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Conic optimization: A survey with special focus on copositive optimization and binary quadratic problems

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A conic optimization problem is a problem involving a constraint that the optimization variable be in some closed convex cone. Prominent examples are linear programs (LP), second order cone programs (SOCP), semidefinite problems (SDP), and copositive problems. We survey recent progress made in this area. In particular, we highlight the connections between nonconvex quadratic problems, binary quadratic problems, and copositive optimization. We review how tight bounds can be obtained by relaxing the copositivity constraint to semidefiniteness, and we discuss the effect that different modelling techniques have on the quality of the bounds. We also provide some new techniques for lifting linear constraints and show how these can be used for stable set and coloring relaxations.

# Introduction

where *𝐶, 𝑋, 𝐴𝑖* are matrices (or vectors) of suitable dimension, and

*𝑏* ∈ ℝ for all *𝑖* = 1*,* … *, 𝑚*. In case of matrices, ⋅*,* ⋅

⟨ ⟩

denotes the Frobe-

*𝑖*

∶= ( *𝑇* )

⟨ ⟩

A conic optimization problem is a problem involving a constraint that the optimization variable be in some closed convex cone. The field of conic optimization is a broad one, as any convex optimization prob- lem can be cast as a conic problem, see [Nesterov and Nemirovski (1992)](#_bookmark78). In this paper, we will focus on more specific conic problems which ap- pear naturally when solving quadratic or combinatorial optimization problems. In particular, we will highlight developments in second or- der cone programming (SOCP), semidefinite programming (SDP), and copositive optimization.

* 1. *The general linear conic problem and its dual*

Consider a proper cone G, i.e., a closed convex and full dimensional cone which is also pointed, meaning that G does not contain a straight

line, or equivalently, that G ∩ (−G) = {0}. For example, when consider-

nius inner product *𝐴, 𝐵* trace *𝐴 𝐵* , in case of vectors, it denotes

the Euclidean inner product. Problem [(P)](#_bookmark2) therefore aims to minimize

a linear function over the intersection of a proper cone and an aﬃne subspace.

As in linear programming, a primal problem of the form [(P)](#_bookmark2) always comes with a dual problem which involves the dual cone: given an ar-

G G

bitrary cone *⊆* ℝ*𝑚*×*𝑛*, the dual cone ∗ is defined as

G∗ ∶= {*𝑋* ∈ ℝ*𝑚*×*𝑛* ∣ *𝑋, 𝐾* ≥ 0 for all *𝐾* ∈ G}*.*

⟨ ⟩

As usual, the Lagrangian function *𝐿* ∶ G × ℝ*𝑚* → ℝ is defined as

*𝑚*

∑

*𝐿*(*𝑋, 𝑦*) ∶= *𝐶, 𝑋* + *𝑦* (*𝑏* − *𝐴 , 𝑋* )*.*

⟨ ⟩ *𝑖 𝑖* ⟨ *𝑖* ⟩

*𝑖*=1

This gives the dual problem

orthant = ℝ*𝑛* ∶= {*𝑥* ∈ ℝ*𝑛* ∣ *𝑥* 0}. In this paper, we will mostly con- ing linear optimization problems, the cone involved is the nonnegative

G ≥

G

+

sider optimization problems in matrix variables, in which case is a

*𝑦*∈ℝ*𝑚 𝑋*∈G

*𝑦*∈ℝ*𝑚* [⟨

⟩ *𝑋*∈G⟨

∑*𝑚*

*𝑖 𝑖* ⟩]

S For the inner minimization problem to be finite, we require that

max min *𝐿*(*𝑋, 𝑦*) = max

*𝑏, 𝑦* + min *𝐶* −

*𝑖*=1

*𝑦 𝐴 , 𝑋*

*.*

proper cone in the set

*𝑦 𝐴 , 𝑋*⟩

*𝑖*=1

*𝑖*

*𝑖*

*𝑖*

*𝑛* of symmetric matrices. For this reason, we will

⟨*𝐶* − ∑*𝑚*

≥ 0 for all *𝑋* ∈ G, in other words, we require *𝐶* −

use capital letters for the data matrices and variables, and we say that a

linear conic optimization problem over G is a problem of the form

*𝑖*

∑*𝑚*

*𝑦 𝐴* ∈ G∗. Therefore, we arrive at the dual problem

*𝑝*∗ = min *𝐶, 𝑋*

⟨ ⟩

*𝑖*=1

= *𝑏*

*𝑑*∗ = max

*𝑏, 𝑦*

∑*𝑚*

⟨

*𝑦 𝐴* + *𝑍* = *𝐶*

(D)

*𝑍* ∈ G *, 𝑦* ∈ ℝ *.*

*𝑖*=1

⟩

s.t. ⟨*𝐴𝑖, 𝑋*G⟩ *𝑖*

s.t.

*𝑋* ∈

*,*

(P)

*𝑖 𝑖*

∗

*𝑚*

∗ Corresponding author.

(*𝑖* = 1*,* … *, 𝑚*)

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– *𝑏, 𝑦* equals the inner *1.2. LP, SOCP, and SDP*

It is easy to see that the duality gap ⟨*𝐶, 𝑋*⟩

⟨

– *𝑏, 𝑦*

=

⟩

product≥of the primal and dual variables: ⟨*𝐶, 𝑋*⟩ ⟨ ⟩ ⟨ G ⟩ G∗ G

*𝑍, 𝑋* . Since

0 for any pair of primal/dual feasible points *𝑋* ∈ *, 𝑍* ∈ , Depending on which cone

⟨*𝑍, 𝑋*⟩

is considered, conic optimization in-

we immediately get weak duality.

+

points *𝑋* ∈ and (*𝑦, 𝑍*) ∈ ℝ*𝑚* × ∗, then *𝑋* is optimal for [(P)](#_bookmark2) and (*𝑦, 𝑍*) Clearly, if the duality gap is zero for a pair of primal/dual feasible

is optimal for [(D)](#_bookmark3). The converse is, however, not true in general: a posi- tive duality gap may exist, or the optimal value of [(P)](#_bookmark2) or [(D)](#_bookmark3) may not be attained. Examples for this phenomenon in second order cone program- [ming can be found in](#_bookmark65) [Alizadeh and Goldfarb (2003)](#_bookmark46) [or in Ben-Tal and Nemirovski (2001, Section 2.4.1). For the SDP case, examples can be](#_bookmark65) found in [Helmberg (2002)](#_bookmark83), and a thorough analysis of this behavior can

cludes various classes of problems: If G = ℝ*𝑛* , then [(P)](#_bookmark2) is a linear prob-

lem, a well studied class which appears in numerous applications. LPs

G G

constraints involving 𝓁1- or 𝓁∞- norms or absolute values. are used to model not only straightforward linear constraints, but also

G

a second order cone problem (SOCP). The second order cone in ℝ*𝑛* If the cone in [(P)](#_bookmark2) is the second order cone, then [(P)](#_bookmark2) is called

*𝑛* ∶= {(*𝑥*0 *, 𝑥*) ∈ ℝ × ℝ*𝑛*−1 ∣ *𝑥*0 ≥ *𝑥* 2 }. It appears in optimization prob-

G(sometimes also called Lorentz cone or ice cream cone) is defined as

lems≤ involving Euclidean norms: for example, theGconstraint ‖*𝐴𝑥*

+

‖ ‖

be found in [Pataki (2019)](#_bookmark88).

*𝑏*‖2

*𝑐𝑇 𝑥*

+ *𝑑* can be written as (*𝑐𝑇*

*𝑥* + *𝑑, 𝐴𝑥* + *𝑏*) ∈

*𝑛*+1

. This is often

In order to get strong duality, we need constraint qualifications:

**Definition 1.1.** A point *𝑋* is called strictly feasible for [(P)](#_bookmark2) if *𝑋* is feasible

used in robust optimization when an ellipsoidal uncertainty set is con- sidered ([Ben-Tal et al., 2009](#_bookmark63)). Other applications of SOCP can be found

in [Alizadeh and Goldfarb (2003)](#_bookmark46), [Lobo et al. (1998)](#_bookmark112). Certain risk mea-

for [(P)](#_bookmark2) and *𝑋* ∈ intG. A pair (*𝑦, 𝑍*) is called strictly feasible for [(D)](#_bookmark3) if

(*𝑦, 𝑍*) is feasible for [(D)](#_bookmark3) and *𝑍* ∈ intG∗. If such points exist, then we say

over the so called *𝑝*-order cone G*𝑝* ∶= [{](#_bookmark107)(*𝑥 , 𝑥*) ∈ ℝ × ℝ ∣ *𝑥* ≥ [‖](#_bookmark74)*𝑥*‖ }

that the problem fulfills the primal (resp. dual) Slater condition.

Note that strict feasibility can always be enforced by consider- ing the so called skew-symmetric embedding of the original problem, see [de Klerk et al. (1997)](#_bookmark91). Assuming strict feasibility gives us strong duality:

*strictly feasible solution* (*𝑦, 𝑍*)*. Then the primal and dual optimal values are* **Theorem 1.2** (Strong Duality Theorem)**.** *Assume that problem* [(*D*)](#_bookmark3) *has a equal: 𝑝*∗ = *𝑑*∗*, and if 𝑝*∗ *<* +∞*, then 𝑝*∗ *is attained, i.e., there exists a primal feasible solution 𝑋*∗ *with 𝑝*∗ = *𝐶, 𝑋 .*

∗

⟨ [⟩](#_bookmark2)

*Conversely, assume that problem* [(*P*)](#_bookmark2) *has a strictly feasible solution 𝑋. Then the primal and dual optimal values are equal: 𝑝*∗ = *𝑑*∗*, and if 𝑑*∗ *>* −∞*,*

with *𝑝* ≥ 1, see e.g. [Vinel and Krokhmal (2014)](#_bookmark107). Any *𝑝*-order cone with

*𝑝* rational can be modeled as SOCP. We refer to [Mosek (2021-10-11)](#_bookmark74) for

sures in stochastic optimization may also lead to optimization problems

*𝑛*

0

*𝑛*−1

0

*𝑝*

details.

A third prominent setting is semidefinite programming (SDP), where is considered to be the cone of symmetric positive semidefinite matri-

G

S

ces + ∶= {*𝑋* ∈ ℝ*𝑛*×*𝑛* ∣ *𝑋* = *𝑋𝑇 , 𝑋* ⪰ 0}. SDPs are used to model prob-

*𝑛*

lems with linear matrix inequalities. They appear in eigenvalue op-

timization and control theory, see [Vandenberghe and Boyd (1996)](#_bookmark106), [Helmberg (2002)](#_bookmark83). Arguably two of the most important areas of applica- tion for SDP are robust and combinatorial optimization. For an in depth [discussion of SDP in robust optimization, we refer to the book (Ben- Tal et al., 2009) and the recent survey paper (](#_bookmark63)[Yan](#_bookmark114)[ıkoğlu et al., 2019).](#_bookmark63) The role of SDP in relaxations of combinatorial problems will be covered in more detail below.

G S G

The cones ℝ*𝑛* , , and + are self-dual, whereas the dual of *𝑝* is

*then 𝑑*∗ *is attained, i.e., there exists a dual feasible solution* (*𝑦*∗*, 𝑍* ∗) *with*

G*𝑞*

+ *𝑛 𝑛*

1 1

*𝑛* G

S+ *𝑛*

*𝑑*∗ = ∗

⟨ ⟩

*𝑛* with *𝑞* such that *𝑝* + *𝑞* = 1. We mention that ℝ+, *𝑛* , and *𝑛* are

*𝑏, 𝑦 .*

A proof of this theorem along with a thorough discussion of conic du- ality can be found for example in [Ben-Tal and Nemirovski (2001)](#_bookmark65). It has been shown in [Dür et al. (2017)](#_bookmark61) that the Slater condition (and hence strong duality) is a generic property of conic problems which loosely speaking means that Slater’s condition is fulfilled (and hence strong du-

ized in the space of data (*𝐶, 𝐴*1 *, ..., 𝐴𝑚, 𝑏*1 *, ..., 𝑏𝑚* ). We stress that a con- ality holds) for almost all feasible conic problems which are parameter-

G

linear programming, where = ℝ*𝑛* . The reason why a positive duality straint qualification is unnecessary if the cone is polyhedral like in

+

G

gap may occur in general conic programming lies in the geometry of the problem and happens if the feasible set is contained in a nontrivial face of the cone.

Note that existence of strictly feasible points is important not only for theoretical purposes to ensure strong duality, but also many optimiza- tion algorithms require this property. In the absence of strictly feasible points, a solver may not terminate or may produce a “solution” with no useful meaning. This is a feature that distinguishes general conic op- timization from linear programming. Consequently, very careful mod- elling is needed, since often existence of strictly feasible points can be guaranteed if the problem is modelled in a proper way. We will return to this point in [Section 5](#_bookmark22).

