*, and ∂φ*(*a, b*) *is a non-zero computable number. Then, there exist computable open intervals*

*∂y*

*∂y*

∈

*U, V* R *with a U and b V such that there exists a unique f* : *U V such that φ*(*x, f* (*x*)) = 0 *for all x U and f* (*a*) = *b. Furthermore, f is computable.*

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We give two proofs of Theorem [2.2](#_bookmark3). The first is a simple, non-uniform proof.

First proof of Theorem [2.2](#_bookmark3) Let *U* , *V* , and *f* be as given by Theorem [2.1](#_bookmark2). Let *V* ' be an open interval such that *V* ' *V* , *b V* ', and the endpoints of *V* ' are rational. By the continuity of *f* , there is an open interval *U* ' such that *U* ' *U* , *a U* ', the endpoints of *U* ' are rational, and *f* [*U* '] *V* '. Let *g* be the restriction of *f* to *U* '. It follows that for each *x U* ', *g*(*x*) is the unique number in *V* ' such that *φ*(*x, g*(*x*)) = 0. It now follows from Corollary 6.3.5 of

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[[5](#_bookmark11)] that *g* is computable.

Our second proof of Theorem [2.2](#_bookmark3) is uniform. It uses a computable version of the Contraction Mapping Theorem which we state and prove below. We

will need the following definition, which is essentially that given in [[1](#_bookmark13)]. Here, and throughout this paper, *d* denotes the Euclidean distance function.

Definition 2.3 Suppose *f* :⊆ R*n* → R*m* and *k* ∈ R. We say that *k* is a *contraction constant for f* if 0 *< k <* 1 and *d*(*f* (*x*)*,f* (*y*)) ≤ *kd*(*x, y*) for all *x, y* ∈ *dom*(*f* ).

The following is well-known.

Theorem 2.4 (Contraction Mapping Theorem) *If U is a closed interval and f* ∈ *C*(*U* ) *has a contraction constant, then f has a unique ﬁxed point.*

Theorem 2.5 (Uniformly Computable Contraction Mapping Theo- rem) *Suppose U* R *is a closed interval. There is a computable* Ψ : *C*(*U* ) (0*,* 1) *U such that for all* (*f, k*) *C*(*U* ) (0*,* 1)*, if k is a con- traction constant for f, then* (*f, k*) *dom*(Ψ) *and* Ψ(*f, k*) *is a ﬁxed point for f.*

∈

× → ∈ ×

⊆ ⊆

Proof. Fix a rational number *p*0 ∈ *U* . For all *f* ∈ *C*(*U* ) and *k* ∈ (0*,* 1), let

Ψ(*f, k*) = lim

*m*→∞

*fm*(*p*0)*.*

If *k* is a contraction constant for *f* , then Ψ(*f, k*) is defined and is a fixed point for *f* . It remains to show that Ψ is computable.

For all *f* ∈ *C*(*U* ), let

*S*(*f* )= (*p*0*,f* (*p*0)*,f* 2(*p*0)*,.. .*)*.*

It follows from Theorem 3.1.7.2 of [[5](#_bookmark11)] that *S* is (*δU ,* [*ρ*]*ω*) computable. For all

*co*

*f* ∈ *C*(*U* ), *k* ∈ (0*,* 1), and *n* ∈ *N*, let *e*(*f, k, n*) be the least number *m* such that

|*p* − *f* (*p* )|*k*  1

*m*

0 0

≤ *.*

1 − *k* 2*n*

It follows that *e* is (*δU , ρ, ν*N*, νN* )-computable.

*co*

Suppose *f* ∈ *C*(*U* ), and suppose *k* ∈ (0*,* 1) is a contraction constant for *f* .

For all *m*, let *pm* = *fm*(*p*0). It follows that for all *m* ∈ N

|*pm* − *pm*+1|≤ *km*|*p*0 − *p*1|*.*

It now follows that when *n > m*,

|*pm* − *pn*|≤ |*pm* − *pm*+1| + |*pm*+1 − *pm*+2| + *...* + |*pn*−1 − *pn*|

≤ *km*|*p*0 − *p*1| + *km*+1|*p*0 − *p*1| + *...* + *kn*−1|*p*0 − *p*1|

*km*

≤ |*p*0 − *p*1| 1 *k.*

—

It now follows from Theorem 4.3.7 of [[5](#_bookmark11)] that Ψ is computable.

