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Computable Riesz Representation for the Dual of *C*[0; 1]

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Abstract

By the Riesz representation theorem for the dual of *C*[0; 1], for every continuous linear operator

*F* : *C*[0; 1] → R there is a function *g* : [0; 1] → R of bounded variation such that

*F* (*f* )= Z *f dg* (*f* ∈ *C*[0; 1]) *.*

The function *g* can be normalized such that *V* (*g*) =  *F *. In this paper we prove a computable version of this theorem. We use the framework of TTE, the representation approach to computable analysis, which allows to define natural computability for a variety of operators. We show that

there are a computable' operator *S* mapping *g* and an upper bound of its variation to *F* and a

computable operator *S* mapping *F* and its norm to some appropriate *g*.

*Keywords:* Computable analysis, integration, Riesz representation theorem

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# Introduction

The Riesz representation theorem is one of the fundamental theorems in Func- tional Analysis and General Topology.

Theorem 1.1 (Riesz representation theorem[[2](#_bookmark23)]) *For every continuous linear operator F* : *C*[*a, b*] → R *there is a function g* : [*a, b*] → R *of bounded*

*variation such that*

*and*

*F* (*f* )= ∫ *f dg* (*f* ∈ *C*[*a, b*])

*V* (*g*)= *F .*

As usual, *C*[*a, b*] is the set of continuous functions *h* : [*a, b*] → R on the real interval [*a, b*], equipped with the norm  *h * = max*a*≤*x*≤*b* |*h*(*x*)|. Its dual *C*'[*a, b*] is the set of continuous linear functions *F* : *C*[*a, b*] → R. The norm of *F* ∈ *C*'[*a, b*] is defined by *F * = sup{|*F* (*h*)| | *h* ∈ *C*[*a, b*]*, h * = 1}. *f dg* is the Riemann-Stieltjes integral and *V* (*g*) is the total variation of *g* : [*a, b*] → R. Let BV[a*,* b] be the set of functions *g* : [*a*; *b*] → R of bounded variation.

∫

On the other hand, for every function *g* : [*a, b*] → R of bounded variation the operator *f* '→ *f dg* is linear and continuous on *C*[*a, b*]. Therefore, the dual space of the space *C*'[*a, b*] can be identified with a space of (appropriately normalized) functions of bounded variation on [*a, b*].

∫

There are more abstract versions of the Riesz representation theorem, for example, for complex valued continuous functions with compact support on a locally compact Hausdorff space instead of *C*[*a, b*] and linear positive operators *F* [[6](#_bookmark27)]. In this article we study aspects of computability of the above simple version which can be found e.g. in [[2](#_bookmark23)]. We prove a computable version of this theorem in the framework of TTE. For given natural representations of the spaces we prove that there are computable operators mapping *F* to *g* and mapping *g* to *F* . For formulating and proving we use the concepts of Type-2 Theory of Effectivity, the representation approach to Computable Analysis [[9](#_bookmark28)]. Some aspects of computability of functions of bounded variation have been already studied in [[5](#_bookmark26),[11](#_bookmark29)]

For convenience we consider only functions on the unit interval [0; 1]. The generalization to arbitrary intervals is straightforward.

In Section [2](#_bookmark1) we estimate the rate of convergence of a sequence of finite sums approximating the Riemann-Stieltjes integral. Section [3](#_bookmark3) contains the construction of a function *g* of bounded variation from *F* . In Section [4](#_bookmark19) we outline shortly some concepts of TTE and define the (multi-)representations of the sets we will use. The last section contains the main theorems. Because of the detailed preparations their proofs ar short.

# Riemann-Stieltjes Integral

In this section we consider the definition of the Riemann-Stieltjes Integral (see for example [[7](#_bookmark30)]) and estimate the rate of convergence of a sequence of finite sums converging to the integral. We will need this rate for proving computability.

Let *a, b* be real numbers such that *a < b*. A *partition* of the interval [*a*; *b*] is a sequence *Z* = (*x*0*, x*1*,... , xn*) such that *a* = *x*0 *< x*1 *< ... < xn* = *b*. The partition *Z* has *precision k*, if *xi* − *xi*−1 ≤ 2−*k* for 1 ≤ *i* ≤ *n*. A partition

*Z*' = (*x*' *, x*' *,... , x*' ) is finer than *Z*, if {*x*0*, x*1*,... , xn*}⊆{*x*' *, x*' *,... , x*'

## }. A

0 1 *m* 0 1 *m*

*selection* for *Z* is a sequence *T* = (*t*1*,... , tn*) such that *xi*−1 ≤ *ti* ≤ *xi*. For a

real function *g* : [*a*; *b*] → R define

*n*

Σ

*S*(*g, Z*) := |*g*(*xi*) − *g*(*xi*−1)|*.* (1)

*i*=1

The *variation* of *g* is defined by

*V* (*g*) := sup{*S*(*g, Z*)|*Z* is a partition of [*a*; *b*]}*.* (2)

A function *g* : [*a*; *b*] → R is of *bounded variation* if its variation *V* (*g*) is finite. In the following let *f* : [*a*; *b*] → R be continuous function and let *g* : [*a*; *b*] →

R be a function of bounded variation. For any partition *Z* = (*x*0*, x*1*,... , xn*) of [*a*; *b*] and any selection *T* for *Z* define

*n*

Σ

*S*(*g, f, Z, T* ) := *f* (*ti*)(*g*(*xi*) − *g*(*xi*−1))*.* (3)

*i*=1

Every continuous function *f* : [*a*; *b*] → R has a (uniform) *modulus of continuity*, i.e., a function *m* : N → N such that |*f* (*x*) − *f* (*y*)| ≤ 2−*k* if

|*x* − *y*|≤ 2−*m*(*k*).

Lemma 2.1 *Let f* : [*a*; *b*] → R *be continuous function with modulus of conti- nuity m* : N → N*. Let g* : [*a*; *b*] → R *be a function of bounded variation. Then there is a number I* ∈ R *such that*

|*I* − *S*(*g, f, Z, T* )|≤ 2−*kV* (*g*)

*for each partition Z of* [*a*; *b*] *with precision m*(*k* + 1) *and each selection T for*

*Z.*

Proof: First, we prove that for any two partitions *Z*1*, Z*2 of [*a*; *b*] with precision

*m*(*k* + 1) and selections *T*1 and *T*2, respectively,

|*S*(*g, f, Z*1*, T*1) − *S*(*g, f, Z*2*, T*2)|≤ 2−*kV* (*g*) *.*

Let *Z*1 = (*x*0*, x*1*,... , xn*) with selection *T*1 = (*t*1*,... , tn*) and let *Z*' be a refinement of *Z*1 with selection *T* '. Then *Z*' can be written as

*x*0 = *y*1*, y*1*,... , y*1

= *x*1 = *y*2*, y*2*,... , y*2

= *x*2 *... ...* = *yn, yn,... , yn*

= *xn*

0 1 *j*1

0 1 *j*2

* 1. 1 *jn*

(*j*1*,... , jn* ≥ 1) and *T* ' as

*t*1*, t*1*,... , t*1 *, t*2*, t*2*,... , t*2 *,... ... t*1 *, t*1 *,... , tn .*

* 1. 2 *j*1 1 2 *j*2 *n n jn*

such that *yi* ≤ *ti* ≤ *yi*. Then

*l*−1 *l l*

|*S*(*g, f, Z*1*, T*1) − *S*(*g, f, Z*'*,T* ')|

*n*

## = Σ

*n*

*f* (*t* )

*g*(*x* ) − *g*(*x*

## ) − Σ

*ji*

*f* (*ti*)

Σ

*g*(*yi*) − *g*(*yi* )

*i* *i*

*i*=1

*i*−1

*i*=1

*l l*

*l*=1

*n*

*l*−1

## = Σ

*n*

*ji*

*f* (*t* )