Two approaches have been developed to tackle conic problems that fail to fulfill a constraint qualification: (i) Facial reduction attempts to

T

identify the so called minimal cone min for problem [(P)](#_bookmark2), such that prob-

instances of so called symmetric cones that can be studied in the unifying framework of Euclidean Jordan algebras, see [Faybusovich (2008)](#_bookmark67) and references therein.

These three problem classes have been studied for decades because of their countless applications and because they can be solved eﬃciently: it has been shown in the vast literature on interior point methods pi- oneered by [Nesterov and Nemirovskii (1994)](#_bookmark80) (see also [Renegar, 2001](#_bookmark100)) that these algorithms are able to solve LPs, SOCPs, and SDPs in polyno- mial time. A different class of algorithms that solves SDPs is conic bundle methods, see [Helmberg and Rendl (2000)](#_bookmark85), [Helmberg et al. (2014)](#_bookmark84). Nu- merous software implementations are available, and we refer the reader to Hans Mittelmann’s website ([Mittelmann, 2021](#_bookmark121)) for an up-to-date list of the various packages.

*1.3. Variants of SDP and SOCP*

So far, we discussed linear conic optimization problems. However, the enormous modelling power of semidefinite and second order cone programming only unfolds if we allow for nonlinearities or integer vari- ables:

*Mixed integer conic optimization problems* are linear conic problems with a constraint that some of the variables are integer valued:

min *𝐶, 𝑋*

⟩

s.t. ⟨*𝐴𝑖 , 𝑋*⟩ = *𝑏𝑖* (*𝑖* = 1*,* … *, 𝑚*)

*𝑋* ∈ G*,*

lem [(P)](#_bookmark2) with G replaced by Tmin is strictly feasible and has the same opti-

[mal solution as](#_bookmark30) [(P)](#_bookmark2)[. This facial reduction technique goes back to Borwein](#_bookmark30)

[and Wolkowicz (1981a,b). (ii) Other approaches (e.g.](#_bookmark30) [Ramana,](#_bookmark98) [1997)](#_bookmark30) work on the dual side and construct an extended dual which achieves strong duality without assuming a constraint qualification. A good ex- position of these two approaches can be found in [Pataki (2013)](#_bookmark87).

*𝑋𝑖𝑗* ∈ ℤ ((*𝑖, 𝑗*) ∈ *𝐽* )*.*

Sometimes binary constraints *𝑋𝑖𝑗* ∈ {0*,* 1} for (*𝑖, 𝑗*) ∈ *𝐽* are used instead.

G S

constraint, mostly a semidefiniteness constraint ( = +) or an SOCP *Nonlinear conic problems* are nonlinear problems that involve a cone constraint ( = *𝑛*). Naturally, mixed integer nonlinear conic problems

*𝑛*

G G

have been studied likewise.

It would be beyond the scope of this paper to discuss the develop- ment in mixed integer nonlinear conic optimization here. We mention just a few applications:

[Nonlinear SOCPs appear for example in facility location (Chen et al., 2011). Mixed integer SOCPs appear in engineering (e.g. turbine bal-](#_bookmark39) ancing problems), in service system design ([Góez and Anjos, 2019](#_bookmark75)), in finance (e.g. cardinality-constrained portfolio optimization), or in combinatorial problems like the Euclidean Steiner Tree Problem, see ([Gally et al., 2018](#_bookmark70)) and references therein. Solution approaches [for these problems include semismooth Newton methods (Chen et al., 2011), outer approximation algorithms (](#_bookmark39)[Drewes](#_bookmark57) [and Ulbrich, 2012),](#_bookmark39) [cutting plane algorithms (Atamtürk and Narayanan, 2007; Drewes and Pokutta, 2014; Kobayashi and Takano, 2020), and Branch-and-Bound](#_bookmark58) algorithms ([Gally et al., 2018](#_bookmark70)).

Mixed integer SDPs have applications in truss topology optimiza- [tion (](#_bookmark48)[Gally et al., 2018](#_bookmark70)[), in certain clustering problems (Aloise and](#_bookmark48) [Hansen,](#_bookmark109) [2009), or in sparse principal component analysis (Li and](#_bookmark48) [Xie, 2020). References to numerous fields of application of nonlinear](#_bookmark109) SDPs in engineering, (robust) control, finance and others can be found in [Andreani et al. (2020)](#_bookmark50) and [Yamashita and Yabe (2015)](#_bookmark110).

Solution algorithms for these problems include augmented La- grangian methods, sequential SDP methods, and primal-dual inte- [rior point methods, see e.g.,](#_bookmark95) [Andreani et al. (2020)](#_bookmark50)[, Kocvara and Stingl (2012),](#_bookmark95) [Yamashita](#_bookmark110) [and Yabe (2015). For pointers to soft-](#_bookmark95) ware implementations, we refer again to Hans Mittelmann’s web- site ([Mittelmann, 2021](#_bookmark121)).

# Conic reformulations of quadratic problems

optimization problems. [Bomze et al. (2000)](#_bookmark31) introduced the term “copos- itive optimization” and showed for the first time equivalence of a non- convex quadratic optimization problem and a linear problem over resp. : They considered standard quadratic optimization problems,

i.e., nonconvex quadratic problems over the standard simplex Δ ∶= {*𝑥* ∈

CP

CUP

ℝ*𝑛* ∣ *𝑒𝑇 𝑥* = 1} where *𝑒* ∈ ℝ*𝑛* denotes the all-ones vector. Given a symmet- ric matrix *𝑄* ∈ ℝ*𝑛*×*𝑛*, a standard quadratic problem is of the form

+

min *𝑥𝑇 𝑄𝑥*

s.t. *𝑒𝑇 𝑥* = 1 (StQP)

*𝑥* ≥ 0*.*

In spite of its simple structure, ([StQP](#_bookmark5)) is an NP-hard problem if *𝑄* has a

negative eigenvalue, see [Pardalos and Vavasis (1991)](#_bookmark85). Alternatively, NP-

problem can be formulated as ([StQP](#_bookmark5)): consider a graph with *𝑛* vertices. hardness of ([StQP](#_bookmark5)) can be seen from the fact that the maximum clique Denote its adjacency matrix by *𝐴*, its clique number by *𝜔*, and define

*𝐽* ∶= *𝑒𝑒𝑇* . It was shown by [Motzkin and Straus (1965)](#_bookmark76) that

1 = min{*𝑥𝑇* (*𝐽* − *𝐴*)*𝑥* ∶ *𝑥* ∈ Δ}*.* (2)

*𝜔*

The max clique problem is a particularly diﬃcult NP-hard problem, and even computing an approximation of any reasonable quality is NP- hard ([Håstad, 1999](#_bookmark82)). We will see below how copositive optimization can be used to tackle this and other NP-hard problems.

By squaring the constraint in ([StQP](#_bookmark5)) and applying the lifting trans- formation outlined in [(1)](#_bookmark8), it is easy to see that the following problem is a relaxation of ([StQP](#_bookmark5)):

min *𝑄, 𝑋*

s.t. ⟨*𝐽 , 𝑋*⟩⟩ = 1

Conic optimization problems play a particularly fruitful role in the theory of quadratic and binary quadratic optimization problems. This is accomplished by a technique called lifting, which was pioneered

*𝑋* ∈ CP *,*

as is its dual problem

(3)

seen as follows: consider a quadratic expression *𝑥𝑇 𝑄𝑥* with a symmetric by [Shor (1987)](#_bookmark101) and [Lovász and Schrijver (1991)](#_bookmark116). The main idea can be matrix *𝑄* ∈ ℝ*𝑛*×*𝑛* and *𝑥* ∈ ℝ*𝑛*. If we introduce a new variable *𝑋* ∈ ℝ*𝑛*×*𝑛* to represent the rank-1 matrix *𝑥𝑥𝑇* , then we get

*𝑥𝑇 𝑄𝑥* = trace(*𝑥𝑇 𝑄𝑥*) = trace(*𝑄𝑥𝑥𝑇* ) = *𝑄, 𝑥𝑥𝑇* = *𝑄, 𝑋 .* (1)

⟨ ⟩ ⟨ ⟩

By this technique, quadratic terms in *𝑥* ∈ ℝ*𝑛* become linear terms in

*𝑋* ∈ ℝ*𝑛*×*𝑛*. Since many optimization problems considered in the sequel contain nonnegativity constraints *𝑥* 0, this leads to the definition of a

≥

convex matrix cone that turns out very useful for modelling purposes: The cone of completely positive matrices is defined as

max{*𝜆* ∈ ℝ ∣ *𝜆𝐽* − *𝑄* ∈ CUP}*.* (4)

It is easy to verify that Slater’s condition and hence strong duality holds for [(3)](#_bookmark6) and [(4)](#_bookmark7). Since the objective function of [(3)](#_bookmark6) is linear and the fea- sible set is convex, it follows that the optimal solution is attained at an extreme point of the feasible set, which can be shown to be the matri-

ces of the form *𝑥𝑥𝑇* with *𝑥* ∈ Δ, cf. [Bomze et al. (2000)](#_bookmark31). This implies

that [(3)](#_bookmark6) and [(4)](#_bookmark7) are not merely relaxations but exact reformulations of

The optimal solutions of ([StQP](#_bookmark5)) and [(3)](#_bookmark6) fulfill the following relation: if *𝑥*∗ ([StQP](#_bookmark5)) in the sense that all three problems have the same optimal value.

is optimal for ([StQP](#_bookmark5)), then *𝑋*∗ ∶= *𝑥*∗(*𝑥*∗)*𝑇* is optimal for [(3)](#_bookmark6). Conversely,

## CP ≥

if *𝑋*∗ is optimal for [(3)](#_bookmark6), then it can be decomposed as *𝑋*∗ = ∑*𝑝 𝑥𝑖* (*𝑥𝑖* )*𝑇*

∶= conv{*𝑥𝑥𝑇* ∣ *𝑥*

0}*,*

for some *𝑥𝑖* ∈ ℝ*𝑛* (*𝑖* = 1*,* … *, 𝑝*). Then each *𝑥*∗ ∶=

1

*𝑒𝑇 𝑥𝑖*

*𝑖*=1

*𝑥𝑖* is an optimal so-

and its dual cone, the cone of copositive matrices, is defined as

CUP ∶= {*𝑋* ∈ S ∣ *𝑧𝑇 𝑋𝑧* ≥ 0 for all *𝑧* ≥ 0}*.*

For the ease of notation, we will omit the index *𝑛* in notations like CUP

CUP

lution of ([StQP](#_bookmark5)).

These reformulations [(3)](#_bookmark6) and [(4)](#_bookmark7) are interesting because they show that the NP-hard ([StQP](#_bookmark5)) can be reformulated equivalently as a linear problem over the convex cones or . In these formulations, all lo-

+

CP CUP

*𝑖*

or CP

*𝑛* unless it is necessary to stress the dimension. Both

*𝑛*

and

cal minima vanish, and the complexity of the problem is entirely moved

CP are proper cones and have been studied for decades in the linear algebra literature, see [Berman and Shaked-Monderer (2003)](#_bookmark71) and refer- ences therein. They have numerous interesting properties but are still not fully understood, cf. [Berman et al. (2015)](#_bookmark68). Note that the two cones are given in different form: is given by its extreme rays which are

[CP](#_bookmark68)

≥ CUP

precisely the rank-1 matrices *𝑥𝑥𝑇* with *𝑥* 0, whereas is given as

the solution set of (infinitely many) inequalities. This fact plays a role

acterization of the extremal rays of has only been given for *𝑛* 6, when considering approximations of these cones, see [Section 3](#_bookmark11). A char-

*𝑛*

CUP ≤

into the cone constraint. This indicates that CP and CUP must be in- tractable. Indeed, it was shown in [Dickinson and Gijben (2014)](#_bookmark49), that checking membership in is NP-hard. Checking membership in

is co-NP-complete, as was shown in [Murty and Kabadi (1987)](#_bookmark77). Whether or not checking membership in is also in NP is still one of the many open problems related to these cones, cf. [Berman et al. (2015)](#_bookmark68).

CP

CP CUP

cency matrix *𝐴*, it follows from [Bomze et al. (2000)](#_bookmark31) that the clique num- Returning to the maximum clique problem for a graph with adja- ber *𝜔* equals the optimal value of the following copositive problem:

cf. [Afonin et al. (2021)](#_bookmark41). Likewise, only limited knowledge is available about the facial structure of and , cf. [Dickinson (2011)](#_bookmark43).

CP CUP

The earliest use of these cones in optimization was a paper by [Preisig (1996)](#_bookmark95) who studied a particular fractional quadratic problem, and by [Quist et al. (1998)](#_bookmark96), who were the first to introduce a conic op- timization perspective while deriving relaxations for general quadratic

*𝜔* = min{*𝜆* ∣ *𝜆*(*𝐽* − *𝐴*) − *𝐽* ∈ CUP}*.*

Many other graph parameters have a representation as a copositive or completely positive problem. We refer to [Dür (2010)](#_bookmark59) for references.