Second proof of Theorem [2.2](#_bookmark3) We follow the proof in [[3](#_bookmark14)]. Define

*φ*(*x, y*) *F* (*x, y*)= *y* − *∂φ* (*a, b*) *.*

*∂y*

Thus, *F* : *E* → R2 is computable, *∂F* (*a, b*) = 0, and *φ*(*x, y*) = 0 if and only if

*∂y*

*F* (*x, y*)= *y*. Also, *∂F* is continuous.

*∂y*

There is a rational *r >* 0 such that *∂F <* 1 on the open disk in R2

*∂y*

2

centered at (*a, b*) and with radius *r*. Fix√a rational number *k* in (0*, r*). Fix

a rational number *h* such that 0 *< h < r*2 − *k*2 and |*F* (*x, b*) − *b*| *< k/*2 if

|*x* − *a*| *< h*. Define *U* to be (*a* − *h, a* + *h*) and *V* to be (*b* − *k, b* + *k*).

Fix *x* ∈ *U* . We first note that if |*y* − *b*| ≤ *k*, then *d*((*x, y*)*,* (*a, b*)) *< r*. The key claim is that if *y, y*' ∈ *V* , then |*F* (*x, y*) − *F* (*x, y*')| ≤ 1 |*y* − *y*'|. For, suppose *y, y*' are distinct elements of *V* . By the Mean Value Theorem, there is a number *y*'' between *y* and *y*' such that

2

*F* (*x, y*) *F* (*x, y*')= *∂F* (*x, y*'')(*y y*')*.*

− −

*∂y*

The claim then follows from the previously imposed bound on *∂F* . We note that we do not need a computable version of the Mean Value Theorem to establish this claim.

*∂y*

We now note that *F* (*x,* ·): *V* → *V* is a contraction map with contraction

constant 1. By Lemma 6.1.7 of [[5](#_bookmark11)], the representations *δU* and [*ρ* → *ρ*]*U* are

2

*co*

computably equivalent. Since *F* is computable, it follows from Theorem 2.3.13

of [[5](#_bookmark11)] that the map *x* '→ *F* (*x,* ·) is computable. Let Ψ be as in Theorem [2.5](#_bookmark5). Define *f* (*x*)= Ψ(*F* (*x,* ·)*,* 1). It follows from Theorem [2.5](#_bookmark5) that *f* is computable. Hence, *f* is continuous. The uniqueness of *f* follows from the uniqueness clause of Theorem [2.4](#_bookmark4).

2

Theorem 2.6 *In Theorem* [*2.2*](#_bookmark3)*, if φ is differentiable, and if φ*' *is computable, then U, V can be chosen so that f* ' *is computable.*

Proof. In the proof of Theorem [2.2](#_bookmark3), choose *r* so that *∂φ >* 0 on the open disk in R2 with center (*a, b*) and radius *r*. Let *B* be this disk.

*∂y*

Let *x*0*,x* be distinct elements of *U* . Then, (*x*0*,f* (*x*0))*,* (*x, f* (*x*)) *U V*

∈ × ⊆

*B*. By the multivariable version of the Mean Value Theorem, there is a point

*z* on the line segment between (*x*0*,f* (*x*0)) and (*x, f* (*x*)) such that

*φ*(*x*0*,f* (*x*0)) − *φ*(*x, f* (*x*)) = *φ*'(*z*) · (*x*0 − *x, f* (*x*0) − *f* (*x*))*.*

(Again, we are not using, nor do we need, a computable version of this theo- rem.) Since *φ*(*x*0*,f* (*x*0)) = *φ*(*x, f* (*x*)) = 0, it follows that

− *∂x* (*z*) = *f* (*x*0) − *f* (*x*) *.*

*∂φ*

*∂φ*(*z*)

*∂y*

*x*0 − *x*

As *x* approaches *x*0, *z* approaches (*x*0*,f* (*x*0)). Since *φ*' is computable, *∂φ* and

*∂x*

*∂y*

*∂φ* are computable. It then follows that *∂φ*

*∂x*

*∂y*

and *∂φ*

are continuous. It then

follows that

*f* '(*x* )= − *∂x* (*x*0*,f* (*x*0)) *.*

0 *∂φ*(*x ,f* (*x* ))

*∂φ*

*∂y* 0 0

Since *∂φ*, *∂φ*, and *f* are computable, it follows that *f* ' is computable.