Σ

*g*(*yi*) − *g*(*yi*

## ) − Σ

*ji*

*f* (*ti*)

Σ

*g*(*yi*) − *g*(*yi* )

*i*=1

*i l*

*l*=1

*l*−1

*i*=1

*l l*

*l*=1

*l*−1

## = Σ

*n*

*ji*

*f* (*t* ) − *f* (*ti*)

Σ

*g*(*yi*) − *g*(*yi* )

*i*=1 *n*

*i l l*

*l*=1 *ji*

*l*−1

≤ Σ Σ *f* (*ti*) − *f* (*ti*) *g*(*yi*) − *g*(*yi* )

*i*=1

*l*=1

*l*

*l*

*l*−1

≤ 2−*k*−1 Σ Σ *g*(*yi*) − *g*(*yi* ) since |*ti* − *ti*|≤ 2−*m*(*k*+1)

*n*

*ji*

*l*

*l*−1

*l*

*i*=1

*l*=1

≤ 2−*k*−1*V* (*g*)

Now let *Z*' be a common refinement of *Z*1 and *Z*2 and let *T* ' be a selection for

*Z*'. Then

|*S*(*g, f, Z*1*, T*1) − *S*(*g, f, Z*2*, T*2)|

≤ |*S*(*g, f, Z*1*, T*1) − *S*(*g, f, Z*'*,T* ')| + |*S*(*g, f, Z*2*, T*2) − *S*(*g, f, Z*'*,T* ')|

≤ 2−*kV* (*g*)

Next, for each *i* ∈ N let *Zi* be a partition of [*a*; *b*] with precision *m*(*i* + 1) and a selection *Ti*. Then for *i > j*,

|*S*(*g, f, Zi, Ti*) − *S*(*g, f, Zj, Tj*)|≤ 2−*jV* (*g*) *.*

Therefore, the sequence (*S*(*g, f, Zi, Ti*))*i* is a Cauchy sequence converging to some *I* ∈ R. If *Z* is a partition with precision *m*(*k* + 1) and selection *T* , then for each *i > k*

|*I* − *S*(*g, f, Z, T* )|≤ |*I* − *S*(*g, f, Zi, Ti*)| + |*S*(*g, f, Zi, Ti*) − *S*(*g, f, Z, T* )|

≤ 2−*iV* (*g*)+ 2−*kV* (*g*) *,*

hence |*I* − *S*(*g, f, Z, T* )|≤ 2−*kV* (*g*).

Definition 2.2 [Riemann-Stieltjes integral]

∫ *f dg* := *I* (the real number defined in Lemma [2.1](#_bookmark2))

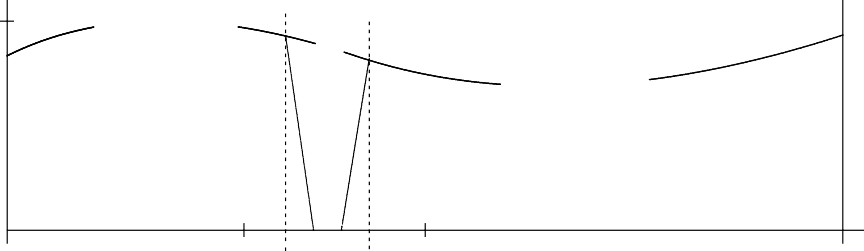
# Construction of a Function of Bounded Variation

In this section for a given continuous linear operator *F* : *C*[0; 1] → R we construct a function *g*' : ⊆[0; 1] → R of variation *F * such that *F* (*h*)= *h dg* for every *h* ∈ *C*[0; 1] and every extension *g* : [0; 1] → R of *g*' of bounded variation.

∫

Let *F* : *C*[0; 1] → R be a linear continuous operator on the set *C*[0; 1] of continuous functions *f* : [0; 1] → R. For a function *h* ∈ *C*[0*,* 1], and 0 ≤ *a < b* ≤ 1 define the function *hab* ∈ *C*[0*,* 1] as follows. The graph of *hab* is the union of the graph of *h* from 0 to *a*, the line from the point (*a, h*(*a*)) to (*a* + (*b* − *a*)*/*3*,* 0), the line from this point to the point (*b* − (*b* − *a*)*/*3*,* 0), the line from this point to (*b, h*(*b*)) and the graph of *h* from *b* to 1 (see Figure [1](#_bookmark4)).

*hab*



1

ˆ

/ \

ˆ

/

/r/

/ \

\

\\1

*h*

0 。

*c a*

*b*

*d*

)1

Fig. 1. The (*a, b*)-cut *hab* of *h*

Lemma 3.1 *Suppose h* ∈ *C*[0*,* 1]*, ε >* 0 *and* 0 ≤ *c < d* ≤ 1*. Then there are*

*a, b* ∈ Q *such that c < a < b < d and* |*F* (*h* − *hab*)| *< ε.*

Proof: Suppose this is false. Then there are infinitely many pairwise disjoint intervals (*ai*; *bi*) in the interval (*c*; *d*) such that |*F* (*h* − *haibi* )| ≥ *ε*. For each

*i* ≤ *N* define

*h* := ⎧ *h* − *haibi* if *F* (*h* − *haibi* ) ≥ 0

⎨

*i*

⎩ −(*h* − *haibi* ) otherwise.

Since *haibi * ≤  *h* , *hi * ≤ 2  *h * . Choose *N >* 2  *F * *h * */ε*. Since

Σ

*N*

*i*=0

*hi * = max*N*

*hi* ≤ 2  *h * , |*F* (Σ*N*

*hi*)|≤*F * Σ*N*

*hi* ≤ 2  *F h * .

On the other hand, since *F* (*hi*) ≥ *ε*, |*F* (Σ*N*

*i*=0

*i*=0

*i*=0

*i*=0

*i*=0

*hi*)| = | Σ*N*

*F* (*hi*)| =

*N*

Σ

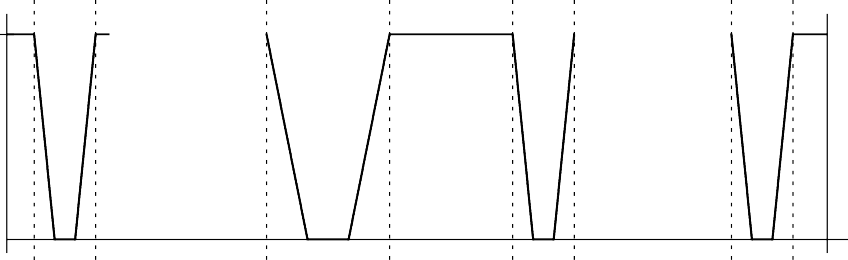
*i*=0

*F* (*hi*) ≥ *N* · *ε >* 2 *F h* . Contradiction.

The function *dab* := *h*−*hab* has a support in [*a*; *b*] and a very small “weight”

|*F* (*dab*)|. It cuts the function *h* into two pices *ha* and *hb* with disjoint supports such that *F* (*h*) and *F* (*ha* + *hb*) are almost the same. Such a cut is possible eveywhere in the interval [0; 1].

Let an *approximate partition* be a sequence *π* = (*a*1*, b*1*,... , an, bn*) (*n* ≥ 1) of rational numbers such that 0 *< a*1 *< b*1 *< ... < an < bn <* 1. Let *b*0 := 0 and *an*+1 := 1. An approximate partition *π* induces an approximate decomposition of the function 1I, 1I(*x*) = 1 for 0 ≤ *x* ≤ 1, into continuous functions *f*0*,... , fn* ∈ *C*[0*,* 1], which are polygons defined by the vertices of their graphs as follows (see Figure [2](#_bookmark6)).



1 ˆ

*fi*

*fn*

ˆ

*f*0

. . . . . . .

. . . . . . .