Copositive optimization experienced a breakthrough with Burer’s 2009 paper [Burer (2009)](#_bookmark35). He showed that every quadratic problem with

linear and binary constraints can be rewritten as such a problem. More precisely, he showed that a quadratic binary problem of the form

min *𝑥𝑇 𝑄𝑥* + 2*𝑐𝑇 𝑥*

It goes without saying that generalized completely positive and copositive cones are even harder to work with than or . The ap- peal of the formulations discussed above lies in the fact that by this tech-

s.t. *𝑎𝑇 𝑥* = *𝑏𝑖*

≥

*𝑥* 0

*𝑖*

(*𝑖* = 1*,* … *, 𝑚*)

(5)

nique diﬃcult, NP-hard problems can be reformulated as linear prob-

lems over a convex cone. Hence these reformulations are convex prob- lems which do not possess local minima, and the hardness of the problem

CP CUP

*𝑥𝑗* ∈ {0*,* 1} (*𝑗* ∈ *𝐵*)

with *𝑄* ∈ S , *𝑐, 𝑎* ∈ ℝ*𝑛* (*𝑖* = 1*,* … *, 𝑚*), *𝑏* ∈ ℝ*𝑚*, and *𝐵 ⊆* {1*,* … *, 𝑛*} can

is completely captured by the cone constraint. Therefore, any progress made in understanding the cones can be used to help solving a whole

*𝑛 𝑖*

equivalently be reformulated as the following completely positive prob- lem:

min *𝑄, 𝑋* + 2*𝑐𝑇 𝑥*

s.t. *𝑎𝑇 𝑥* = *𝑏𝑖* (*𝑖* = 1*,* … *, 𝑚*)

⟨ ⟩

range of different problems. As a first step, the approximation schemes for and discussed in [Section 3](#_bookmark11) can be extended to G and

. However, more research on algorithmic approaches for problems

CUP CP CUP

CP G

over generalized copositive and completely positive cones is needed to

make these approaches work numerically for bigger problems.

⟨*𝑎𝑖 𝑎𝑖 , 𝑋*⟩

*𝑖*

*𝑇*

=

2

( = 1 …

)

*𝑏𝑖 𝑖*

*, , 𝑚*

(6)

*𝑥𝑗* = *𝑋𝑗𝑗* (*𝑗* ∈ *𝐵*)

( ) ∈ *,*

*𝑋 𝑥* CP

*𝑥𝑇* 1

provided that [(5)](#_bookmark9) satisfies the so-called key condition, i.e., *𝑎𝑇 𝑥* = *𝑏𝑖* for

≤ 1 for all ∈

* 1. *Extensions: polynomial optimization and infinite dimensional conic problems*

It is easy to see that any polynomial optimization problem can be

all *𝑖* and *𝑥* ≥ 0 implies

*𝑥𝑗*

*𝑖*

*𝑗 𝐵*. As noted by Burer, this condi-

rewritten as a quadratic problem by introducing extra variables and

tion can be enforced without loss of generality. Doing so may, however, [have consequences when relaxing the cone constraint (Bomze et al., 2017; Bomze and Jarre, 2010; Jarre, 2012).](#_bookmark28)

Similar techniques can be used to derive copositive or completely

replacing the constraint *𝑥* ∈ ℝ*𝑛* in [(5)](#_bookmark9) by other closed convex cones. positive formulations for problems involving quadratic constraints or

+

*completely positive cones*: given a closed convex cone *⊂* ℝ*𝑛*, one can This leads to reformulations involving so called *generalized copositive and*

G

define

CPG ∶= conv{*𝑥𝑥𝑇* ∣ *𝑥* ∈ G}*,*

and its dual cone of generalized copositive matrices

CUPG ∶= {*𝑋* ∈ S ∣ *𝑧𝑇 𝑋𝑧* ≥ 0 for all *𝑧* ∈ G}*.*

These cones were introduced in [Quist et al. (1998)](#_bookmark96) and studied also in [Eichfelder and Jahn (2008)](#_bookmark64). As shown by [Burer (2012)](#_bookmark36), the problem where the nonnegativity constraints in [(5)](#_bookmark9) are replaced by the constraint

G

*𝑥* ∈ is (under mild conditions) equivalent to a linear conic program

G

over the cone of matrices which are completely positive over ℝ+ × , i.e., over the cone conv{*𝑦𝑦𝑇* ∣ *𝑦* ∈ ℝ+ × }. [Eichfelder and Povh (2013)](#_bookmark66),

[G](#_bookmark66)

G

[Dickinson et al. (2013)](#_bookmark47) generalize this even more to the case where is an arbitrary set, and they also give a formulation for problems involving one quadratic constraint.

When considering quadratic constraints, reformulating the problem as a conic problem becomes more involved. [Burer (2009)](#_bookmark35) already con- sidered certain special cases, namely binary constraints (which can

be viewed as quadratic equations *𝑥*2 = *𝑥𝑖* ), and complementarity con-

*𝑖*

straints. For more general quadratically constrained quadratic problems,

similar reformulations have been obtained: consider a quadratically con- strained quadratic problem of the form

constraints. For example, by defining an extra variable and constraint

*𝑦* = *𝑥𝑗 𝑥𝑘* , the cubic term *𝑥𝑖 𝑥𝑗 𝑥𝑘* becomes the quadratic term *𝑥𝑖 𝑦*. So the

reformulations discussed above can in principle be applied to polyno-

mial problems, as well. A different line of research has worked with the *cone of completely positive tensors*. This concept was originally intro-

CP

[Peña et al. (2015)](#_bookmark92) consider optimization problems involving *𝑛*-variate duced by [Dong (2013)](#_bookmark54) and consititutes another natural extension of .

polynomials and show that under certain assumptions these can be re-

sors of order *𝑑* and dimension *𝑛* + 1. They also show that in case of a formulated as linear problems over the cone of completely positive ten- compact feasible set order *𝑑* = 4 is suﬃcient. The approach has been ex-

tended in [Kuang and Zuluaga (2018)](#_bookmark97), [Xia and Zuluaga (2017)](#_bookmark109). We also refer to [Anjos and Lasserre (2012)](#_bookmark55) for more discussion on the connection between conic and polynomial optimization.

alently, the stable set problem) on a graph with vertex set *𝑉* = {1*,* … *, 𝑛*} We mentioned above that the maximum clique problem (and, equiv-

can be formulated as an ([StQP](#_bookmark5)) and consequently as a copositive or com- pletely positive problem [(3)](#_bookmark6) and [(4)](#_bookmark7). This can be generalized to infinite

metric space *𝑉* equipped with a probability measure *𝜇*. The problem of graphs, i.e., to the setting where the vertex set is not finite but a compact

determining the stability number of an infinite graph appears e.g. in the kissing number problem ([Dobre et al., 2016](#_bookmark51)) and other packing prob- lems, see [DeCorte et al. (2021)](#_bookmark44). [Dobre et al. (2016)](#_bookmark51) generalized the con- cept of copositive matrices to the infinite dimensional setting by defining

*copositive kernels*: a kernel is a continuous function *𝐾* ∶ *𝑉* × *𝑉* → ℝ. Such

a kernel *𝐾* is called a *copositive kernel* if for all continuous nonnegative functions *𝑓* ∶ *𝑉* → ℝ+ we have

*𝐾*(*𝑥, 𝑦*)*𝑓* (*𝑥*)*𝑓* (*𝑦*)*𝑑𝜇*(*𝑥*)*𝑑𝜇*(*𝑦*) ≥ 0*.*

∫

∫

*𝑉 𝑉*

min *𝑥𝑇 𝑄*0*𝑥* + 2(*𝑐*0)*𝑇 𝑥*

≤

s.t. *𝑥𝑇 𝑄𝑖𝑥* + 2(*𝑐𝑖*)*𝑇 𝑥 𝑏𝑖*

(*𝑖* = 1*,* … *, 𝑚*) (QCQP)

choice of *𝜇*. This cone as well as its dual (the cone of completely positive It can be shown that the cone of copositive kernels is independent of the

measures) can be used to derive exact copositive and completely posi-

[with *𝑄𝑖* ∈](#_bookmark37) S [, *𝑐𝑖* ∈ ℝ*𝑛* (*𝑖* = 0*,* … *, 𝑚*), and *𝑏* ∈ ℝ*𝑚*. Burer and](#_bookmark37)

[*𝑛*](#_bookmark37)

[Dong (2012) show two different ways of formulating a (](#_bookmark37)[QCQP](#_bookmark10)[) as](#_bookmark37)

G CP

is a direct product of ℝ*𝑛* and second-order cones, and another where such a generalized completely positive problem over G: one where is the direct product of ℝ*𝑛* and semidefinite cones (viewed as vectors

+

G

+

by stacking the columns on top of each other). [Bai et al. (2016)](#_bookmark60) and [Arima et al. (2013)](#_bookmark56) derive similar formulations under milder assump- tions.

CP

We remark that a different generalization of is the cone of com- pletely positive semidefinite matrices, i.e., the cone consisting of all

*𝑛* × *𝑛* matrices that admit a Gram representation by positive semidef-

inite matrices. This cone appears when studying quantum analogues

of graph parameters like the stability or chromatic numbers. We refer to [Laurent and Piovesan (2015)](#_bookmark108) for an in-depth discussion.

tive reformulations of the stability number problem for infinite graphs, see [Dobre et al. (2016)](#_bookmark51). Since these cones are intractable, approxima- tions have been proposed in [Kuryatnikova and Vera Lizcano (2017)](#_bookmark100), [Kuryatnikova and Vera (2018)](#_bookmark99). This in turn has been used to derive good bounds for the underlying problems.

# Approximation hierarchies for CUP and CP

Since the cones CUP and CP are computationally intractable, ap- proximations have to be used in order to solve an optimization problem

CP

the cone was by introducing a symmetric matrix *𝑋* to represent the over one of these cones. As outlined in [(1)](#_bookmark8), the motivation to introduce rank-1 matrix *𝑥𝑥𝑇* . So a first straightforward relaxation is to replace the constraint *𝑋* = *𝑥𝑥𝑇* by *𝑋* ⪰ *𝑥𝑥𝑇* (meaning that *𝑋* − *𝑥𝑥𝑇* ∈ +), which by

*𝑛*

S

Schur’s complement lemma is equivalent to rapidly, resulting in problems that are beyond the range of current SDP-

( )

*𝑋 𝑥*

∈ S

+

*𝑥𝑇* 1

*𝑛*+1 *.*

solvers even for moderate values of *𝑟* and *𝑛*.

[Ahmadi and Majumdar (2019)](#_bookmark45) developed a more general theory

for nonnegativity of polynomials which when applied in our context

This relaxation goes back to [Shor (1987)](#_bookmark101) and corresponds to the sim-

boils down to relaxing the sos-condition by requiring that *𝑃𝐴* (*𝑥*) resp.

ple fact that CP *⊆* S+ for any *𝑛*. It is interesting to note that the SDP-

*𝑛*

*𝑛*

*𝑃* (*𝑥*)(∑*𝑛 𝑥*2)*𝑟* has a decomposition as a *sum of squares of binomials*.

relaxation of a quadratic problem corresponds to the Lagrangian dual of that problem, whereas considering partial Lagrangian duals (i.e., du- alizing the problem only with respect to a subset of the constraints) leads to various copositive relaxations, cf. [Bomze (2015)](#_bookmark29). The Shor re- laxation can be improved by adding more constraints to the SDP, or by using some relaxation-linearization techniques, yielding stronger SDP- relaxations. This has been discussed in detail in [Bao et al. (2011)](#_bookmark62).

Shor’s approximation can be strengthened by using better approx- imations to and : Denote by the set of symmetric entry-

*𝑛*

CUP CP 

This is clearly a weaker suﬃcient condition for nonnegativity of *𝑃𝐴* (*𝑥*),

but the advantage is that this condition can be verified by solving an

*𝐴*

*𝑖*=1

*𝑖*

they call *𝑟𝐷𝑆𝑂𝑆𝑛* and *𝑟𝑆 𝐷𝑆𝑂𝑆𝑛* , referring to the *𝑟*th level of the hierar- SOCP. They also consider scaled versions and obtain hierarchies that chies corresponding to *diagonally dominant sum of squares (𝐷𝑆𝑂𝑆)* and *scaled diagonally dominant sum of squares* (*𝑆𝐷𝑆𝑂𝑆*).