*∂x ∂y*

We now state uniform versions of these results. We will need the following definitions.

Definition 2.7 Let *n, m* ≥ 1.

1. *Cn,m* is the set of all functions *φ* : R*n* R*m* such that *dom*(*φ*) is open and *φ* is continuous.

⊆ →

1. *C*1 is the set of all functions *φ* :⊆ R*n* → R*m* such that *dom*(*φ*) is open

*n,m*

and *φ*' is continuous.

The following is a straightforward modification of the naming system for

*C*1(R*n*) in [[6](#_bookmark15)].

Definition 2.8 Let *n, m* ≥ 1. Let *a, b, r*1*, r*2 ∈ Q, and suppose *r*1*, r*2 *>* 0.

∈ ⊆

1. *Ra,r*1*,b,r*2 is the set of all functions *φ Cn,m* such that *φ*[*B*(*a, r*1)] *B*(*b, r*2).
2. Fix *i, j* such that 1 ≤ *i, j* ≤ *n*. We define *Ri,j* to be the set of all

*a,r*1*,b,r*2

*∂φj*

functions *φ* ∈ *C*1 such that [*B*(*a, r*1)] ⊆ *B*(*b, r*2).

*n,m*

*∂xi*

1. We define *σn,m* to be {*Ra,r*1 *,b,r*2 | *a, b, r*1*, r*2 ∈ Q ∧ *r*1*, r*2 *>* 0}.
2. We define *σ*1 to be

*n,m*

{*Ra,r*1*,b,r*2 ∩ *Cn,m* | *a, b, r*1*, r*2 ∈ Q ∧ *r*1*, r*2 *>* 0}

1

∪ {*Ra,r*1 *,b,r*2 | *a, b, r*1*, r*2 ∈ Q ∧ *i, j* ∈ {1*,..., n*} ∧ *r*1*, r*2 *>* 0}*.*

*i,j*

1. Let *νn,m* be a standard notation of *σn,m*.
2. Let *ν*1 be a standard notation of *σ*1 .

*n,m*

*n,m*

1. *δn,m* is the representation of *Cn,m* given by *νn,m*.
2. *δ*1

*n,m*

is the representation of *C*1

given by *ν*1 .

Thus, the *δ*1 name of a function *φ* ∈ *C*1 provides the information to

*n,m*

*n,m*

*n,m*

*n,m*

compute *φ* as well as *φ*'.

Theorem 2.9 (Non-Differentiable Uniformly Computable Single- Variable Implicit Function Theorem) *There is a* (*δ*2*,*1*, δ*2*,*1*, ρ*2*, ρ*2*, δ*1*,*1)*- computable function* Ψ :⊆ *C*2*,*1 × *C*2*,*1 × R2 → R2 × *C*1*,*1 *such that if φ* ∈ *C*2*,*1*,*

*φ*(*a, b*) = 0*, ∂φ is continuous, and ∂φ*(*a, b*) /= 0*, then* (*φ, ∂φ, a, b*) ∈ *dom*(Ψ)*.*

*∂y*

*∂y*

*∂y*

*Furthermore, if* (*r*1*, r*2*,f* )= Ψ(*φ, ∂φ, a, b*)*, then f is the unique function such that f* : (*a*−*r*1*, r*+*r*1) → (*b*−*r*2*, b*+*r*2)*, φ*(*x, f* (*x*)) = 0 *for all x* ∈ (*a*−*r*1*, a*+*r*1)*,*

*∂y*

*f* (*a*)= *b*

*and .*

Proof sketch Most of the work has been done in the proof of Theorem [2.2](#_bookmark3). In the proof of Theorem [2.2](#_bookmark3), use the information provided by the *δ*2*,*1 name of

*∂φ* to find *r, h, k*. In Theorem [2.5](#_bookmark5), the function Ψ can be obtained uniformly from the interval *U* . The proof of Theorem [2.6](#_bookmark6) shows that we can compute a *δ*1*,*1 name of *f* from a *δ*2*,*1 name of *φ*.