0 。

*b*0*a*1 *b*1

*ai*

*bi*

*ai*+1 *bi*+1

)1

*an bn an*+1

Fig. 2. Decomposition of 1I by a partition (*a*1*, b*1 *,..., an, bn*)

For 1 ≤ *i < n*,

*b*1 − *a*1

*f* : (0*,* 1)*,* (*a ,* 1)*,* (*a* + )*,* (1*,* 0)*,*

0 1 1 3

*f* : (0*,* 0)*,* (*b* − *bi* − *ai ,* 0)*,* (*b ,* 1)*,* (*a*

*bi*+1 − *ai*+1

*,* 1)*,* (*a* + *,* 0)*,* (1*,* 0)*,*

*i i* 3

*i i*+1

*i*+1 3

*f* : (0*,* 0)*,* (*b* − *bn* − *an* )*,* (*b*

*,* 1)*,* (1*,* 1)*.*

*n n* 3 *n*

By the next lemma the function 1I can be partitioned into finitly many functions *fi* of Norm 1 with disjoint support, such that |*F* (*fi*)| is arbitrarily close to *F *, and, in addition, for a given interval *J* ∈ *L* there is some *i* such that (*ai*; *bi*)⊆*J* .

Σ

Lemma 3.2 *Let F* : *C*[0; 1] → R *be continuous. For every ε >* 0 *and every open interval in J* ⊆[0; 1] *there is an approximate partion π* = (*a*1*, b*1*,... , an, bn*) *such that*

*n*

Σ

 *F * − *ε <* |*F* (*fi*)|≤*F * *,* (4)

*i*=0

(∀ *i,* 1 ≤ *i* ≤ *n*) *bi* − *ai < ε* (5)

*and* (∃ *i,* 1 ≤ *i* ≤ *n*) [*ai*; *bi*]⊆*J.* (6)

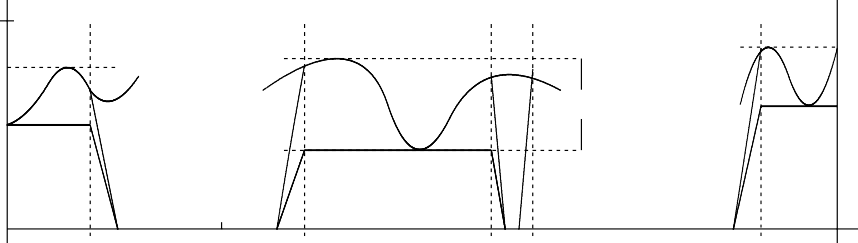
Proof: Let *ε*' := *ε/*(2 + *F* ). Since  *F * = sup{*F* (*h*)| *h * = 1}, there is some *h* ∈ *C*[0; 1] such that *h* =1 and

 *F * − *ε*' *< F* (*h*)*.* (7)

Since *h* is uniformly continuous there is some *ε*1 *>* 0 such that

*ε*1 *< ε*' and |*h*(*x*) − *h*(*y*)| *< ε*' for |*x* − *y*|≤ *ε*1*.* (8) Divide the interval (0; 1) into consecutive intervals (*cj*; *dj*) (*j* = 1*,... , n*) such that *c*1 = 0, *dj* = *cj*+1 and *dn* = 1 of length ≤ min(*ε*1*,* length(*J* ))*/*3. Ap-

ply Lemma [3.1](#_bookmark5) in turn to each of these intervals (*cj*; *dj*) (*j* = 1*,... , n*) with precision *ε*'*/n*. The result is a partition as shown in Figure [3](#_bookmark8).



1 ˆ

ˆ

*h*0

*C*

*h*

*i*

*D*

*g*0

.. .. .. .

ˆ

*< ε*'

v . .. .. ..

*hn*

*g*

*n*

*A gi*

*B*

0 。

*a*1

*ai*

*E bi*

*ai*+1*F bi*+1

*bn*

)1

Fig. 3. Approximate decomposition of 1I via *h*.

Notice that the ranges from *ai* to *bi* correspond to the range from *a* to *b* in Figure [1](#_bookmark4) and that the distance from *Ei* to (*bi,* 0) is (*bi* − *ai*)*/*3 and the distance from *ai*+1 to *Fi* is (*bi*+1 − *ai*+1)*/*3. For 1 ≤ *i* ≤ *n* − 1 define *hi* and *gi* as follows. The graph of *hi* is the union of the line segments from (0*,* 0) to *Ei*, from *Ei* to *Ci*, from *Di* to *Fi* and from *Fi* to (1*,* 0) and the section of graph(*h*) from *Ci* to *Di*. The graph of *gi* is the union of the line segments from (0*,* 0) to *Ei*, from *Ei* to *Ai*, from *Ai* to *Bi*, from *Bi* to *Fi* and from *Fi* to (1*,* 0), where the ordinate of *Ai* and *Bi* is min{*h*(*x*) | *bi* ≤ *x* ≤ *ai*+1}. The functions *h*0*, g*0*, hn* and *gn* are defined accordingly.

By the construction and Lemma [3.1](#_bookmark5) for the approximate partition *π* = (*a*1*, b*1*,... , an, bn*),

(∃ *i*)[*ai*; *bi*] ∈ *J,* (9)

*ai*+1 − *bi < ε*1 for *i* = 1*,... , n* (10)

*N*

Σ

and |*F* (*h*) − *F* (*hi*)| *< ε*'*.* (11)

*i*=0

It remains to prove ([4](#_bookmark7)). By ([10](#_bookmark8)) and ([8](#_bookmark7)),  *hi* − *gi * ≤ *ε*' for 0 ≤ *i* ≤ *n* and

hence  Σ*n* (*hi* − *gi*)  ≤ *ε*' (since the (*hi* − *gi*) have disjoint supports). We

*i*=0

obtain

*n*

Σ

and

|*F* ( (*hi* − *gi*))|≤ *ε*'  *F * (12)

*i*=0

 *F * − *F* Σ *gi* ≤ *F* (*h*) − *F* Σ *gi* + *ε*' by ([7](#_bookmark7))

≤ |*F* (*h*) − *F* (Σ *hi*)| + |*F* (Σ *hi*) − *F* Σ *gi*| + *ε*'

Σ

*n*

*< ε*' + |*F* ( (*hi* − *gi*))| + *ε*' by ([11](#_bookmark8))

*i*=0

≤ *ε*'(2 + *F* ) ≤ *ε* by ([12](#_bookmark8))*.*

For *i* = 0*,... , n* let *fi* be the function from the decomposition of 1I induced by the approximate partition *π* = (*a*1*, b*1*,... , an, bn*). If *gi* =0 then |*F* (*gi*)| = 0 ≤ |*F* (*fi*)|. Otherwise,

|*F* (*g* )| = |*F* (|*g* |)| = *g*

|*gi*|

|*F* ( )|*g*

|*F* (|*f* |)|≤ |*F* (*f* )|

*i i i i i* *i i*

*g*

Since *F * − *F* Σ *gi < ε* (see above),

*F* − *ε < F* Σ *gi* = Σ *F* (*gi*) ≤ Σ |*F* (*gi*)|≤ Σ |*F* (*fi*)| *.*

Finally, for each *i* there is some *αi* ∈ {−1*,* 1} such that |*F* (*fi*)| = *F* (*αifi*). Since  *αifi * = 1,

Σ

Σ |*F* (*fi*)| = Σ *F* (*αifi*)= *F* (Σ *αifi*) ≤ *F * *.*

Thus we have proved ([4](#_bookmark7)).

Since the adjacent intervals (*cj, dj*) have length ≤ length(*J* )*/*3, there is some *i* such that [*ai*; *bi*]⊆*J* . This proves ([6](#_bookmark7)). Finally *bi* − *ai* ≤ *di* − *ci < ε*1 *< ε*' *< ε*.

In tΣhe proof the differences *ai*+1 − *bi* are made small in order to get Σ *hi* close

to

*gi*. Also the differences *bi* − *ai* are made small so that the errors by

cutting remain small according to Lemma [3.1](#_bookmark5).