An alternative suﬃcient condition for nonnegativity of a polyno- mial is that all of its coeﬃcients are nonnegative. Exploiting this idea, [de Klerk and Pasechnik (2002)](#_bookmark89), cf. also [Bomze and De Klerk (2002)](#_bookmark32),

wise nonnegative *𝑛* × *𝑛* matrices. Then it is obvious from the definition

 CUP  S CUP

that *⊆* . We therefore get that + + *⊆* , and by dual-

*𝑛 𝑛 𝑛 𝑛*

define the cones

{

*𝑃𝐴 𝑥*

( *𝑛*

)*𝑟* }

ity CP

*𝑛 ⊆*

 ∩ S+

*𝑛* ≤ 4

C *𝑟* ∶=

## ∈ S ∣

( ) ∑ 2

and aSre strict for *𝑛* ≥ 5, cf. [Maxfield and Minc (1962/63](#_bookmark120)). Matrices in

*𝑛*

*𝑛* . Interestingly, both inclusions are equalities for *𝑛*

*𝐴*

*𝑥𝑖*

has nonnegative coeﬃcients

*.*

*𝑖*=1

*𝑛* ∩ + are sometimes called doubly nonnegative (both the entries and

*𝑛*

⋃

They showed that

the eigenvalues are nonnegative). This cone has been frequently used to obtain bounds for certain combinatorial problems, with the most promi-

nent case being the Lovász–Schrijver bound *𝜗*+(*𝐺*) (sometimes called

*𝜗*′(*𝐺*)) on the clique number of a graph *𝐺*, see [Schrijver (1979)](#_bookmark102).

CU[P CP](#_bookmark102)

In order to get better approximations of and , a number of

techniques have been developed which often lead to so called approxi- mation hierarchies, i.e., monotonic sequences of inner or outer approx- imations of or which are, in some sense, exact in the limit. The approximating cones are constructed in such a way that optimizing over them amounts to solving an LP, an SOCP, or an SDP, all of which can be done in polynomial time. Several of these hierarchies have been proposed, and we discuss the most important ones next. Note that these hierarchies were originally designed to approximate either or . However, it should be clear that any inner (resp. outer) approximation hierarchy of one cone by duality yields an outer (resp. inner) approxi- mation hierarchy for the dual cone.

CUP CP

CUP CP

* 1. *Inner approximation hierarchies for* CUP

[Parrilo (2000)](#_bookmark86) was the first to propose a hierarchy approximating from the interior. The basic idea is to reformulate the copositivity condition as a nonnegativity condition for certain polynomials, and then to use the suﬃcient condition that a polynomial is nonnegative if it can

CU[P](#_bookmark86)

 = C0 *⊂* C1 *⊂* … *⊂* CUP and int(CUP) *⊆* C *𝑟 .*

*𝑟*∈ℕ

C

Each of the cones *𝑟* is polyhedral, so optimizing over one of them is

solving an LP.

erarchy of cones *𝑟* which in a sense sits between *𝑟* and *𝑟*, i.e., it [Peña et al. (2007)](#_bookmark90) refined the above approaches and derived a hi- fulfills *𝑟 ⊆ 𝑟 ⊆ 𝑟* for all *𝑟* ∈ ℕ. These cones can be described by LMIs

[g](#_bookmark90) C G

C g G

gas well, so optimizing over g*𝑟* is again an SDP, however, optimizing over

*𝑟* provides SDPs of smaller size compared to G*𝑟* .

Each of the above hierarchies provides a uniform inner approxima- tion to , i.e., the approximation quality is independent of which part of the cone is considered. However, this may not be desirable when considering a specific optimization problem over . In this case, one would rather like to obtain a good approximation of in the vicinity of the optimal solutions, whereas in other parts of the feasi- ble set a coarse approximation is suﬃcient. This idea gave rise to the

CUP

CUP

CUP

CUP [CUP](#_bookmark33) S ≥

nition of is equivalent to = {*𝐴* ∈ ∣ *𝑥𝑇 𝐴𝑥* 0 for all *𝑥* ∈ Δ}. approach by [Bundfuss and Dür (2009)](#_bookmark33): It is easy to see that the defi- Now [Bundfuss and Dür (2009)](#_bookmark33) consider partitions = {*𝑆* 1*,* … *, 𝑆 𝑚*} of Δ into subsimplices and give conditions ensuring nonnegativity of *𝑥𝑇 𝐴𝑥* over each *𝑆 𝑖*. Let *𝑆* = conv{*𝑣*1 *,* … *, 𝑣𝑛* } *⊆* Δ be such a simplex. Then *𝑥* ∈ *𝑆* can be written as a convex combination *𝑥* = *𝑛 𝜆𝑖 𝑣𝑖* with *𝑛 𝜆𝑖* = 1

P

*𝑖 𝐴* then means that

and *𝜆* ≥ 0 for all . Copositivity of a matrix *𝑖*=1

*𝑖*=1

∑ ∑

we are given a matrix *𝐴* ∈ S and we would like to determine whether

be represented as a sum of squares (sos) of other polynomials. Suppose

or not *𝐴* ∈ CUP . To this end, consider the polynomial

*𝑛*

*𝑛*

*𝑖*

≤ (∑*𝑛* )*𝑇* (∑*𝑛* ) ∑*𝑛*

0

*𝑥𝑇 𝐴𝑥* =

*𝜆𝑖 𝑣𝑖*

*𝐴*

*𝜆𝑗 𝑣𝑗*

=

(*𝑣𝑇 𝐴𝑣𝑗* )*𝜆𝑖 𝜆𝑗 .* (8)

*𝑖*

*𝑃𝐴* (*𝑥*) ∶=

∑ ∑ *𝑎𝑖𝑗 𝑥*2*𝑥*2 (7)

*𝑖*=1

*𝑗*=1

*𝑖,𝑗*=1

*𝑖*=1 *𝑗*=1

*𝑖 𝑗*

*𝑛*

*𝑛*

Since *𝜆* ≥ 0 by construction, a suﬃcient condition for [(8)](#_bookmark13) is that *𝑣𝑇 𝐴𝑣* ≥

*𝑖 𝑖 𝑗*

and observe that *𝐴* ∈ CUP if and only if *𝑃* (*𝑥*) ≥ 0 for all *𝑥* ∈ ℝ*𝑛*. A

*𝐴*

*𝑛*

0 for all *𝑖, 𝑗*. Note that this constitutes a system of linear inequalities for

the entries of *𝐴*. Therefore,

suﬃcient condition for this is that *𝑃𝐴* (*𝑥*) is sos. Parrilo showed that the

S 

set of matrices *𝐴* for which *𝑃* (*𝑥*) is sos equals + + .

S ∶= {*𝐴* ∈ S ∣ *𝑣𝑇 𝐴𝑣* ≥ 0 for all vertices *𝑣* of simplices in P*,*

*𝐴 𝑛 𝑛* P

[Pólya (1928)](#_bookmark93) and considering higher order polynomials. For any *𝑟* ∈ ℕ, Moreover, he was able to refine this by using a result by

define the cone

(∑

)

}

*𝑢𝑇 𝐴𝑣* ≥ 0 for all edges {*𝑢, 𝑣*} of simplices in P}*.*

[is a polyhedral inner approximation of](#_bookmark33) CUP[. It is shown in Bundfuss and Dür (2009) how the partition P can be refined in order to obtain a se-](#_bookmark33)

CUP

G*𝑟* ∶=

{*𝐴* ∈ S

∣ *𝑃𝐴* (*𝑥*)

*𝑛*

*𝑖*=1

*𝑟*

2 has an sos decomposition *.*

*𝑥*

*𝑖*

quence of inner approximations that can either be tailored to yield a uniform approximation of or an adaptive approximation with good quality in the vicinity of the optimal solution of the underlying coposi-

Parrilo showed that

S+ +  = G0 *⊂* G1 *⊂* … *⊂* CUP and int(CUP) *⊆* G*𝑟,*

⋃

*𝑟*∈ℕ

so the cones G*𝑟* approximate CUP from the interior. The sos condition can be written as a system of linear matrix inequalities (LMIs), and there-

G

fore optimizing over *𝑟* amounts to solving an SDP. However, it should

be noted that for increasing values of *𝑟*, the size of these SDPs increases

tive optimization problem.

* 1. *Outer approximation hierarchies for* CUP

Since we can write CUP = {*𝐴* ∈ S ∣ *𝑥𝑇 𝐴𝑥* ≥ 0 for all *𝑥* ∈ Δ}, outer

infinite) subsets *𝐼 ⊂* Δ and considering {*𝐴* ∈ S ∣ *𝑥𝑇 𝐴𝑥* ≥ 0 for all *𝑥* ∈ *𝐼* }. approximations of CUP can be obtained by picking suitable (possibly

One option studied by [Yıldırım (2012)](#_bookmark117) is to consider regular grids of and the latter condition is equivalent to the second order cone constraint

+

rational points on the unit simplex defined as

(*𝑎* + *𝑐,* 2*𝑏, 𝑎* − *𝑐*)*𝑇* ∈ G3. Therefore, optimizing over SDD*𝑛*

amounts to

*𝑟*

⋃

*𝛿*(*𝑟*) ∶= {*𝑥* ∈ Δ ∣ (*𝑘* + 2)*𝑥* ∈ ℕ*𝑛*}*.* (9)

0

*𝑘*=0

Then the set U*𝑟* ∶= {*𝐴* ∈ S ∣ *𝑥𝑇 𝐴𝑥* ≥ 0 for all *𝑥* ∈ *𝛿*(*𝑟*)} is clearly a poly-

Uhedral outer approximation of CUP for any *𝑟* ∈ ℕ, and one can show that

0 *⊃* U1 *⊃* ⋯ *⊃* CUP and CUP = ⋂*𝑟*∈ℕ U*𝑟* . This approximation scheme

gives again uniform approximations of CUP and allows for exact assess- ment of the quality of the approximation.

Alternatively one can use an approach developed by Lasserre in a series of papers which makes use of the vast body of theory on positive

(resp. nonnegative) polynomials and polynomial optimization. Denote

by P

fined by considering scaled variants of SDD*𝑛* which can be tailored to solving an SOCP. In [Gouveia et al. (2020)](#_bookmark79), this approach is further re-

obtain either uniform or problem-dependent approximation schemes.

+

# Examples of binary quadratic problems

In this section, we discuss a few combinatorial problems which can be formulated as binary quadratic optimization problems. These prob- lems are typically NP-hard, so it will be useful to consider reformula- tions and relaxations which are tractable. It will turn out that relaxations based on conic optimization are particularly useful. We will mostly focus on relaxations in the cone of positive semidefinite matrices. Throughout

+

*𝑛,𝑑*

the cone of *𝑛*-variate polynomials of total degree ≤ *𝑑* which are

*𝑛*

this section, assume we are given an undirected graph *𝐺* = (*𝑉 , 𝐸*) with

nonnegative on ℝ (note that such a polynomial necessarily has even

*𝑉* = {1*,* … *, 𝑛*} and adjacency matrix *𝐴* ∈ S .

degree). Then *𝐴* ∈ S is copositive if and only if the polynomial *𝑃 𝑛*

from [(7)](#_bookmark14) fulfills *𝑃* ∈ P+ . The Riesz–Haviland Theorem tells us that the

* 1. *Unconstrained binary quadratic optimization and MaxCUt*

dual of P+

*𝑛 𝐴*

*𝐴 𝑛,*4

*𝑛,𝑑* is the so called moment cone. Exploiting this, one can obtain

CUP CP

another hierarchy of cones approximating resp. which is a spe-

cial case of the Lasserre–hierarchy applied to the setting of copositivity and complete positivity. We refer to the book by [Lasserre (2010)](#_bookmark103) and the survey by [Laurent (2009)](#_bookmark106) which both give excellent introductions to the general moment approach for polynomial optimization. The pa- per by [Lasserre (2014)](#_bookmark104) explicitly describes how to construct hierar- chies of outer approximations of and inner approximations of

CUP CP

by using this moment approach. It should be noted that these hierar- chies are based on conditions that can be expressed as LMIs, and hence optimizing over these hierarchies amounts to solving SDPs.

A third option is the adaptive approximation approach by [Bundfuss and Dür (2009)](#_bookmark33) detailed in [Section 3.1](#_bookmark12) which gives the outer approximation

UP ∶= {*𝐴* ∈ S ∣ *𝑣𝑇 𝐴𝑣* ≥ 0 for all vertices *𝑣* of simplices in P}*.*

This yields a hierarchy of polyhedral approximations that can again be tailored to either yield a uniform outer approximation of or a finer approximation in the vicinity of the set of optimal solutions but only a coarse approximation in the remaining parts.

CUP

*3.3. Inner approximation hierarchies for* CP

Recall that CP = conv{*𝑥𝑥𝑇* ∣ *𝑥* ∈ ℝ*𝑛* } = cone conv{*𝑥𝑥𝑇* ∣ *𝑥* ∈ Δ}.