*∂y*

Theorem 2.10 (Differentiable Uniformly Computable Single-

Variable Implicit Function Theorem) *There is a* (*δ*1 *, ρ*2*, ρ*2*, δ*1 )*-*

2*,*1

1*,*1

2 2 1

*computable function* Ψ :⊆ *C*1 × R → R × *C*1*,*1 *such that if φ* ∈ *C*1 *,*

2*,*1

2*,*1

*φ*(*a, b*) = 0*, and ∂φ*(*a, b*) /= 0*, then* (*φ, a, b*) ∈ *dom*(Ψ)*. Further-*

*∂y*

*more, if* (*r*1*, r*2*,f* ) = Ψ(*φ, a, b*)*, then f is the unique function such that*

*f* : (*a* − *r*1*,r* + *r*1) → (*b* − *r*2*,b* + *r*2)*, φ*(*x, f* (*x*)) = 0 *for all x* ∈ (*a* − *r*1*,a* + *r*1)*, f* (*a*)= *b*

*and .*

Proof sketch Most of the work has already been done. The only addition is

that the proof of Theorem [2.6](#_bookmark6) shows we can compute a *δ*1

1*,*1

name of *f* from a

1

*δ*

1*,*1

name of *φ* once we have first computed a *δ*1*,*1 name of *f* .

# The multivariable case

If *A* is a square matrix, then *det*(*A*) denotes the determinant of *A*.

Theorem 3.1 (Multivariable Implicit Function Theorem) *Let a* ∈ R*m, and let b* ∈ R *. Let E* ⊆ R *be an open set that contains* (*a, b*)*. Let φ* : *E* → R *be continuous. Suppose the following hold.*

*n*

*n m*+*n*

* *φ*(*a, b*)= *→*0*.*
* *For all i, j* ∈ {1*,... , n*}*,*  *∂φi*

*∂xm*+*j*

*is continuous on E.*

* *det* *∂φi* (*a, b*)

*∂xm*+*j*

*i,j*=1*,...,n*

/= 0*.*

*Then, there exist open U* ⊆ R*m and V* ⊆ R*n such that a* ∈ *U, b* ∈ *V , and there is a unique continuous function f* : *U* → *V such that φ*(*x, f* (*x*)) = *→*0 *for* *all x* ∈ *U.*

Definition 3.2 An open or closed ball *B* R*n* is *computable* if its center and radius are computable.

⊆

When *B* is an open or closed ball in R*n*, let *C*(*B*) denote the set of all continuous functions from *B* into *B*. A function Ψ : *C*(*B*) (0*,* 1) *B* is *computable* if it is (*δB , ρ, ρn*)-computable.

*co*

⊆ × →

Theorem 3.3 (Uniformly Computable Multivariable Contraction Mapping Theorem) *Let B be a closed ball in* R*n. There is a computable* Ψ : *C*(*B*) (0*,* 1) *B such that for all* (*f, k*) *C*(*B*) (0*,* 1)*, if k is a contraction constant for f, then* Ψ(*f, k*) *is a ﬁxed point for f.*

⊆ × → ∈ ×

The proof is basically identical to the proof of Theorem [2.5](#_bookmark5).

Theorem 3.4 (Computable Multivariable Implicit Function Theo- rem) *Let a* R*m, and let b* R*n be computable. Let E* R*m*+*n be an open set that contains* (*a, b*)*. Let φ* : *E* R*n be computable. Suppose the following hold.*

→

∈ ∈ ⊆

* *φ*(*a, b*)= *→*0*.*
* *For all i, j* ∈ {1*,... , n*}*,*  *∂φi*

*∂xm*+*j*

*is continuous on E.*

* *det* *∂φi* (*a, b*)

*∂xm*+*j*

{1*,... , n*}*.*

*i,j*=1*,...,n*

*∂φi*

/= 0*, and ∂xm*+*j* (*a, b*) *is computable for all i, j* ∈

*Then, there exist computable open U* ⊆ R*m and V* ⊆ R*n such that a* ∈ *U, b* ∈ *V , and there is a unique function f* : *U* → *V such that φ*(*x, f* (*x*)) = *→*0 *for all x* ∈ *U and f* (*a*)= *b. Furthermore, f is computable.*

As with Theorem [2.2](#_bookmark3), we can give a simple non-uniform proof of Theorem

[3.4](#_bookmark8). We show later that the proof below is uniform.