We introduce some terminology. For *d* ∈ *C*[0; 1] let supp(*d*) (the *support* of *d*) be the closure of the set {*x* | *d*(*x*) /= 0}. For 0 ≤ *a < b* ≤ 1 let (*a*; *b*)*/*3 := (*a* + (*b* − *a*)*/*3; *b* − (*b* − *a*)*/*3). The *slanted step* at (*a, b*) is the function *s* ∈ *C*[0; 1] the graph of of which is a polygon with the vertices (0*,* 1)*,* (*a,* 1)*,* (*b,* 0)*,* (1*,* 0). Let *v*(*s*) := (*a*; *b*)⊆[0*,* 1].

In Lemma [3.2](#_bookmark7) the operator *F* has small values for every function the sup- port of which does not intersect the supports of the functions *fi*, see also Figure [2](#_bookmark6).

Corollary 3.3 *Let π be the approximate partition from Lemma* [*3.2*](#_bookmark7)*.*

1. *If d* ∈ *C*[0; 1] *such that* supp(*d*)⊆ *n* (*ai*; *bi*)*/*3 *then* |*F* (*d*)|≤ *ε * *d * *.*

*i*=1

1. *If s, s*' *are slanted steps s.th. v*(*s*)*, v*(*s*')⊆(*ai*; *bi*)*/*3 *for some* 1 ≤ *i* ≤ *n, then* |*F* (*s*) − *F* (*s*')|≤ *ε.*

Proof: [i](#_bookmark9). This is true for *d* = 0. Assume *d * = 1. There are signs

*σ, σi* ∈ {−1*,* 1} such that |*F* (*fi*)| = *F* (*σifi*) and *F* (*σd*) = |*F* (*d*)|. Since

 *σd* + Σ*n*

*i*=0

(*σifi*)  = 1,

*n n*

|*F* (*d*)| + Σ |*F* (*fi*)| = *F* (*σd*)+ Σ *F* (*σifi*))

*i*=0

= *F σd* +

*i*=0

Σ*i*=0

*n*

(*σifi*)

Since  *F* − *ε* ≤ Σ*n*



*d*' := *d/ d* .

≤ *F * *.*

|*F* (*fi*)| by ([4](#_bookmark7)), |*F* (*d*)| ≤ *ε*. If  *d * *>* 0, consider

*i*=0

1. Apply [i](#_bookmark9). to *d* := (*s* − *s*').

Lemma 3.4 *For every linear and continuous F* : *C*[0; 1] → R *and every open interval J* ⊆[0; 1] *there are a sequence* (*πk*)*k*∈N *, πk* =

(*ak, bk, ak, bk,... , ak , bk*

)*, of approximate partitions, a sequence* (*ik*)*k*∈N*,* 1 ≤

1 1 2 2 *nk nk*

*ik* ≤ *nk, of indices and a sequence* (*sk*)*k*∈N *of slanted steps such that for all k,*

*nk*

*F * − 2 *<* Σ |*F* (*f* )|≤ *F * *,* (13)

−*k k*

*i*

*i*=0

(∀ *i*) *bk* − *ak <* 2−*k ,* (14)

*i* *i*

(*a*0 ; *b*0 )⊆*J ,* (15)

*i*0 *i*0

[*ak*+1 ; *bk*+1 ] ⊆ (*ak* ; *bk* )*/*3 (16)

*ik*+1

*ik*+1

*ik ik*

*v*(*sk*)⊆(*ak* ; *bk* )*/*3 *.* (17)

*ik ik*

Proof: For *π*0 and *i*0 apply Lemma [3.2](#_bookmark7) to *ε* = 2−0 =1 and *J* . For *πk*+1 and

*ik*+1 apply Lemma [3.2](#_bookmark7) to *ε* = 2−*k*−1 and *J* ' := (*ak* ; *bk* )*/*3. The slanted steps

*ik ik*

*sk* can be chosen appropriately.

Lemma 3.5 *For the slanted steps sk in Lemma* [*3.4*](#_bookmark12)*,* |*F* (*sm*) − *F* (*sl*)| ≤ 2−*k*

*if k* ≤ *l* ≤ *m.*

Proof: This follows from Corollary [3.3](#_bookmark11).[i](#_bookmark9) and ([16](#_bookmark12),[17](#_bookmark12)).

Definition 3.6 For the operator *F* and the interval *J* let (*πk*)*k*∈N, (*ik*)*k*∈N and (*sk*)*k*∈N be the sequences from Lemma [3.4](#_bookmark12). Define

*xJ* := [*ak* ; *bk* ]*, yJ* := lim *F* (*sk*) *.* (18)

*ik*

*ik*

*k*→∞

By ([16](#_bookmark12)) and Lemma [3.5](#_bookmark13), the numbers *xJ* and *yJ* are well-defined and

(∀ *k*) |*yJ* − *F* (*sk*)| ≤ 2−*k .* (19)

Let (*K**i*)*i*∈N be a canonical numbering of the set of all open subintervals (*c, d*)⊆[0; 1] with *c, d* ∈ Q. For each *i* let *xKi* and *yKi* be real numbers defined via sequences (*πk*)*k*∈N and (*ik*)*k*∈N according to Lemma [3.4](#_bookmark12) and ([18](#_bookmark14)). Then the set of all *xKi* is dense in [0; 1]. Let

*G*0 := {(*xKi, yKi* ) | *i* ∈ N} *,* (20)

*G*' := *G*0 ∪ {(0*,* 0)*,* (1*,F* (1I))} *.* (21)

Lemma 3.7 (i) *The set G*0 *is the graph of a continuous function g*0*.*

(ii) *The function g*' *with graph G*' *has variation V* (*g*')= *F .*

Here, as a generalization of ([2](#_bookmark1)), we define the variation *V* (*g*') of the function

*g*' with dom(*g*')⊆[0; 1] by

*V* (*g*') := sup{*S*(*g*'*, Z*)|(∃ *x*0*,... , xn* ∈ dom(*g*'))

*Z* = (*x*0*,... , xn*) is a partition of [0; 1]} *.*

Proof: First we show:

lim

*i*→∞

*yi* = *y* if (*x, y*)*,* (*x*0*, y*0)*,* (*x*1*, y*1)*,...* ∈ *G*0 and lim

*i*→∞

*xi* = *x* (22)

Let *ε >* 0. The pair (*x, y*) is determined by some sequence of approximate partitions (*πk*)*k* according to Lemma [3.4](#_bookmark12) and Definition [3.6](#_bookmark14). Therefore, there some number *k* and a slanted step *sk* such that

(*x* − *ε*; *x* + *ε*)⊆(*ak* ; *bk* )*/*3 for some *ε >* 0 *,* (23)

*ik ik*

|*y* − *F* (*sk*)| ≤ 2−*k* and *v*(*sk*)⊆(*ak* ; *bk* )*/*3 *.* (24)

*ik ik*

There is some *j* such that |*x* − *xj*| *< ε/*2. Let (*π*¯*m*)*m* be the sequence of ap- proximate partitions defining (*xj, yj*) and let *s*¯*m* be the slanted steps according to Lemma [3.4](#_bookmark12). Let *i* be a number such that *i > k* and 2−*i < ε/*2. By ([19](#_bookmark14))

|*yj* − *F* (*s*¯*i*)|≤ 2−*i* and *v*(*s*¯*i*)⊆(*x* − *ε*; *x* + *ε*) *.* (25)

By ([23](#_bookmark15),[24](#_bookmark15),[25](#_bookmark15)),

*v*(*sk*)*, v*¯(*si*)⊆(*ak* ; *bk* )*/*3 *.*

*ik ik*

By Corollary [3.3](#_bookmark11), |*F* (*sk*) − *F* (*s*¯*i*)|≤ 2−*k* Therefore,

|*y* − *yj*|≤ |*y* − *F* (*sk*)| + |*F* (*sk*) − *F* (*s*¯*i*)| + |*F* (*s*¯*i*) − *yj*|

≤ 2−*k* + 2−*k* + 2−*i*

≤ 2−*k*+2 *.*

This proves ([22](#_bookmark15)).