*𝑛*

+

Therefore, inner approximations of CP can be constructed analogous

input a symmetric *𝑛* × *𝑛* matrix *𝑄* and asks to find An unconstrained binary quadratic optimization problem takes as

min *𝑥𝑇 𝑄𝑥*

s.t. *𝑥* ∈ {0*,* 1}*𝑛.* (10)

Since *𝑥𝑖* = *𝑥*2, a possible linear term in the objective function could be integrated in the main diagonal of *𝑄*, so it is not necessary to explicitly

*𝑖*

include a linear term in this model.

through its adjacency matrix *𝐴*. Hence *𝐴* is symmetric, but we do not The **MaxCut problem** is defined by an edge weighted graph, given impose any further restrictions to the entries of *𝐴*. In particular, *𝑎𝑖𝑗 <* 0 is possible. If [*𝑖, 𝑗*] is not an edge of the graph, we set *𝑎𝑖𝑗* = 0. The *Laplacian matrix 𝐿* associated to *𝐴* is defined by

∑

*𝑙𝑖𝑗* ∶= −*𝑎𝑖𝑗* for *𝑖* ≠ *𝑗,* and *𝑙𝑖𝑖* ∶= *𝑎𝑖𝑘 .*

*𝑘*

≥

Note that *𝐿𝑒* = 0, and if *𝐴* 0, then *𝐿* is diagonally dominant and hence

*𝐿* ⪰ 0. It is a simple exercise to verify that for *𝑦* ∈ {−1*,* 1}*𝑛* the value of the cut defined by *𝑆* ∶= {*𝑖* ∣ *𝑦𝑖* = 1} is given by

∑ *𝑎𝑖𝑗* (1 − *𝑦𝑖 𝑦𝑗* ) = *𝑦 𝐿𝑦,*

1 1 *𝑇*

2 4

*𝑖<𝑗*

so the MaxCUt problem can be formulated as

max 1 *𝑦𝑇 𝐿𝑦* such that *𝑦* ∈ {−1*,* 1}*𝑛.*

4

Setting *𝑥* ∶= (*𝑦* + *𝑒*) ∈ {0*,* 1} and using *𝐿𝑒* = 0, we get that *𝑦 𝐿𝑦* =

1

*𝑛*

1

*𝑇*

to outer approximations of CUP, namely by chosing suitable subsets

2

*𝑥𝑇 𝐿𝑥* which shows that

MaxCUt

4

and the binary quadratic opti-

*𝐼 ⊂* Δ and considering C(*𝐼* ) ∶= cone conv{*𝑥𝑥𝑇* ∣ *𝑥* ∈ *𝐼* }. A thorough

of the set *𝐼* is given in [Yıldırım (2017)](#_bookmark119). treatment investigating properties of the approximation in dependence

C

If the set *𝐼* stems from a finite discretization of Δ, then (*𝐼* ) is polyhe-

dral. A different approach was developed by [Gouveia et al. (2020)](#_bookmark79) based

on similar work in [Ahmadi and Majumdar (2019)](#_bookmark45). They consider the cone

SDD*𝑛* ∶= conv{*𝑥𝑥𝑇* ∣ *𝑥* ∈ ℝ*𝑛 ,* supp(*𝑥*) ≤ 2} *⊆* CP *,*

+ + | | *𝑛*

where the support of a vector *𝑥* is defined as supp(*𝑥*) ∶= {*𝑖* ∣ *𝑥* ≠ 0}.

*𝑖*

[2020 and references therein) that *𝐴* ∈ SDD*𝑛* if and only if *𝐴* is scaled It can be shown (see Ahmadi and Majumdar, 2019; Gouveia et al.,](#_bookmark45) diagonally dominant, i.e., if there exists a diagonal matrix *𝐷* with pos- itive diagonal entries such that *𝐷𝐴𝐷* is diagonally dominant. From the

[+](#_bookmark45)

mization problem [(10)](#_bookmark15) are indeed equivalent optimization problems. [Lasserre (2016)](#_bookmark105) showed an even more general result: Considering the linearly constrained binary quadratic problem

min{*𝑐𝑇 𝑥* + *𝑥𝑇 𝐹 𝑥* ∣ *𝐴𝑥* = *𝑏, 𝑥* ∈ {0*,* 1}*𝑛* }*,* (11)

a graph with *𝑛* + 1 nodes that can be explicitly constructed from the data Lasserre showed that this can by reformulated as a MaxCUt problem on

of the problem. So in a sense MaxCUt is a canonical model for linear and quadratic binary problems. Note that [(11)](#_bookmark16) is a special instance of the problem [(5)](#_bookmark9) studied by Burer. It can therefore be formulated as a copos- itive problem, and then approximation hierarchies from [Section 3](#_bookmark11) can be used.

Historically, linear and then semidefinite relaxations were studied [before copositivity came into play. In a celebrated paper, Goemans and](#_bookmark74)

definition we get that that *𝐴* ∈ SDD*𝑛* if and only if *𝐴* can be written

+

∑

as *𝐴* =

*𝑖<𝑗 𝑀 𝑖𝑗* , where *𝑀 𝑖𝑗* are symmetric nonnegative and positive

[Williamson (1995) took the following approach: since *𝑦𝑇 𝐿𝑦* =](#_bookmark74)

*𝑇*

they introduce the matrix *𝑌* taking the role of *𝑦𝑦* . Then *𝑌* ⪰ 0 and

⟨[*𝐿, 𝑦𝑦𝑇*](#_bookmark74)⟩

semidefinite matrices whose entries are zero everywhere except at the

diag(*𝑌* ) = *𝑒* must hold, and this yields the semidefinite relaxation

positions *𝑖𝑖, 𝑖𝑗, 𝑗𝑖, 𝑗𝑗*. Observe that positive semidefiniteness of 2 ×2 sym-

metric matrices can be characterized by second order conditions. In-

max 1 *𝐿, 𝑌*

such that diag(*𝑌* ) = *𝑒, 𝑌* ⪰ 0*.* (12)

deed, we have

(*𝑎 𝑏*) ∈ S+

*𝑎* ≥ 0*, 𝑐* ≥ 0*,*

⟺

⟺ *𝑎* ≥ 0*,* ‖( 2*𝑏* )‖

≤ *𝑎* + *𝑐,*

4 ⟨

⟩

celebrated result that the optimal value of [(12)](#_bookmark17) is at most 13*.*83% higher For graphs with nonnegative edge weights, they were able to show the

than the optimal value of MaxCUt. In other words, the SDP relaxation

*𝑏 𝑐* 2

*𝑎𝑐* − *𝑏*2 ≥ 0

*𝑐* ≥ 0*,*

‖

‖2

*𝑎* − *𝑐*

has a performance guarantee of ≈ 87%.

* 1. *Partition and clustering problems*

We briefly discuss various extensions of unconstrained binary quadratic optimization problems which lead to additional linear con- straints on the binary variables.

# *𝒌*-cluster problems

The simplest extension of problem [(10)](#_bookmark15) consists in asking that exactly

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*𝑘* of the variables in [(10)](#_bookmark15) are set to 1. Let *𝐴* 0 be a symmetric *𝑛* × *𝑛* matrix. We may think of *𝑎𝑖𝑗* as a measure for the interaction between *𝑖* and *𝑗*. The problem

max 1 *𝑥𝑇 𝐴𝑥* such that *𝑒𝑇 𝑥* = *𝑘* and *𝑥* ∈ {0*,* 1}*𝑛*

2

asks for a subset of *𝑘* vertices having maximum total pairwise interac-

a set of *𝑘* items with maximum mutual interaction. This type of problem tion. Such a set may be viewed as a “cluster” in the sense that it collects

has found increased interest from applications in data mining, see for in- stance ([Fortunato, 2010](#_bookmark72)). Various SDP relaxations have been discussed in [Malick and Roupin (2012)](#_bookmark119).

set problem. Let *𝐴* be the adjacency matrix of an unweighted graph *𝐺* A different application of this type of problem is related to the stable

The optimal value *𝑧*(*𝑘*) of the following optimization problem can be used to check whether *𝐺* contains a stable set of size *𝑘*. As usual, *𝐴* de- notes the adjacency matrix of *𝐺*. Consider the following problem which

was already briefly discussed in [Section 4.2](#_bookmark18):

*𝑧*(*𝑘*) ∶= min 1 *𝑥𝑇 𝐴𝑥* such that *𝑒𝑇 𝑥* = *𝑘, 𝑥* ∈ {0*,* 1}*𝑛.* (13)

2

If *𝑧*(*𝑘*) = 0, then *𝐺* contains a stable set of size *𝑘* given by *𝑆* = {*𝑖* ∣ *𝑥𝑖* = 1}. On the other hand, *𝑧*(*𝑘*) *>* 0 shows that *𝐺* has no stable set of size *𝑘* and therefore *𝛼*(*𝐺*) *< 𝑘*.

Let us now turn to vertex colorings of *𝐺*. A *𝑘*-coloring of *𝑉* (*𝐺*) can be seen as a vertex partition of *𝑉* (*𝐺*) into *𝑘* stable sets (the color classes). The *chromatic number 𝜒* (*𝐺*) denotes the smallest number *𝑘* such that

*𝐺* is *𝑘*-colorable. A formulation to compute *𝜒* (*𝐺*) as the solution of a

copositive problem was given in [Gvozdenović and Laurent (2008)](#_bookmark81).

Expressing *𝜒* (*𝐺*) as a binary optimization problem is usually done as follows. Let *𝑆* = {*𝑠*1*, 𝑠*2*,* …} be the collection of characteristic vectors of stable sets in *𝐺*.

*𝜒* (*𝐺*) = min ∑ *𝜆* such that ∑ *𝜆 𝑠* = *𝑒, 𝜆* ∈ {0*,* 1}*.*

*𝑖*

*𝑖 𝑖*

*𝑖*

*𝑖*

*𝑖*

and let *𝑘* be given. Consider the minimization problem

*𝑧*(*𝑘*) ∶= min 1 *𝑥𝑇 𝐴𝑥* such that *𝑒𝑇 𝑥* = *𝑘* and *𝑥* ∈ {0*,* 1}*𝑛.*

This is a linear optimization problem in binary variables *𝜆𝑖* . Unfortu-

nately, there may be an exponential number of them. Weakening the

≤ ≤

condition *𝜆* ∈ {0*,* 1} to 0 *𝜆* 1 for all *𝑖* leads us to the *fractional chro-*

2

If the optimal value fulfills *𝑧*(*𝑘*) *>* 0, then we have a proof that *𝐺* has no

*𝑖 𝑖*

*matic number 𝜒𝑓* (*𝐺*):

stable set of size *𝑘*, so that the stability number *𝛼*(*𝐺*) fulfills *𝛼*(*𝐺*) ≤ *𝑘* − 1.

*𝑓*

*𝑖*

*𝑖*

*𝑖*

*𝑖 𝑖*

*𝑖*

This idea will be further exploited in [Section 4.3](#_bookmark21).

*𝜒* (*𝐺*) = min ∑ *𝜆* such that ∑ *𝜆 𝑠* = *𝑒,* 0 ≤ *𝜆* ≤ 1 for all *𝑖.*

# Max-*𝒌*-Cut

The MaxCUt problem may also be seen as a very simple graph par-

tition problem as it asks to separate the vertices of the graph into two

Computing the optimal value *𝜒𝑓* (*𝐺*) of this linear program is again

known to be NP-hard, see for instance ([Lund and Yannakakis, 1994](#_bookmark118)).

Let us now consider testing whether *𝐺* contains a *𝑘*-coloring for some

for all 1 ≤ *𝑖* ≤ *𝑛*, which we write in a slight abuse of notation as *𝑋𝑒* = *𝑒*.

*𝑟*=1

parts such as to maximize the weight of the edges joining the two par- tition blocks. It is a natural generalization to consider vertex partitions

≥

into (at most) *𝑘* partition blocks for some fixed *𝑘* 2.

given *𝑘*. We introduce the *𝑛* × *𝑘* binary matrix *𝑋* and require ∑*𝑘*

*𝑥𝑖𝑟* = 1

We represent *𝑘*-partitions of *𝑉* = {1*,* … *, 𝑛*} by 0-1 matrices *𝑋* of or- der *𝑛* × *𝑘* satisfying *𝑋𝑒* = *𝑒*. This condition simply states that (*𝑋𝑒*)*𝑖* = 1

This condition asks that each row of *𝑋* contains exactly one entry equal

to one, so that the columns of *𝑋* provide a vertex partition of *𝑉* (*𝐺*) into

(at most) *𝑘* partition blocks. Since each row of *𝑋* has exactly one nonzero

for all *𝑖*, meaning that vertex *𝑖* is in exactly one partition block, namely

block *𝑗* in case *𝑥𝑖𝑗* = 1. The sum of the elements of column *𝑗* of *𝑋* equals

entry, we see that ∑*𝑘*

*𝑥𝑖𝑟 𝑥𝑗𝑟*

= 1 for some *𝑖* ≠ *𝑗* is only possible if *𝑖* and

the number of vertices (possibly zero) in partition block *𝑗*. It is a sim-

ple exercise to verify that the total weight of edges joining vertices in

distinct partition blocks is given by

1 trace(*𝐿𝑋𝑋𝑇* )*.*

*𝑗* both belong to the same partition block *𝑟* for some *𝑟* ∈ {1*,* … *, 𝑘*}. As a

consequence

*𝑟*=1

*𝑥𝑖𝑟 𝑥𝑗𝑟* = 1 *𝑋, 𝐴𝑋*

2

∑ ∑ ⟨ ⟩

[*𝑖,𝑗*]∈*𝐸* (*𝐺*) *𝑟*

2

Therefore, the Max-*𝑘*-CUt problem may be formulated as

⟨ ⟩

counts the number of edges joining vertices in the same partition block. We introduce

max 1 trace(*𝐿𝑋𝑋𝑇* ) such that *𝑋𝑒* = *𝑒* and *𝑋* ∈ {0*,* 1}*𝑛*×*𝑘.*

*𝑧*(*𝑘*) ∶= min 1

*𝑋, 𝐴𝑋*

such that *𝑋𝑒* = *𝑒, 𝑋* ∈ {0*,* 1}

*𝑛*×*𝑘*

*.* (14)

2

The ***𝒌*-partition problem** is obtained by further constraining the par- titions to have exactly *𝑘* partition blocks, and to require that partition block *𝑗* contains exactly *𝑚𝑗* ∈ ℕ vertices, where *𝑗 𝑚𝑗* = *𝑛*. We collect the cardinalities *𝑚𝑗* in the vector *𝑚* ∈ ℕ*𝑘*, so that feasible partitions are represented by matrices *𝑋* ∈ {0*,* 1}*𝑛*×*𝑘* satisfying

∑

*𝑋𝑒* = *𝑒* and *𝑋𝑇 𝑒* = *𝑚.*

The special case where all the *𝑚𝑖* are equal is of special interest

[in certain telecommunication problems, and we refer to Lisser and](#_bookmark111)

[Rendl (2003) for a discussion of these applications and relaxations based](#_bookmark111) on semidefinite optimization.