Proof. Let *yj* denote *xm*+*j*. Let

*D* = *∂φi* (*a, b*) *.*

*∂yj*

*i,j*=1*,...,n*

Let (*ci,j*)*i,j*=1*,...,n* = *C* = *D*−1. For each *i* ∈ {*i,... , n*}, let

*n*

Σ

*Fi*(*x, y*)= *yi* − *ci,kφk*(*x, y*)*.*

*k*=1

Let *F* = (*F*1*,... , Fn*). It follows that *F* is continuously differentiable on *E*. Since the entries in *D* are computable, it follows from Proposition 6 of [[7](#_bookmark16)] that *C* is computable. Hence, *F* is computable. Also, *φ*(*x, y*) = *→*0 if and only if *F* (*x, y*)= *y*. By direct calculation, we have:

*∂Fi* (*a, b*)= 1 − Σ*n c d*

*∂yi*

*∂Fi* (*a, b*)= −Σ*n*

*k*=1

*c*

*i,k*

*d*

*k,i*

if *i* /= *j*

*∂yj*

*k*=1

*i,k*

*k,j*

Since *CD* = *In*, it follows that *∂Fi* (*a, b*) = 0 for all *i, j* ∈ {1*,..., n*}.

*∂yj*

For each *r >* 0, let *Br* be the open ball with center (*a, b*) and radius *r*.

Choose a rational number *r >* 0 so that *Br* ⊆ *E* and the following hold.

* For all *i, j* ∈ {1*,..., n*}, *∂Fi* *<*  1 on *Br*.

*∂yj*

2*n*2

* *det* *∂φi* is non-zero on *Br*.

*∂yj*

*i,j*=1*,...,n*

Now, choose a rational number *k* such th√at 0 *< k < r*. Finally, choose

a rational number *h* such that 0 *< h < r*2 − *k*2 and *d*(*F* (*x, b*)*, b*) *< k/*2

whenever *x Rn* and *d*(*x, a*) *< h*. Let *U* be the open ball in R*m* with center

∈

*a* and radius *h*. Let *V* be the open ball in R*n* with center *b* and radius *k*.

Clearly, *U, V* are computable. If (*x, y*) ∈ *U* × *V* , then

*d*((*x, y*)*,* (*a, b*))2 = *d*(*x, a*)2 + *d*(*y, b*)2

≤ *h* + *r* − *h* = *r .*

2 2 2 2

Hence, *U* × *V* ⊆ *Br*.

We now claim that *d*(*F* (*x, y*)*,F* (*x, y*')) ≤ 1 *d*(*y, y*') whenever (*x, y*)*,* (*x, y*') ∈ *U* × *V* . For, let (*x, y*)*,* (*x, y*) ∈ *U* × *V* . Without loss of generality, suppose *y* /= *y*'. Fix *i* ∈ {1*,..., n*}. By the multivariable version of the Mean Value Theorem, there is a point *y*0 on the line segment between *y* and *y* such that

2

'

*F* (*x, y*) − *F* (*x, y*')= *∂Fi* (*x, y* )(*y* − *y*' )+ *...* + *∂Fi* (*x, y* )(*y*

— *y*' )*.*

0 1

*i i ∂y*1 1

*∂yn*

0 *n n*

Hence,

|*F* (*x, y*) − *F* (*x, y*')|≤ *∂Fi* (*x, y* ) |*y* − *y*' | + *...* + *∂Fi* (*x, y* ) |*y*