Suppose (*x, y*)*,* (*x, y*') ∈ *G*0. Apply ([22](#_bookmark15)) to (*x, y*) and the sequence

(*x, y*)*,* (*x, y*')*,* (*x, y*)*,* (*x, y*')*,... .*

Then the sequence *y, y*'*, y, y*'*,...* converges, hence *y* = *y*'. Therefore, *G*0 is the graph of a function *g*0 which is continuous by ([22](#_bookmark15)).

[ii](#_bookmark15). First we show *S*(*g*'*, Z*) ≤  *F * for any partition *Z* = (*x*0*, x*1*,... , xn*) in dom(*g*'). Let *yi* := *g*'(*xi*) and *ε >* 0. Let *c <* (*xi* − *xi*−1)*/*2 for *i* = 1*,... , n*.

For every *i* there is some slanted steps *si* such that

*ε*

*v*(*si*)⊆(*xi* − *c*; *xi* + *c*) and |*F* (*si*) − *yi*|≤ 2*n .* (26)

Then

|*y* − *y* | = |*F* (*s* )| + |*F* (*s* ) − *y* | ≤ |*F* (*s* )| + *ε ,*

1 0 1

1 1 1 2*n*

*ε*

|*yn* − *yn*−1| = |*F* (1I) − *F* (*sn*)| + |*F* (*sn*) − *yn*−1|≤ |*F* (1I − *sn*)| + 2*n*

and for 1 *< i < n*,

|*yi* − *yi*−1|≤ |*yi* − *F* (*si*)| + |*F* (*si*) − *F* (*si*−1)| + |*F* (*si*−1) − *yi*−1|

Therefore,

*n*

≤ |*F* (*si*

— *si*−1

*ε*

)| +2 *.*

2*n*

*n*−1

Σ |*yi* − *yi*−1| ≤ |*F* (*s*1)| + Σ |*F* (*si* − *si*−1)| + |*F* (1I − *sn*)| + *ε*

*i*=1

*i*=2

There are signs *αi* ∈ {−1*,* 1} such that |*F* (*s*1)| = *F* (*α*1*s*1), |*F* (1I − *sn*)| = *F* (*αn*(1I − *sn*)) and |*F* (*si* − *si*−1)| = *F* (*αi*(*si* − *si*−1)) for 1 *< i < n*. Since *α*1*s*1 + *n*−1(*αi*(*si* − *si*−1)) + *αn*(1I − *sn*)  = 1,

*i*=2

Σ

*n*

Σ

*S*(*g*'*, Z*)= |*g*'(*xi*) − *g*'(*xi*−1)|

*i*=1

*n*−1

Σ

= |*F* (*s*1)| + |*F* (*si* − *si*−1)| + |*F* (1I − *sn*)| + *ε*

*i*=2

*n*−1

Σ

= *F* (*α*1*s*1)+ *F* (*αi*(*si* − *si*−1)) + *F* (*αn*(1I − *sn*)) + *ε*

*i*=2

= *F* *α*1*s*1 +

*n*−1

*i*=2

Σ

(*αi*(*si* − *si*−1)) + *αn*(1I − *sn*) + *ε*

≤ *F * + *ε.*

Since this is true for all *ε >* 0 and all *Z*, *V* (*g*') ≤ *F *.

For the other direction it suffices to show that (∀*ε >* 0)(∃*Z*)  *F * −

*ε* ≤ *S*(*g*'*, Z*). By Lemma [3.2](#_bookmark7) there is an approximate partition *π* =

(*a*1*, b*1*,... , an, bn*) such that *F * − *ε/*3 ≤ Σ*n* |*F* (*fi*)| (Figure [2](#_bookmark6)). For

*i*=0

1 ≤ *i* ≤ *n* define slanted steps *ui* and *vi* by the vertices of their graphs as follows:

*ui* : (0*,* 1)*,* (*ai,* 1)*,* (*ai* + (*bi* − *ai*)*/*3*,* 0)*,* (1*,* 0)

*vi* : (0*,* 1)*,* (*bi* − (*bi* − *ai*)*/*3*,* 1)*,* (*bi,* 0)*,* (1*,* 0) *.*

Then

*f*0 = *u*1*, fi* = *ui*+1 − *vi* (for 1 ≤ *i < n*) and *fn* = 1I − *vn* (27)

Since the first projection of *G*0 is dense in (0; 1) ([20](#_bookmark14)), for 1 ≤ *i* ≤ *n* there are pairs (*xi, yi*) ∈ *G*0 and slanted steps *si* such that

*xi* ∈ (*ai*; *bi*)*/*3*, v*(*si*)⊆(*ai*; *bi*)*/*3 and |*F* (*si*) − *yi*|≤ *ε*' (28)

for *ε*' := *ε/*(6*n*). We consider the partition *Z* := (0 = *x*0*, x*1*,... , xn, xn*+1 = 1).

Let *αi, βi, γi* ∈ {−1*,* 1} be signs and let

*h* := *β*0*u*1 + *γ*1(*s*1 − *u*1)

*n*−1

Σ

+ (*αi*(*vi* − *si*)+ *βi*(*ui*+1 − *vi*)+ *γi*(*si*+1 − *ui*+1))

*i*=1

+*αn*(*vn* − *sn*)+ *βn*(1I − *vn*)

Choose the signs such that *F* (*β*0*u*1) ≥ 0, *F* (*γ*1(*s*1 − *u*1)) ≥ 0, ...,

*F* (*βn*(1I − *vn*)) ≥ 0. It is seen easily that *h * = 1. Since |*F* (*fi*)| = *F* (*βifi*),

*F* (*h*) := |*F* (*f*0)| + |*F* (*s*1 − *u*1)|

*n*−1

Σ

+ (|*F* (*vi* − *si*)| + |*F* (*fi*)| + |*F* (*si*+1 − *ui*+1)|)

*i*=1

+|*F* (*vn* − *sn*)| + |*F* (*fn*)| *.*

We obtain

*n*

Σ

 *F * − *ε/*3 ≤ |*F* (*fi*)|≤ *F* (*h*) ≤ *F * *,*

*i*=0

and therefore,

*n*−1

Σ

|*F* (*s*1 − *u*1)| + (|*F* (*vi* − *si*)| + |*F* (*si*+1 − *ui*+1)|)+ |*F* (*vn* − *sn*)|≤ *ε/*3 *.*(29)

*i*=1

Finally,

*n*

Σ

 *F * − *ε/*3 ≤ |*F* (*fi*)

*i*=0

*n*−1

Σ

= |*F* (*u*1)| + |*F* (*ui*+1 − *vi*)| + |*F* (1I − *vn*)| by ([27](#_bookmark15))

*i*=1

≤ |*y*1| + |*F* (*s*1) − *y*1| + |*F* (*u*1) − *F* (*s*1)|

*n*−1

Σ

+ (|*F* (*ui*+1 − *si*+1)| + |*F* (*si*+1) − *yi*+1| + |*yi*+1 − *yi*|

*i*=1

+|*yi* − *F* (*si*)| + *F* (*si* − *vi*)|)

+|*F* (1I) − *yn*| + |*yn* − *F* (*sn*)| + |*F* (*sn*) − *F* (*vn*)|

*n*+1

Σ

≤ |*yi* − *yi*−1| + 2*nε*' + *ε/*3 by ([28](#_bookmark15)*,* [29](#_bookmark15))

*i*=1

= *S*(*g*'*, Z*)+ 2*nε*' + *ε/*3 *.*

We obtain *F * − *ε* ≤ *S*(*g*'*, Z*).

Let *g* : [0*,* 1] → R be a function of bounded variation which extends *g*'.