* 1. *Stable sets and graph coloring*

We briefly look at formulations for the stability and the chromatic number of a graph in connection with quadratic binary optimization.

A subset *𝑆* of vertices of a graph *𝐺* is called *stable* (or *independent*),

if the subgraph of *𝐺* induced by *𝑆* is empty. The *stability number 𝛼*(*𝐺*)

denotes the cardinality of a largest stable set in *𝐺*. Determining *𝛼*(*𝐺*) is

considered an extremely diﬃcult combinatorial optimization problem,

cf. [Håstad (1999)](#_bookmark82). The following binary quadratic optimization problem

determines *𝛼*(*𝐺*):

2

In a slight abuse of notation, we use *𝑧*(*𝑘*) again for the optimal value of the relaxation with exactly *𝑘* columns in *𝑋*. If *𝑧*(*𝑘*) = 0, then the optimal

*𝑋* provides a partitioning of *𝑉* (*𝐺*) into (at most) *𝑘* stable sets and there- fore *𝜒* (*𝐺*) *𝑘*. On the other hand *𝑧*(*𝑘*) *>* 0 implies that no *𝑘*-partition exists where all partition blocks are stable sets, and therefore *𝜒* (*𝐺*) *> 𝑘*.

≤

tigate connections to the *𝜗* number in [Section 5.2](#_bookmark23). We will come back to SDP relaxations for these problems and inves-

* 1. *Quadratic set cover*

set *𝐶* ∶= {*𝑣*1*,* … *, 𝑣𝑛* } of *𝑛* elements and a collection *𝑆* of *𝑚* subsets of *𝐶* The (linear) set cover problem is defined as follows. We are given a such that their union equals *𝐶*. Each subset *𝑆𝑖* in *𝑆* has cost *𝑞𝑖* . The task is to select subsets in *𝑆* such that their union is *𝐶* and such that the cost

of the selected subsets in minimized. This problem is one of Karp’s 21 NP-complete problems.

To state this problem formally, define an *𝑛* × *𝑚* binary matrix *𝐴* with

*𝑎𝑖𝑗* = 1 if *𝑣𝑖* ∈ *𝑆𝑗* . Row *𝑖* of *𝐴* indicates which subsets *𝑆𝑗* contain *𝑣𝑖* , col- umn *𝑗* of *𝐴* is the incidence vector of subset *𝑆𝑗* . With this, the linear set

cover problem reads:

*𝑚 𝑚*

*𝑗*=1

*𝑗*=1

( ) = max ∑

*𝛼 𝐺*

*𝑥𝑖* such that *𝑥𝑖 𝑥𝑗*

for all *𝑖, 𝑗*

*𝐸 𝐺 , 𝑥*

*,*

*𝑛 .*

= 0 [

] ∈ ( )

∈ {0 1}

*𝑧*∗ ∶= min ∑ *𝑞𝑗 𝑥𝑗* s.t. ∑ *𝑎𝑖𝑗 𝑥𝑗* ≥ 1 for all *𝑖* = 1*,* … *, 𝑛* and *𝑥* ∈ {0*,* 1}*𝑚.*

Let us denote the largest row sum of *𝐴* by *𝑓* and the largest col- umn sum of *𝐴* by *𝑔*. The following approximation results go back to the

1980’s. [Hochbaum (1982)](#_bookmark86) introduces a primal-dual LP-rounding heuris- tic which gives (in polynomial time) a feasible solution to set cover

≤

with value *𝑧* at most *𝑓𝑧*∗, i.e. *𝑧 𝑓𝑧*∗. [Chvátal (1979)](#_bookmark42) proposes a greedy

≤

to set cover with value *𝑧* at most (1 + log(*𝑔*))*𝑧*∗, i.e., *𝑧* (1 + log(*𝑔*))*𝑧*∗. rounding heuristic which yields (in polynomial time) a feasible solution

The quadratic set cover problem differs from the linear one only in the objective function which now also may contain quadratic terms:

*𝑚 𝑚*

∑ ∑

*𝑞𝑖𝑗 𝑥𝑖 𝑥𝑗 .*

*𝑖*=1 *𝑗*=1

where *𝐽* ∶= *𝑒𝑒𝑇* and *𝑀* ∈ S is an arbitrary symmetric matrix of order

*𝑛* − 1. Note in particular that *𝑉 𝑀𝑉 𝑇* lies in the linear space of matrices

having row and column sums equal to 0.

[Povh and Rendl (2009)](#_bookmark94) proposed a copositive formulation of QAP

rewrite the objective function of QAP in terms of *𝑥* ∶= vec(*𝑋*), where and semidefinite relaxations based upon it. To this end, it is useful to vec(*𝑋*) is a vector obtained from the matrix *𝑋* by stacking the columns of *𝑋* on top of each other. Using the Kronecker product *𝐵 ⊗ 𝐴* of the matrices *𝐵* and *𝐴*, it is not diﬃcult to see that

= *𝑇* (*𝐵 ⊗ 𝐴*)*𝑥*

⟨*𝐴𝑋𝐵, 𝑋*⟩ *𝑥*

We also set *𝑐* ∶= vec(*𝐶*) and derive

It is clear that if *𝑞*

⟨ ⟩

= 0 for all *𝑖* ≠ *𝑗*, then we recover the linear set cover

2

*𝑖𝑗*

+ = *𝑥𝑇* (*𝐵 ⊗ 𝐴*)*𝑥* + *𝑐𝑇 𝑥.*

problem (as *𝑥𝑗* = *𝑥𝑗* ).

A good summary on complexity issues related to quadratic set cover

to deciding whether a graph has chromatic number 3: Let *𝐺* be a (nonbi- is given by [Escoﬃer and Hammer (2007)](#_bookmark69). They relate quadratic set cover partite) graph on *𝑛* vertices and construct a quadratic set cover instance as follows. The ground set is *𝑉* = {1*,* … *, 𝑛*} and we have 3*𝑛* subsets *𝑆𝑖𝑟*

with

*𝑆𝑖𝑟* ∶= {*𝑖*} for *𝑟* = 1*,* 2*,* 3*,*

*𝐴𝑋𝐵 𝐶, 𝑋*

We are now interested in the set

P ∶= conv{*𝑥𝑥𝑇* ∶ *𝑥* = vec(*𝑋*)*, 𝑋* ∈ Π}*.*

We have just seen that

*𝑥* = vec(*𝑋*) = vec( 1 *𝐽* + *𝑉 𝑀𝑉 𝑇* ) = 1 *𝑒 ⊗ 𝑒* + (*𝑉 ⊗ 𝑉* )*𝑚*

*𝑛 𝑛*

using *𝑚* ∶= vec(*𝑀* ). Let *𝑧* ∶= (1 ) and set *𝑊* ∶= ( 1 *𝑒 ⊗ 𝑒, 𝑉 ⊗ 𝑉* ). Then

so each set consists of only one element, and we have three copies of

*𝑚*

*𝑛*

each set. The *𝑛* covering conditions ask that

*𝑇*

*𝑇*

*𝑇*

*𝑇*

*𝑥* = *𝑊 𝑧* and *𝑥𝑥* = *𝑊 𝑧𝑧 𝑊* . The definition of *𝑧* implies (*𝑧𝑧* )1*,*1 = 1.

*𝑥𝑖*1

+ *𝑥*

*𝑖*2

+ *𝑥*

*𝑖*3

≥ 1 for all *𝑖* = 1*,* … *, 𝑛.*

matrix *𝑅* with (*𝑅*)1*,*1 = 1 in place of *𝑧𝑧𝑇* . For ease of notation we intro- The SDP relaxation of QAP is now obtained by allowing any semidefinite duce *𝑌* ∶= *𝑊 𝑅𝑊 𝑇* , and we get the following semidefinite relaxation of

We may think of these constraints as asking that each vertex should re- ceive color 1 or 2 or 3. Thus we do not allow that all three of these vari- [ables are zero but more than one of them may be set to one. Escoﬃer and](#_bookmark69)

[Hammer (2007) show the following theorem.](#_bookmark69)

## QAP:

min

⟨*𝐵 ⊗ 𝐴*

+ diag(*𝑐*)*, 𝑌*

such that *𝑌* = *𝑊 𝑅𝑊 𝑇 ,* (*𝑅*)

1*,*1

= 1*,*

**Theorem 4.1.** *Let 𝐺 be a nonbipartite graph and consider*

*𝑦* = diag(*𝑌* )*, 𝑌* − *𝑦𝑦𝑇* ⪰ 0*.*

Since *𝑌* takes the role of *𝑥𝑥𝑇* we may think of the *𝑛*2 × *𝑛*2 matrix *𝑌* as

= 0 for *𝑘* ≠ *𝑙*, it follows immediately

*𝑧*∗ ∶= min ∑

[*𝑖,𝑗*]∈*𝐸* (*𝐺*)

*𝑥𝑖*1 *𝑥𝑗*1 + *𝑥𝑖*2 *𝑥𝑗*2 + *𝑥𝑖*3 *𝑥𝑗*3

being partitioned into *𝑛* × *𝑛* matrices *𝑌 𝑖,𝑗* such that *𝑌 𝑖,𝑗* corresponds to

⋅ *𝑋𝑇* . Since *𝑋*

the matrix *𝑋*

*.,𝑖*

*.,𝑗*

*𝑖,𝑘*

*𝑋*

*𝑖,𝑙*

*𝑖*1

s.t. *𝑥*

+ *𝑥*

*𝑖*2 *𝑖*3

that the submatrix *𝑌* is 0 outside its main diagonal, i.e., (*𝑌* )*𝑘,𝑙* = 0 for

*𝑥𝑖𝑟*

+ *𝑥*

≥ 1 for all *𝑖* = 1*,* … *, 𝑛*

*𝑖,𝑖*

*𝑖,𝑖*

∈ {0*,* 1} for all *𝑖* = 1*,* … *, 𝑛, 𝑟* = 1*,* 2*,* 3*.*

all *𝑘* ≠ *𝑙*. In a similar way we conclude that diag(*𝑌 𝑖,𝑗* ) = 0 for *𝑖* ≠ *𝑗*. We

refer to [Povh and Rendl (2009)](#_bookmark94) for further details.

*Then 𝑧*∗ = 0 *if and only if 𝜒* (*𝐺*) = 3*.*

As a consequence, it is NP-hard to decide whether a quadratic set cover problem has optimal value 0 or greater than 0. The covering prob- lem in this construction is quite simple. Each set consists of only one

element (*𝑆𝑖𝑟* = {*𝑖*}), and each cover constraint involves only three ele-

ments from the ground set. This problem would therefore be trivial to

solve with a linear objective function.

* 1. *Quadratic assignment problem*

The Quadratic Assignment Problem (QAP) asks to minimize a quadratic objective function over the set of permutation matrices. It contains many prominent NP-hard problems as special cases, see for

ces *𝐴, 𝐵* and *𝐶* of order *𝑛* × *𝑛* and assume that *𝐴* and *𝐵* are symmetric. instance ([Pardalos et al., 1994](#_bookmark83)). We define it through three data matri- The set of *𝑛* × *𝑛* permutation matrices is denoted by Π*𝑛* or Π for short.