— *y*' |

0 1

*i i*  *∂y*1 1

*∂yn*

0 *n n*

≤ 1 (|*y* − *y*' | + *...* + |*y*

1

2*n*2

1

— *y*' |)

≤ *n d*(*y, y*')= 1 *d*(*y, y*')*.*

*n*

*n*

2*n*2

2*n*

It now follows that *d*(*F* (*x, y*)*,F* (*x, y*')) ≤ 1 *d*(*y, y*'). In addition, by taking

2

*y*' = *b*, it also follows that

*d*(*F* (*x, y*)*, b*) ≤ *d*(*F* (*x, y*)*,F* (*x, b*)) + *d*(*F* (*x, b*)*, b*)

1 *k*

*< d*(*y, b*)+ 

2 2

 ≤ *k.*

Hence, for each *x* ∈ *U* , *F* (*x,* ·): *V* → *V* , and *F* (*x,* ·) has a contraction constant

of 1. Let Ψ be as in Theorem [3.3](#_bookmark7). Let *f* (*x*)= Ψ(*F* (*x,* ·)*,* 1). It follows that *f*

2

2

is computable and *φ*(*x, f* (*x*)) = 0 for all *x* ∈ *U* .

We now discuss differentiability.

Theorem 3.5 *If in Theorem* [*3.4*](#_bookmark8) *we assume φ is differentiable on E and that φ*' *is computable, then we may conclude that f is differentiable and f* ' *is computable.*

Proof. Let *r*, *U* , *V* , *etc.* be as in the proof of Theorem [3.4](#_bookmark8).

There are two parts to this proof. The first is to show that *f* is differen- tiable. The second is to show that *f* ' is computable. The first part is not the main concern here and in any case is well-established. A thorough proof may be found in [[3](#_bookmark14)]. So, we give the second part only.

To show that *f* ' is computable, it suffices to show that *∂fi*

*∂x*

*j*

is computable

for all *i* ∈ {1*,... , n*} and *j* ∈ {1*,..., m*}. Fix *j* ∈ {1*,... , m*}. For each *x* ∈ *U*

let

*g*(*x*)= (*x, f* (*x*))*.*

Now, fix *i* ∈ {1*,..., n*}. We note that *φi* ◦ *g* = 0. If we apply the chain rule to *φi* ◦ *g* for the purpose of calculating its partial derivative with respect to *xj*, we obtain

*∂*(*φi* ◦ *g*) = (*φ*' ◦ *g*) · *∂g .*

*∂xj*

*i ∂xj*

(We note that we are not applying, nor do we need, a computable version of the multivariable chain rule.) We have,

*∂g*

= 0*,... ,* 1*,... ,* 0*, ∂f*1 *,... , ∂fn*

*∂xj*

*∂xj*

*∂xj*

where the 1 appears in the *j*-th position. We also have

*φ*' = *∂φi ,... ,*  *∂φi , ∂φi ,... , ∂φi* *.*

*i ∂x*1

*∂xm*

*∂y*1

*∂yn*

By combining these results, we obtain

0= *∂φi* ◦ *g* = *∂φi* ◦ *g* + *∂φi* ◦ *g* *∂f*1 + *...* *∂φi* ◦ *g* *∂f*1 *.*

*∂xj*

*∂xj*

*∂y*1

*∂xj*

*∂yn*

*∂xj*

We now allow *i* to vary, but keep *j* fixed. For each *x* ∈ *U* , let

*A* (*x*)= *∂φi* (*g*(*x*)) *,*

*j*

*∂yk*

*i,k*=1*,...,n*

and let

⎛ − *∂φ*1 (*g*(*x*)) ⎞

*Bj*(*x*)= ⎜

⎜

⎜

*∂xj*

*.*

*.* ⎟ *.*

⎟

*.* ⎟

⎝ − *∂φn* (*g*(*x*)) ⎠

*∂xj*

It now follows that for each *x* ∈ *U* , *∂f* (*x*)= *∂f*1 (*x*)*,... , ∂fn* (*x*) is a solution

*∂xj*

*∂xj*

*∂xj*

to the system of linear equations in the *n* variables *ω*1*,... , ωn* given by

⎛ *ω*1 ⎞

*.*

⎜

*Aj*(*x*) · *.*

⎜

⎜ *.*

⎟ = *Bj*(*x*)*.*

⎟

⎟

⎝ *ωn* ⎠

However, by the choice of *r*, *det*(*Aj*(*x*)) /= 0 for all *x* ∈ *U* . Hence, *∂f* (*x*) is

*∂xj*

the unique solution to this system. It now follows from Proposition 6 of [[7](#_bookmark16)]

that *∂f*

*∂x*

*j*

is computable for each *j* ∈ {1*,..., m*}. Hence, *f* ' is computable.