∫

Lemma 3.8 *For every continuous function h* : [0*,* 1] → R*, F* (*h*)= *h dg.*

Proof: Let *K* ∈ N. There is some *a* ∈ N such that *V* (*g*) ≤ 2*a*. Let *m* : N → N be an increasing modulus of continuity of the function *h*. We construct a partition *Z* of precision *m*(*K* +2+ *a*) and a selection *T* for *Z* such that

|*F* (*h*) − *S*(*g, h, Z, T* )|≤ 2−*K*−1 *.* (30)

Then by Lemma [2.1](#_bookmark2), |*F* (*h*) − ∫ *h dg*|≤ |*F* (*h*) − *S*(*g, h, Z, T* )| + |*S*(*g, h, Z, T* ) −

*h dg*| ≤ 2−*K*−1 + 2−*K*−1−*aV* (*g*) ≤ 2−*K*. Since this is true for all *K*, *F* (*h*) =

∫

*h dg*.

Let *ε* := 2−*K*−1*/*((2*n* + 1)  *h * +  *F *). Since *h* is unifomly continuous there is some *ε*' *>* 0 such that |*h*(*x*) − *h*(*x*')| ≤ *ε* if |*x* − *x*'| ≤ *ε*'. By Corollary [3.3](#_bookmark11),

Lemma [3.4](#_bookmark12) and ([19](#_bookmark14)) there are

– (*x*0*, y*0)*,* (*x*1*, y*1)*,... ,* (*xn*+1*, yn*+1) ∈ *G*',

* rational numbers *ci < di* (1 ≤ *i* ≤ *n*)
* and slanted steps *ui, vi* (1 ≤ *i* ≤ *n*)

such that *Z* = (0= *x*0*, x*1*,... , xn*+1 = 1) is a partition with

*xi* − *xi*−1 *< ε*'*/*2 for *i* = 1*,... ,n* + 1 (31) and for *i* = 1*,... , n*,

*ci < xi < di, di* − *ci <* (*xj* − *xj*−1)*/*2 for 1 ≤ *j* ≤ *n* + 1*,* (32)

*v*(*ui*)*, v*(*vi*) ∈ (*ci*; *di*)*, v*(*ui*) *< v*(*vi*)*,* (33)

|*F* (*ui*) − *yi*| *< ε,* |*F* (*vi*) − *yi*| *< ε,* (34)

|*F* (*d*)| *< ε * *d * if supp(*d*)⊆[*ci*; *di*] *.* (35)

ˆ *v*. .1..

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*B fi C*

*v*.*i*. .

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*v*. .*n*.. ˆ

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*u*1 ..

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*c*

*ui*−1

*D*

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. . .*v*.

. .*i*−1

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.. )1

*c*1 *x*1

ˆ

*d*1 /

*i*−1

*xi*−1

//’ *ci*

*i*−1

*d*

*xi di cn xn dn*

ˆ

*f* (*ti*)

。

*h G H*

*L hi*

*I J gi*

*F*

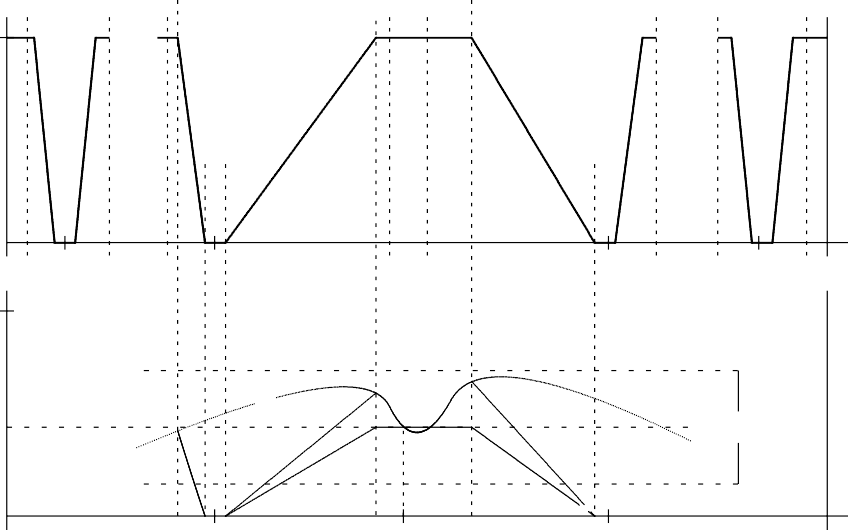
ˆ

*<* 2*ε*1

v

)

*M D ti* *xi*

Fig. 4. Approximate decomposition of 1I via *h*.

In Figure [4](#_bookmark18) the slanted step *vi*−1 is given by the line segments via the points (0*,* 1)*, A, E,* (1*,* 0) and *ui* by (0*,* 1)*, C, F,* (1*,* 0). Let

*f*1 := *u*1*, fi* := *ui* − *vi*−1 (2 ≤ *i* ≤ *n*)*, fn*+1 := 1I − *vn .* (36) For example, *fi* is given by the points (0*,* 0)*, D, B, C, F,* (1*,* 0).

In each interval (*ci*−1; *di*−1) (*i* = 2*,... , n*+1) we “pull” the function *h* down

as shown in the lower part of Figure [4](#_bookmark18) where the arc from *L* to *G* is pulled

down to *L, M, D, G*. Let *ei*−1 be the continuous function such that *ei*−1(*x*)= 0 for *x* left to *L* and right to *G* and *ei*−1(*x*) = 0 is the length the function *h* has been pulled down at *x* otherwise. Then

supp(*ei*)⊆(*ci*; *di*) and *ei * ≤ *h * for 1 ≤ *i* ≤ *n.* (37)

The function *h* − Σ*n ei* can be written as Σ*n*+1 *hi* with pairwise disjoint

*i*=1

*i*=0

supports. In Figure [4](#_bookmark18) the function *hi* is given by the sequence of vertices (0*,* 0)*, D, G, H, F,* (1*,* 0).

Let *T* = (*t*1*,... , tn*+1) be a selection for *Z*. Define

*gi* := *h*(*ti*)*fi* [0 ≤ *i* ≤ *n* + 1] *.* (38)

In Figure [4](#_bookmark18) the function *gi* is given by the sequence of vertices (0*,* 0)*, D, I, J, F,* (1*,* 0).

*i*=1

By ([35](#_bookmark17),[37](#_bookmark18)), |*F* (*ei*)| ≤ *ε * *h * . Since *h* = Σ*n*

*i*=1

*ei* + Σ*n*+1 *hi*

*F* (*h*) − *F*

*n*

*n*

*h*

Σ*n*+1

= Σ

*F* (*e* ) ≤ Σ

|*F* (*e* )|≤ *nε h .* (39)

*i*

*i*=1

*i*

*i*=1 

*i*

*i*=1

Since |*xi* − *xi*−1| ≤ *ε*'*/*2,  *hi* − *gi* ≤ *ε*, hence  Σ*n*+1 *hi* − Σ*n*+1 *gi * ≤ *ε*.