The QAP then reads:

min *𝐴𝑋𝐵* + *𝐶, 𝑋* such that *𝑋* ∈ Π*.*

⟨

⟩

# Modelling linear equalities and inequalities in SDP relaxations

In this section we take a closer look at modelling issues related to combinatorial optimization problems. We stress that formulations which are equivalent in the binary setting may give different results when we move to conic relaxations. For instance, suppose we have binary vari-

ables *𝑥𝑖* ∈ {0*,* 1} and we would like to express the constraint that for a

given pair *𝑖, 𝑗* at most one of the associated variables *𝑥𝑖* and *𝑥𝑗* is al-

≤

inequality *𝑥𝑖* + *𝑥𝑗* 1 or by requiring the quadratic equation *𝑥 𝑥* = 0 to lowed to be equal to 1. This could be done either by imposing the linear

*𝑖 𝑗*

hold. In the binary setting, both conditions are equivalent, but once we move to relaxations, they may yield different results.

Moreover, various ways of constructing SDP relaxations have been proposed in the literature which may also lead to bounds of varying quality. We refer to [Anjos et al. (2021)](#_bookmark52) for a recent discussion of various SDP models related to the stable set problem.

Finally, the idea of using approximation hierarchies as detailed in [Section 3](#_bookmark11) has found a lot of interest. Applied to combinatorial optimiza-

tion problems, these hierarchies typically have the property that the

£ ∶= {

Note that Π is contained in the aﬃne space

*𝑋* ∈ ℝ

*𝑛*×*𝑛*

∣ *𝑋𝑒* = *𝑋𝑇*

*𝑒* = *𝑒*}

quality of the relaxation gets tighter as one moves up in the hierarchy, yielding the integer optimum as one moves up high enough in the hi-

of all matrices having row and column sums equal to 1. Let the (*𝑛* − 1) ×

⟩

*𝑛* matrix *𝑉* represent a basis of *𝑒*⟂, the orthogonal complement to the vector *𝑒* ∈ ℝ*𝑛*. It is well known that any *𝑋* ∈ may be written as

£

*𝑋* = 1 *𝐽* + *𝑉 𝑀𝑉 𝑇*

*𝑛*

the first few levels in these hierarchies. If the initial problem has *𝑛* bi- erarchy. Unfortunately, it is computationally challenging to tackle even

in matrices of order *𝑛* + 1, but already the second level uses matrices of nary variables, the SDP in the first level of the hierarchy is formulated order *𝑛* which is prohibitive once *𝑛* is much larger than 100.

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* 1. *Lifting linear constraints*

Next we investigate the situation where a linear term *𝑎𝑇 𝑥* is required

0. The con-

≤

| |

Here we focus on the modelling issue and investigate how linear constraints may be lifted into the SDP relaxation. The precise method of lifting linear constraints can be a subtle issue and can, if not done

to be contained in some interval, say *𝑎𝑇 𝑥*

straints *𝑋* − *𝑥𝑥𝑇* ⪰ 0 and diag(*𝑋*) = *𝑥* imply

*𝑎𝑇 𝑋𝑎* ≥ (*𝑎𝑇 𝑥*)2 *.*

*𝑎*0, with *𝑎*0 *>*

carefully, destroy strict feasibility of the lifted problem. For given *𝑎*0 *>* 0,

we consider the SDP relaxation of

max *𝑥𝑇 𝐶𝑥* such that *𝑥* ∈ {0*,* 1}*𝑛, 𝑎𝑇 𝑥* = *𝑎*0 *.*

equality constraint *𝑎𝑇 𝑥* = *𝑎*0. How should this equation be included in This is a binary quadratic optimization problem with a single linear

the SDP relaxation?

The semidefiniteness constraint *𝑋* − *𝑥𝑥𝑇* ⪰ 0 with diag(*𝑋*) = *𝑥* im-

mediately shows that

*𝑎𝑇 𝑋𝑎* ≥ (*𝑎𝑇 𝑥*)2 *.*

We conclude that *𝑎*2 ≥ *𝑎𝑇 𝑋𝑎* is at least as strong as the original inequal- ity *𝑎𝑇 𝑥* ≥ *𝑎* . 0

Finally, observe that the situation is different for one-sided linear inequalities of the form

we

| | 0

*𝑎𝑇 𝑥* ≥ *𝑎*0

with *𝑎*0 *>* 0. Arguing as before we see that

*𝑎𝑇 𝑋𝑎* ≥ (*𝑎𝑇 𝑥*)2 *,*

hence the original inequality *𝑎𝑇 𝑥* ≥ *𝑎*0 is at least as strong as *𝑎𝑇 𝑋𝑎* ≥ *𝑎*2.

0

Since we should have equality it seems plausible to optimize over the set

T1 ∶= {(*𝑋, 𝑥*) ∣ ( *𝑋 𝑥*) ⪰ 0*,* diag(*𝑋*) = *𝑥, 𝑎𝑇 𝑥* = *𝑎*0 *, 𝑎𝑇 𝑋𝑎* = *𝑎*2}*.*

*𝑥𝑇* 1

0

* 1. *Stable set and coloring relaxations*

We recall the formulation for the stability number *𝛼*(*𝐺*) for a graph

*𝐺*:

Unfortunately, this construction makes feasible matrices singular, as we show in the next lemma:

**Lemma 5.1.** *Let* (*𝑋, 𝑥*) ∈ T1*. Then the matrix* ( *𝑋 𝑥*) *is singular and*

*𝑥𝑇* 1

*𝑋𝑎* = *𝑎*0*𝑥.*

**Proof.** We first note that

( *𝑎* )*𝑇* ( *𝑋 𝑥*)( *𝑎* )

0

*𝛼*(*𝐺*) ∶= max{*𝑒𝑇 𝑥* ∣ *𝑥𝑖 𝑥𝑗* = 0 for all [*𝑖, 𝑗*] ∈ *𝐸*(*𝐺*)*, 𝑥* ∈ {0*,* 1}*𝑛* }*.*

by [Lovász (1979)](#_bookmark115). It may be derived from nonzero binary vectors *𝑥* by One of its first relaxations using SDP was introduced in a seminal paper introducing *𝑋* ∶= 1 *𝑥𝑥𝑇* . Note that *𝑒𝑇 𝑥* = *𝑒𝑇 𝑋𝑒* holds for any feasible

*𝑥𝑇 𝑥*

*𝑥*. Let

*𝜗*(*𝐺*) ∶= max{*𝑒𝑇 𝑋𝑒* ∣ *𝑋* ⪰ 0*, 𝑥𝑖𝑗* = 0 for all [*𝑖, 𝑗*] ∈ *𝐸*(*𝐺*)}*.*

−*𝑎*0

*𝑥𝑇*

1 −*𝑎*0

= *𝑎𝑇 𝑋𝑎* − 2*𝑎*0 *𝑎𝑇 𝑥* + *𝑎*2 = 0*.*

As any characteristic vector *𝑥* of a stable set leads to a feasible ma-

[trix *𝑋* for this problem, it is clear that *𝛼*(*𝐺*) *𝜗*(*𝐺*). Lovász and Schri-](#_bookmark116)

≤

Since ( *𝑋 𝑥*) ⪰ 0, this implies ( *𝑋 𝑥*)( *𝑎* ) = 0, and therefore

[jver (1991) showed that *𝜗*(*𝐺*) can also be obtained as the optimal value](#_bookmark116)

*𝑥𝑇* 1

*𝑋𝑎* = *𝑎*0*𝑥*. □

*𝑥𝑇* 1

−*𝑎*0

of the following SDP:

*𝜗*(*𝐺*) = max{*𝑒𝑇 𝑥* ∣ *𝑥* = diag(*𝑋*)*, 𝑋𝑖𝑗* = 0 ∀ [*𝑖, 𝑗*] ∈ *𝐸*(*𝐺*)*,*

*𝑋 𝑥*

*𝑇*

( )

⪰ 0}*.*

It is well known that the set of singular positive semidefinite matri-

S

ces equals the boundary of + and the set of positive definite matrices

*𝑥* 1

equals the interior of S+. *𝑛* [Lemma 5.1](#_bookmark24) implies that if we work with the set 1 in an SDP-relaxation, then the Slater condition is nec-

T

*𝑛* Therefore,

essarily violated, and we already saw that this is disadvantageous both from a theoretical perspective (strong duality may not hold) and from a practical perspective (solvers may not be able to handle the problem). As an alternative, we propose the set

T2 ∶= {(*𝑋, 𝑥*) ∣ *𝑋* ⪰ 0*,* diag(*𝑋*) = *𝑥, 𝑎𝑇 𝑥* = *𝑎*0 *, 𝑋𝑎* = *𝑎*0 *𝑥*}*.*

A formulation similar to T2 appeared in [Burer, 2010](#_bookmark38), Proposition 1.

We next show that the two sets are actually equal:

We now return to the parameter *𝑧*(*𝑘*) from [(13)](#_bookmark19) and derive upper bounds on *𝛼*(*𝐺*) based on it. We denote by *𝐴* the adjacency matrix of our graph *𝐺*. For given *𝑘* ∈ ℕ we have

*𝑧*(*𝑘*) ∶= min 1 *𝑥𝑇 𝐴𝑥* such that *𝑒𝑇 𝑥* = *𝑘* and *𝑥* ∈ {0*,* 1}*𝑛.*

2

If *𝑧*(*𝑘*) *>* 0, then clearly *𝐺* has no stable set of size *𝑘* and there- fore *𝛼*(*𝐺*) *< 𝑘*. Computing *𝑧*(*𝑘*) is an NP-complete problem, see for in-

tractable relaxation denoted by *𝑃* (*𝑡*) for some *𝑡 >* 1: stance ([Billionnet and Roupin, 2008](#_bookmark73)), so we consider the following

*𝑃* (*𝑡*) min 1 *𝐴, 𝑌* such that *𝑌* ⪰ 0*, 𝑌* ≥ 0*,* trace(*𝑌* ) = *𝑡, 𝑌 𝑒* = *𝑡*diag(*𝑌* )

2 ⟨

⟩

**Lemma 5.2.** *We have that* T1 = T2*.*

**Proof.** We first take (*𝑋, 𝑥*) ∈ T1. Then *𝑋* ⪰ 0*,* diag(*𝑋*) = *𝑥* and *𝑎𝑇 𝑥* = *𝑎*0 .

[Lemma 5.1](#_bookmark24) shows that *𝑋𝑎* = *𝑎*0*𝑥* and therefore (*𝑋, 𝑥*) ∈ T2.

Note that this is a doubly nonnegative relaxation, since the constraints

*𝑌* ⪰ 0*, 𝑌* 0 mean that *𝑌* should be in the cone of doubly nonnega-

≥

tive matrices. We denote the optimal value of *𝑃* (*𝑡*) by val(*𝑃* (*𝑡*)). Note

that val(*𝑃* (*𝑡*)) *>* 0 implies *𝛼*(*𝐺*) ≤ *𝑡* , so we are interested in finding the

Conversely, let (*𝑋, 𝑥*) ∈ T2. We immediately get that *𝑎𝑇 𝑥* = *𝑎*0 and

*𝑎𝑇 𝑋𝑎* = *𝑎*2. It remains to show that

smallest such that val( ( )) 0⌊. ⌋It turns out that the answer to this

0 question is closely related to Schrijver’s refinement *𝜗*

*𝑡*

*𝑃 𝑡*

*>*

+

+

1 ′

= *𝜗* (*𝐺*) (some-

( *𝑋 𝑥*) ⎛ *𝑋*

*𝑌* ∶=

*𝑇* 1

= ⎜( 1

*𝑋𝑎*)*𝑇*

*𝑥*

*𝑎 𝑋𝑎*⎞

times denoted by *𝜗* (*𝐺*)) of the original theta function *𝜗*(*𝐺*):

*𝜗*

+

*𝑎*0

⎝

( )

0

1

⪰ 0*.*

⎟

Let *𝑋𝑣* = 0. Then *𝑤* ∶=

*𝑣*

0

is in the null-space of *𝑌* , as *𝑌 𝑤* =

*𝑋𝑣*

1 *𝑎𝑇 𝑋𝑣*

*𝑎*0

⎠

s.t. *𝑌* − *𝑦𝑦𝑇* ⪰ 0*,* diag(*𝑌* ) = *𝑦,*

*𝑖𝑗*

(*S*1)

0. This shows that the null-space of *𝑋* (extended with an additional component equal to 0) is contained in the null-space of *𝑌* . We also have *𝑌 𝑎* = 0, so that *𝑋* and *𝑌* have the same rank. If *𝑋* ⪰ 0 then

( )

∶= max trace(*𝑌* )

=

*𝑦* = 0 for all [*𝑖, 𝑗*] ∈ *𝐸*(*𝐺*)*, 𝑌* ≥ 0*.*

−*𝑎*0

all nonzero eigenvalues of *𝑌* are positive by the interlacing property

formulation of *𝜗*+: Before we establish this connection we recall the following alternative

*𝜗*+ = max *𝐽, 𝑋*

⟨ ⟩

| | | |

s.t. *𝑋* ⪰ 0*,* trace(*𝑋*) = 1*, 𝑥*

*𝑖𝑗*

= 0 for all [*𝑖, 𝑗*] ∈ *𝐸*(*𝐺*)*, 𝑋* ≥ 0*.*

(*S*2)

between the eigenvalues of *𝑋* and *𝑌* . □

( )

We set *𝑛* ∶= *𝑉* (*𝐺*)

and *𝑚* ∶= *𝐸*(*𝐺*) . The next two theorems can be

**Remark 5.3.** The matrix *𝑌* above is sometimes called a *flat extension* of

*𝑋*. It is a well known fact that flat extensions of semidefinite matrices

are also semidefinite.

used to get bounds for *𝛼*(*𝐺*) and *𝜒* (*𝐺*). Contrary to the computation of

*𝜗*+(*𝐺*) which requires the solution of an SDP with more than *𝑚* equality constraints, the SDP relaxation *𝑃* (*𝑡*) contains *𝑛* + 1 equality constraints,

independent of *𝑚*. As a drawback, one has to guess the proper value *𝑡*

which might require solving several SDPs for different values of *𝑡*.