We now discuss uniformity.

Theorem 3.6 (Non-Differentiable Uniformly Computable Multivariable Implicit Function Theorem) *There is a*

*m,n*

(*δm*+*n,n*

*,* (*δm*+*n,n*

)*n*2 *, ρm, ρn, ρ*2*,δ*

)*-computable* Ψ :⊆ *C*

*m*+*n,n*

× (*C*

*m*+*n,n*

)*n*2 ×

R*m* × R*n* → R × R × *Cm,n such that if φ* ∈ *Cm*+*n,n, a* ∈ R*m,*

*b* ∈ R*n, φ*(*a, b*) = 0*, ∂φi is continuous for all i, j* ∈ {1*,... , n*}*, and*

*∂xm*+*j*

*det* *∂φi* (*a, b*)

*∂xm*+*j*

*i,j*=1*,...,n*

/= 0*, then* (*φ,* *∂φi*

*i,j*=1*,...,n*

*, a, b*) ∈ *dom*(Ψ)*.*

*Furthermore, if* (*r*1*, r*2*,f* ) = Ψ(*φ,* *∂φi*

*∂xm*+*j*

*∂xm*+*j*

*i,j*=1*,...,n*

*, a, b*)*, then f is the unique*

*function such that f* : *B*(*a, r*1) *B*(*b, r*2)*, φ*(*x, f* (*x*)) = *→*0 *for all x B*(*a, r*1)*, and f* (*a*)= *b.*

→ ∈

Proof sketch In the proof of Theorem [3.4](#_bookmark8), *r, h, k* can be computed from the

*δm*+*n,n* names of *∂φi*

*∂x* +

*m j*

for *i, j* ∈ {1 *... , n*}. In Theorem [3.3](#_bookmark7), the function Ψ

can be obtained uniformly from the ball *B*. The proof of Theorem [3.4](#_bookmark8) shows

that we can compute a *δm,n* name of *f* from a *δm*+*n,n* name of *φ* and the *δm*+*n,n*

names of *∂φi* for *i, j* ∈ {1 *... , n*}.

*∂xm*+*j*

Theorem 3.7 (Differentiable Uniformly Computable Multivari-

able Implicit Function Theorem) *There is a* (*δ*1 *, ρm, ρn, ρ*2*, δ*1 )*-*

*m*+*n,n*

*m,n*

*m n* 1

*computable* Ψ :⊆ *C*1 × R × R → R × R × *C such that if φ* ∈ *C*1 *,*

*m*+*n,n*

*m*+*n,n*

*m,n*

*a* ∈ R*m, b* ∈ R*n, φ*(*a, b*) = 0*, and det* *∂φi* (*a, b*)

*∂xm*+*j*

*i,j*=1*,...,n*

/= 0*, then*

(*φ, a, b*) ∈ *dom*(Ψ)*. Furthermore, if* (*r*1*, r*2*,f* ) = Ψ(*φ, a, b*)*, then f is the*

*unique function such that f* : *B*(*a, r*1) → *B*(*b, r*2)*, φ*(*x, f* (*x*)) = *→*0 *for all*

*x* ∈ *B*(*a, r*1)*, and f* (*a*)= *b.*

Proof sketch Most of the work has already been done. The proof of Theorem

[3.5](#_bookmark9) shows that we can compute a *δ*1

*m,n*

name of *f* from a *δ*1

name of *φ*

once we have computed a *δm,n* name of *f* .

*m*+*n,n*

# Acknowledgement

Many thanks to Iraj Kalantari, Ted Mahavier, Alec Matheson, Susan McNi- choll, and the referees.

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