Therefore,

*i*=1

*i*=1

*n*+1

[¨](#_bookmark18)

Σ

Σ

*F hi* − *F*

[¨](#_bookmark18) *i*=1

*n*+1

*i*=1

*gi* ¨

≤ *F ε.* (40)

By ([36](#_bookmark18),[38](#_bookmark18)),

*F* (*g*1)= *h*(*t*1)*F* (*u*1)*,*

*F* (*gi*)= *h*(*ti*)(*F* (*ui*) − *F* (*vi*−1)) (2 ≤ *i* ≤ *n*)*, F* (*gn*+1)= *h*(*tn*+1)*F* (1I − *vn*) *.*

By ([34](#_bookmark17)),

Σ

Σ*n*+1

*F*

— *S*(*g, h, Z, T* )

*n*+1

=

*n*+1

*F* (*g* ) − *h*(*t* )(*y* − *y* )

Σ

*i*

*g*

*i*=1

*i*=1

*i i i*

*i*=1

*i*−1

As a summary,

= |*h*(*t*1)(*F* (*u*1) − *y*1)

*n*

Σ

+ *h*(*ti*)(*F* (*ui*) − *F* (*vi*−1) − (*yi* − *yi*−1))

*i*=2

+*h*(*tn*+1)(*F* (1I − *vn*) − (*F* (1I) − *yn*))|

*n*

Σ

≤ |*h*(*t*1)|*ε* + 2|*h*(*ti*)|*ε* + |*h*(*tn*+1)|*ε*

*i*=2

≤ (*n* + 1) *h ε.*

|*F* (*h*) − *S*(*g, h, Z, T* )|≤ *nε h* + *F ε* + (*n* + 1) *h ε* = 2−*K*−1 *.*

# The Computability Background

For studying computability we use the representation approach (TTE) to Computable Analysis [[9](#_bookmark28)]. Let Σ be a finite alphabet. Computable functions on Σ∗ (the set of finite sequences over Σ) and Σ*ω* (the set of infinite sequences over Σ) are defined by Turing machines which map sequences to sequences (fi- nite or infinite). On Σ*ω* finite or countable tupling will be denoted by ⟨ ⟩ [[9](#_bookmark28)]. Sequences are used as “names” of abstract objects. We generalize the concept of representations in [[9](#_bookmark28)] to multi-representations and consider computability of multi-functions w.r.t. multi-representations (see [[10](#_bookmark31)] for the definition, which differs from that in [[8](#_bookmark32)], and [[3](#_bookmark24)] for an application).

A *multi-function* is a triple *f* = (*A, B, Rf* ) such that *Rf* ⊆*A* × *B*, which we will denote by *f* : ⊆*A* ⇒ *B*. For *X*⊆*A* let *f* [*X*] := {*b* ∈ *B* | (∃*a* ∈ *X*)(*a, b*) ∈

R} and for *a* ∈ *A* define *f* (*a*) := *f* [{*a*}]. Notice that *f* is well-defined by the

values *f* (*a*)⊆*B* for all *a* ∈ *A*. We define dom(*f* ) := {*a* ∈ *A* | *f* (*a*) /= ∅}. For muli-functions *f* : ⊆ *A* ⇒ *B* and *g* : ⊆ *C* ⇒ *D* we define the composition

*g* ◦ *f* : ⊆ *A* ⇒ *D* by

*a* ∈ dom(*g* ◦ *f* ): ⇐⇒ *a* ∈ dom(*f* ) and *f* (*a*)⊆dom(*g*) *,* (41)

*g* ◦ *f* (*a*) := *g*[*f* (*a*)] *.* (42)

Notice that ([42](#_bookmark19)) without ([41](#_bookmark19)) corresponds to ordinary relational composition of *Rf* and *Rg*. For a multi-function *f* ⊆*M*1 ⇒ *M*0 we will usually interpret

*f* (*x*)⊆*B* as the set of “acceptable” values for the argument *x* ∈ *M*1.

Definition 4.1 [multi-representation]

A multi-representation of a set *M* is a surjective multi-function *δ* : ⊆Σ*ω* ⇒ *M* .

A multi-representation *δ* : ⊆Σ*ω* ⇒ *M* can be considered as a naming system for the points of a set *M* , where each name can encode many points.

Therefore, *x* ∈ *δ*(*p*) is interpreted as “*p* is a name of *x*”. We generalize the concept of realization of a function or multi-function w.r.t. (single-valued) representations [[9](#_bookmark28)] to multi-representations as follows [[10](#_bookmark31)]:

Definition 4.2 [realization]

For multi-representations *γi* : ⊆ *Yi* ⇒ *Mi* (*i* = 0*,... , k*), abbreviate *Y* :=

*Y*1 × *...* × *Yk*, *M* := *M*1 × *...* × *Mk*, and *γ*(*y*1*,... , yk*): *γ*1(*y*1) × *...* × *γk*(*yk*).

Then a function *h* : ⊆ *Y* → *Y*0 is a (*γ, γ*0)-realization of a multi-function

*f* : ⊆ *M* ⇒ *M*0, iff for all *p* ∈ *Y* and *x* ∈ *M* ,

*x* ∈ *γ*(*p*) ∩ dom(*f* ) =⇒ *f* (*x*) ∩ *γ*0 ◦ *h*(*p*) /= ∅ *.* (43)

The multi-function *f* is called (*γ, γ*0)-computable, if it has a computable (*γ, γ*0)-realization.

(We will say (*γ*1*,... γk, γ*0)-computable instead of (*γ, γ*0)-computable, etc.)

Fig. [5](#_bookmark20) illustrates the definition. Whenever *p* is a *γ*-name of *x* ∈ dom(*f* ), then *h*(*p*) (the sequence of symbols computed by a machine for *h*) is a *γ*0-name of some *y* ∈ *f* (*x*).

*p*

/\.

/ \

*h* ) *h*(*p*)

/\.

/ \

/ \

/ \

*γ* / \

/ \

/ \1

/r

ccccs

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/ \*γ*0

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/ \1

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vccc

cz

. zzz

*x* zz

z*f*zzz,

v

.

)

∃ *y* ∈ *f* (*x*) ∩ *γ*0 ◦ *h*(*p*)

Fig. 5. *h*(*p*) is a name of some *y* ∈ *f* (*x*), if *p* is a name of *x* ∈ dom(*f* ).

For two multi-representations *δi*⊆Σ*ω* ⇒ *Mi* (*i* = 1*,* 2), *δ*1 ≤ *δ*2 (“reducible to”) iff (∀ *p* ∈ dom(*δ*1)) *δ*1(*p*)⊆*δ*2*h*(*p*) for some computable function *h* : ⊆Σ*ω* →

Σ*ω*.

If multi-functions on represented sets have realizations, then their com- position is realized by the composition of the realizations. In particular, the computable multi-functions on represented sets are closed under composition. Much more generally, the computable multi-functions on multi-represented sets are closed under flowchart programming with indirect addressing [[10](#_bookmark31)]. This result allows convenient informal construction of new computable func- tions on multi-represented sets from given ones.

For the real numbers we use the Cauchy representation *ρ* : ⊆Σ*ω* → R, for the set of continuous real functions on the unit interval the Cauchy rep- resentation *δC* : ⊆Σ*ω* → *C*[0; 1] defined via the dense set of rational polygons

(Definitions 4.1.5 and 6.1.9 in [[9](#_bookmark28)]). For the space *C*˜ of continuous functions

*F* : *C*[0; 1] → R there is a canonical representation [*δC* → *ρ*] (Definitions 3.1.13 in [[9](#_bookmark28)]). For this representation we have the type conversion lemma (Theorem 3.3.15 in in [[9](#_bookmark28)]).

Lemma 4.3 (type conversion) *For every representation δ of the space C*˜*, the function* eval: (*F, h*) '→ *F* (*h*) *is* (*δ, δC, ρ*)*-computable, iff δ* ≤ [*δC* → *ρ*]*.*

Since the dulal *C*'[0; 1] is a subset of *C*˜, we can use the representation [*δC* → *ρ*] for it. The norm   : *C*'[0; 1] → R is ([*δC* → *ρ*]*, ρ<*)-computable (a *ρ<*-name of *x* ∈ R is an (encoded) increasing sequence of rational numbers converging to *x* [[9](#_bookmark28)]). The multi-function UB : *C*'[0; 1] ⇒ R, *a* ∈ UB(*F* ) ⇐⇒  *F * *< a*,

is ([*δC* → *ρ*]*, ρ*)-computable. But the norm is not ([*δC* → *ρ*]*, ρ*)-computable [[1](#_bookmark22)]

since the space (*C*'[0; 1]*, *  ) is not separable [[4](#_bookmark25)].