The following property of optimal solutions to ([S1](#_bookmark25)) will be used later

on.

Szegedy as a lower bound on the chromatic number of *𝐺*:

*𝜗*−(*𝐺*) ∶= min *𝑡* such that ( *𝑌 𝑒*) ⪰ 0*,* diag(*𝑌* ) = *𝑒,*

*𝑒𝑇 𝑡*

**Lemma 5.4.** *Let* (*𝑌* ∗*, 𝑦*∗) *be optimal for* ([*S*1](#_bookmark25) ). *Then 𝑌* ∗*𝑒* = *𝜗*+*𝑦*∗*.*

*𝑌* ≥ 0*, 𝑦*

*𝑖𝑗*

= 0 for all [*𝑖, 𝑗*] ∈ *𝐸*(*𝐺*)*.*

**Proof.** We first note that *𝑋*∗

*𝜗*+

⟨

*𝐽, 𝑌* ∗

≤

∗

⟩

∶=  1 *𝑌* ∗

is feasible for ([S2](#_bookmark26)) and there-

We have *𝜗*(*𝐺*) ≤ *𝜗*−(*𝐺*) ≤ *𝜒* (*𝐺*).

fore

∗

*𝑌*

– *𝑦*∗(*𝑦*∗)*𝑇* ⪰ 0 so that

⟨*𝐽, 𝑋* ⟩

*𝜗*+. This means that *𝐽, 𝑌*

≤ (*𝜗*+)2 . On the other hand,

+ 2

≥ (*𝑒𝑇 𝑦*∗)2 = (*𝜗* ) . Thus we have shown

**Theorem 5.6.** *For a given graph 𝐺 we have that*

that *𝐽, 𝑌* ∗ = (*𝜗*+)2 . This implies that *𝑒𝑇* (*𝑌* ∗ − *𝑦*∗(*𝑦*∗)*𝑇* )*𝑒* = 0 and to- gether with the semidefiniteness of the matrix *𝑌* ∗ − *𝑦*∗(*𝑦*∗)*𝑇* we get (*𝑌* ∗ − *𝑦*∗(*𝑦*∗)*𝑇* )*𝑒* = 0, showing that *𝑌* ∗*𝑒* = *𝜗*+*𝑦*∗. □

∗

⟨ ⟩ ⟨ ⟩

We are now ready to show the following result.

**Theorem 5.5.** *Let 𝐴 be the adjacency matrix of a graph 𝐺. Then* val(*𝑃* (*𝑡*)) *>*

val(*𝑃* (*𝑡*)) *>* 0 if and only if *𝑡 < 𝜗*−(*𝐺*)*.*

**Proof.** We first consider problem *𝑃* (*𝑡*) for *𝑡* = *𝜗*−(*𝐺*). Let *𝑌* be an optimal solution for *𝜗*−(*𝐺*). Then *𝑌* is also feasible for *𝑃* (*𝜗*−(*𝐺*)) with value 0, so

*𝑃* (*𝜗*−(*𝐺*)) = 0. The solution *𝑌* remains feasible for any *𝑡*′ *> 𝑡* = *𝜗*−(*𝐺*) so that val(*𝑃* (*𝑡*′)) = 0 also in this case. Finally, the definition of *𝜗*−(*𝐺*) shows that for any *𝑡 < 𝜗*−(*𝐺*), the system

0 *if and only if 𝑡 > 𝜗*+(*𝐺*)*.*

( *𝑌 𝑒*) ⪰ 0

( ) =

≥ 0 = 0

[ ] ∈ ()

**Proof.** We first consider the problem *𝑃* (*𝑡*) for the value *𝑡* = *𝜗*+ ∶=

*𝑒𝑇 𝑡*

*,* diag *𝑌*

*𝑒, 𝑌*

*, 𝑦𝑖𝑗*

for all *𝑖, 𝑗*

*𝐸 𝐺*

*𝜗*+(*𝐺*):

*𝑃* (*𝜗*+) min 1 *𝐴, 𝑌*

2 ⟨

0*,* trace(*𝑌* ) = *𝜗 , 𝑌 𝑒* = *𝜗* diag(*𝑌* )*.*

Now take an optimal solution (*𝑌 , 𝑦*) for problem ([S1](#_bookmark25)), so trace(*𝑌* ) =

⟩

s.t. *𝑌* ⪰ 0*, 𝑌* ≥

is infeasible. Therefore, any *𝑌* satisfying *𝑌 𝑒* ⪰ 0*,* diag(*𝑌* ) =

+ + *𝑒, 𝑌* ≥ 0 will have an entry *𝑦 >* 0 for some [*𝑖, 𝑗*] ∈ *𝐸*(*𝐺*), and hence

*𝑒𝑇 𝑡*

( )

*𝑖𝑗*

val(*𝑃* (*𝑡*)) *>* 0. □

*𝜗*+*, 𝑌* − *𝑦𝑦𝑇* ⪰ 0*, 𝑦*

*𝑖𝑗*

= 0 for all [*𝑖, 𝑗*] ∈ *𝐸*(*𝐺*)*, 𝑦*

*𝑖𝑗*

≥ 0 for all [*𝑖, 𝑗*] ∉ *𝐸*(*𝐺*).

[Lemma 5.4](#_bookmark27) shows us that *𝑌 𝑒* = *𝜗*+*𝑦*. We conclude that (*𝑌 , 𝑦*) is feasible for problem *𝑃* (*𝜗*+). Since *𝑦𝑖𝑗* = 0 on *𝐸*(*𝐺*) we conclude that val(*𝑃* (*𝜗*+)) = 0.

Next, suppose that val(*𝑃* (*𝑡*)) = 0 and consider *𝑡*′ with 1 *< 𝑡*′ *< 𝑡*. Sup-

pose that (*𝑌 , 𝑦*) is optimal for *𝑃* (*𝑡*). It is a simple exercise to verify that

*𝑌* ′ ∶= *𝑡*′ [(*𝑡*′ − 1)*𝑌* + (*𝑡* − *𝑡*′)diag(*𝑌* )]

*𝑡*(*𝑡* − 1)

is feasible for *𝑃* (*𝑡*′) with objective value 0, hence val(*𝑃* (*𝑡*)) = 0 for all

≤

1 *< 𝑡 𝜗*+.

Finally, the definition of problem ([S1](#_bookmark25)) shows that *𝜗*+ is the largest possible value for the trace of a matrix *𝑌* which satisfies *𝑌* ⪰ 0*, 𝑌*

≥

0*, 𝑦𝑖𝑗* = 0 for all [*𝑖, 𝑗*] ∈ *𝐸*(*𝐺*), and therefore val(*𝑃* (*𝑡*)) *>* 0 for any *𝑡 >*

*𝜗*+. □

chromatic number *𝜒* . We recall the binary quadratic problem from [(14)](#_bookmark20): A similar approach can also be used to get lower bounds for the

*𝑧*(*𝑘*) ∶= min 1 *𝑋, 𝐴𝑋* such that *𝑋𝑒* = *𝑒, 𝑋* ∈ {0*,* 1}*𝑛*×*𝑘.*

2 ⟨

⟩

If *𝑧*(*𝑘*) *>* 0 for some given *𝑘*, then *𝜒* (*𝐺*) *> 𝑘*. As before, we need a tractable

# Conclusions

We have seen that conic optimization is an extremely versatile tool with an abundance of applications. Depending on the cone in question, different complexities may occur: while linear programming, second or- der cone programming as well as semidefinite programming are solvable in polynomial time, optimizing over the cones of copositive or com- pletely positive matrices is NP-hard.

CUP CP

The cones and are highly useful modelling tools for non- convex quadratic or combinatorial optimization. In many cases, it is possible to reformulate such problems equivalently as linear problems over or . Relaxing the cone constraint to a semidefiniteness or double nonnegativity constraints yields very good bounds which are of- ten provably tighter than LP-based bounds. When using approximation hierarchies, one can often show that some finite level of the hierarchy gives the exact solution of the underlying combinatorial problem.

CUP CP

In this paper, we have discussed various conic approaches to *binary* quadratic problems. It is noteworthy that not much literature is avail- able for conic approaches to nonconvex quadratic problems involving general integer or mixed integer variables, i.e., problems with a vari-

[able *𝑥* ∈ ℝ*𝑝* × ℤ*𝑟* . This type of problems is studied in Burer and Letch-](#_bookmark40)

relaxation for this problem. It is obtained by first extending *𝑋* with an [+ +](#_bookmark40)

additional row of all ones, so we introduce

*𝑋̃ 𝑋*

∶= ( )

*𝑒𝑇*

and observe that

( )

[ford (2014) and](#_bookmark40) [Buchheim](#_bookmark34) [and Traversi (2015), however many open](#_bookmark40) questions remain in this area.

In contrast to linear programming, the existence of strictly feasible solutions plays a crucial role in conic optimization. In the absence of strictly feasible points, strong duality may not hold and algorithms may

fail to solve the problem. This is therefore a point that should be care-

*𝑋̃ 𝑋̃ 𝑇* =

*𝑋𝑋𝑇 𝑒*

*𝑒𝑇 𝑘 .*

fully considered when modelling the problem in question as a conic optimization problem.

At the moment, the main bottleneck for using SDP relaxations or

The main diagonal of the matrix *𝑋𝑋𝑇* clearly equals the all-ones vector,

because each row of the 0-1 matrix *𝑋* has exactly one entry equal to 1. A relaxation is obtained by allowing arbitrary matrices *𝑌* instead of *𝑋𝑋𝑇* .

We get

conic optimization in a broader context of applications is the lack of algorithms that can solve large scale problems in reasonable time.

Semidefinite relaxations of a problem in ℝ*𝑛* clearly involve matrices of

order at least *𝑛* × *𝑛*, so the number of variables is roughly squared. If

(*𝑃* (*𝑡*)) min 1

⟨

( *𝑌 𝑒*) ⪰ 0

*𝑒, 𝑌*

*.*

*𝑒𝑇 𝑡*

( ) = ≥ 0

one works with approximation hierarchies, the the SDPs get larger at

From val(*𝑃* (*𝑡*)) *>* 0 we may conclude that *𝜒* (*𝐺*) ≥

⌈ ⌉

2 *𝐴, 𝑌* ⟩ such that

*,* diag *𝑌*

*𝑡* . Thus we would

A possible remedy when the underlying combinatorial problem is

each level of the hierarchy. Very quickly, these SDPs are out of reach for current computational algorithms.

like to find the largest value *𝑡* such that val(*𝑃* (*𝑡*)) *>* 0. It turns out that the strengthening of the *𝜗* number towards the chromatic number *𝜒* (*𝐺*)

parameter *𝜗*−(*𝐺*) defined for the complement *𝐺* of *𝐺* was introduced by proposed by [Szegedy (1994)](#_bookmark104) provides the answer to this question. The

trix ∗-algebras. Roughly speaking, the idea is to pre-process the SDP highly structured is to exploit the symmetry by using the theory of ma-

by applying a suitable unitary transformation in such a way that the resulting matrices in the SDP exhibit block diagonal structure. This

structure can then be exploited by interior point methods. We refer [to](#_bookmark53) [de Klerk (2010)](#_bookmark88) [for a survey on this approach, and to Dobre and Vera (2015) for a discussion on how it can be used in approximation](#_bookmark53) hierarchies.

Unfortunately however, not all SDPs exhibit symmetries, and so there is a need for faster algorithms. Maybe in the future other al- gorithms than interior point methods will turn out to be eﬃcient. First attempts are a semismooth Newton-CG augmented Lagrangian method ([Yang et al., 2015](#_bookmark113)) and an augmented Lagrangian method com- bined with a suitable randomization technique ([Yurtsever et al., 2021](#_bookmark120)). The ADMM method has already proved successful when applied to SDP relaxations of binary quadratic problems ([Wen et al., 2010](#_bookmark108)), the quadratic assignment problem ([Oliveira et al., 2018](#_bookmark82)), or the quadratic shortest path problem ([Hu and Sotirov, 2020](#_bookmark87)).

As we have outlined, the past decades have seen an enormous progress in understanding conic problems and using them for modelling purposes. The next decades should be particularly devoted to the nu- merical side.

# Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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