For the set B = {*m* | *m* : N → N} we consider the representation *δ*B defined by *δ*B(*p*) = *m*, iff *p* = 1*m*(0)01*m*(1)01*m*(2)0 *...* . By Lemma 6.2.7 in [[9](#_bookmark28)], a modulus of continuity *m* can be computed for every function *h* ∈ *C*[0; 1]:

Lemma 4.4 *The multi-function* MC : *C*[0; 1] ⇒ B *such that m* ∈ MC(*h*) *iff*

*m* : N → N *is a uniform modulus of continuity of h* : [0; 1] → R *is* (*δC, δ*B)*-*

*computable.*

Finally, for the set BV[0; 1] of functions *g* : [0; 1] → R of bounded variation we define a multi-representation *δ*BV by *g* ∈ *δ*BV(*p*) iff *p* =

⟨*r*0*, r*1*, p*0*, q*0*, p*1*, q*1*,.. .*⟩ such that

*g*(0) = *ρ*(*r*0)*, g*(1) = *ρ*(*r*1)*,*

{*ρ*(*pi*) | *i* ∈ N} is dense in [0; 1]*, gρ*(*pi*)= *ρ*(*qi*) for *i* ∈ N *.*

Remember that by Lemma [2.1](#_bookmark2) the values of *g* on a dense set are sufficient to approximate *f dg* for continuous *f* .

∫

# The Main Results

∫

First, we show that Riemann-Stieltjes integration *h dg* is computable in *h* and *g*. As an additional information for the computation we use some upper bound of *V* (*g*), the variation of *g*.

Theorem 5.1 *Deﬁne the operator S* : ⊆BV[0; 1]×R → *C*'[0; 1] *by* dom(*S*) :=

∫

{(*g, b*) | *V* (*g*) *< b*} *and and S*(*g, b*)(*h*)= *h dg for all h* ∈ *C*[0; 1]*. Then S is*

(*δ*BV*, ρ,* [*δC* → *ρ*])*-computable.*

∫

Proof: First we show how *h dg* can be computed from *g, b* and *h*. We assume that the function *g* is given by some *δ*BV-name *p* = ⟨*r*0*, r*1*, p*0*, q*0*, p*1*, q*1*,.. .*⟩, the bound *b* by some *ρ*-name and the continuous functionb *h* by some *δC*-name. For *h* we can compute some uniform modulus *m* of continuity (Theorem 6.2.7 in [[9](#_bookmark28)]). ¿From *b* we can compute some *l* ∈ N such that *b* ≤ 2*l*. ¿From *g, k* and *l* we can compute points

(*x*0*, y*0)*,* (*x*1*, y*1)*,... ,* (*xn, yn*) ∈ graph(*g*) such that *π* = (*x*0*, x*1*,... , xn*) is a partition of precision *m*(*k* +1+ *l*). For the selection *T* := (*x*1*,... , xn*) for *π*

according to ([3](#_bookmark1)) we can compute

*n*

Σ

*S*(*g, h, Z, T* ) := *f* (*xi*)(*yi* − *yi*−1)*.*

*i*=1

By Lemma [2.1](#_bookmark2),

*S*(*g, h, Z, T* ) − ∫ *h dg* ≤ 2−*k*−*lV* (*g*) ≤ 2−*k*−*lb* ≤ 2−*k .*

Therefore, from *g, b* and *h* we can compute a sequence (*zk*)*k*∈N of real numbers such that |*zk* − *h dg*| ≤ 2−*k*. Since the limit o∫f such sequences is

∫

computable (Theorem 4.3.7 in [[9](#_bookmark28)]) the function (*g, b, h*) '→

*h dg* for *V* (*g*) ≤ *b*

is (*δ*BV*, ρ, δC, ρ*)-computable. By type conversion, Theorem 3.3.15 in [[9](#_bookmark28)], the operator *S* is (*δ*BV*, ρ,* [*δC* → *ρ*])-computable.

Theorem 5.2 *Deﬁne the operator S*' : ⊆*C*'[0; 1] × R ⇒ BV[0; 1] *by*

∫

*g* ∈ *S*'(*F, c*)*, iff c* = *F * = *V* (*g*) *and F* (*h*)= *h dg for all h* ∈ *C*[0; 1]*. Then*

*S*' *is* ([*δC* → *ρ*]*, ρ, δ*BV)*-computable.*

Proof: We assume that *F* is given by some [*δC* → *ρ*]-name and *c* by some *ρ*-name. We want to compute some *δ*BV-name *p* = ⟨*r*0*, r*1*, p*0*, q*0*, p*1*, q*1*,.. .*⟩ of some appropriate function *g*. Since by Lemma [4.3](#_bookmark21) (*F, h*) '→ *F* (*h*)

is computable, the function, mapping each approximate partition *π* =

(*a*1*, b*1*,... , an, bn*) to Σ*n* |*F* (*fi*)|, see Section [3](#_bookmark3), is computable. Since ex-

*i*=0

istence is guaranteed by Lemma [3.2](#_bookmark7), for each interval *J* with rational end points and for each *k* by exhaustive search some approximate partiton *π* can be computed such that

*n*

Σ

 *F * − 2−*k <* |*F* (*fi*)|≤ *F * *,* (44)

*i*=0

(∀ *i,* 1 ≤ *i* ≤ *n*) *bi* − *ai <* 2−*k* (45)

and (∃ *i,* 1 ≤ *i* ≤ *n*) [*ai*; *bi*]⊆*J.* (46)

Since existence is guaranteed by Lemma [3.4](#_bookmark12), For each *m* a sequence (*πk*)*k*∈N ,

*πk* = (*ak, bk, ak, bk,... , ak , bk*

), of approximate partitions, a sequence (*ik*)*k*∈N,

1 1 2 2 *nk nk*

1 ≤ *ik* ≤ *nk*, of indices and a sequence (*sk*)*k*∈N of slanted steps can be computed

such that for all *k*,

*nk*

*F * − 2 *<* Σ |*F* (*f* )|≤ *F * *,*

−*k k*

*i*

*i*=0

(∀ *i*) *bk* − *ak <* 2−*k ,*

*i* *i*

(*a*0 ; *b*0 )⊆*Km ,*

*i*0 *i*0

[*ak*+1 ; *bk*+1 ] ⊆ (*ak* ; *bk* )*/*3

*ik*+1

*ik*+1

*ik ik*

*v*(*sk*)⊆(*ak* ; *bk* )*/*3 *.*

*ik ik*

Then according to Lemma [3.5](#_bookmark13) and Definition [3.6](#_bookmark14) numbers *xKi* and *yKi* can be computed.

Therefore, from *F* and *c* = *F * sets

*G*0 := {(*xKi, yKi* ) | *i* ∈ N} *,*

*G*' := *G*0 ∪ {(0*,F* (0))*,* (1*,F* (1I))}

can be computed such that Lemmas [3.7](#_bookmark16) holds true. Computing means to find *r*0*, r*1*, pi, qi* ∈ Σ*ω* such that *ρ*(*r*0) = 0, *ρ*(*r*1) = *F* (1I), *ρ*(*pi*) = *xKi* and *ρ*(*qi*)= *yKi* . Then for any function *g* : [0; 1] → R of bounded variation which extends *g*',

*g* ∈ *δ*BV(*p*)*, p* := ⟨*r*0*, r*1*, p*0*, q*0*, p*1*, q*1*,.. .*⟩

There is an extension *g*[0; 1] → R of *g*' such that *V* (*g*) = *V* (*g*') = *F *. For *x* ∈ [0; 1] \ dom(*g*') define *g*(*x*) := lim{*g*'(*x*') | *x*' *< x*}. By Lemma [3.8](#_bookmark17), *F* (*h*)= *h dg* for all *h* ∈ *C*[0; 1].

∫

Therefore, the operator *S*' is ([*δC* → *ρ*]*, ρ, δ*BV)-computable.

The above proof uses the norm of *F* explicitly. As we have already men- tioned in Section [4](#_bookmark19), *F * cannot be computed from *F* .

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