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De-linearizing Linearity: Projective Quantum Axiomatics From Strong Compact Closure

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Abstract

Elaborating on our joint work with Abramsky in [[2](#_bookmark37),[3](#_bookmark38)] we further unravel the linear structure of Hilbert spaces into several constituents. Some prove to be very crucial for particular features of quantum theory while others obstruct the passage to a formalism which is not saturated with physically insignificant global phases.

First we show that the bulk of the required linear structure is purely multiplicative, and arises from the strongly compact closed tensor which, besides providing a variety of notions such as scalars, trace, unitarity, self-adjointness and bipartite projectors [[2](#_bookmark37),[3](#_bookmark38)], also provides *Hilbert-Schmidt norm*, *Hilbert-Schmidt inner-product*, and in particular, the *preparation-state agreement axiom* which enables the passage from a formalism of the vector space kind to a rather projective one, as it was intended in the (in)famous Birkhoff & von Neumann paper [[7](#_bookmark42)].

Next we consider additive types which distribute over the tensor, from which measurements can be build, and the correctness proofs of the protocols discussed in [[2](#_bookmark37)] carry over to the resulting weaker setting. A full probabilistic calculus is obtained when the trace is moreover *linear* and satisfies the *diagonal axiom*, which brings us to a second main result, characterization of the necessary and sufficient additive structure of a both qualitatively and quantitatively effective categorical quantum formalism without redundant global phases. Along the way we show that if in a category a (additive) monoidal tensor distributes over a strongly compact closed tensor, then this category is always enriched in commutative monoids.

*Keywords:* Strong compact closure, quantum mechanics, global phases, projective geometry, categorical trace, quantum logic.

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# Introduction

The formalism of the most successful physical theory of the previous century has many redundant and operationally insignificant ingredients e.g. the redun- dancy of *global phases*. Its creator himself, John von Neumann [[23](#_bookmark58)], was very aware of this fact [[20](#_bookmark55)]. The key insight of the (in)famous 1936 Birkhoff-von Neumann paper entitled “The Logic of Quantum Mechanics” [[7](#_bookmark42)] is that when eliminating redundant global scalars one passes from a vector space to a *projec- tive space*. Such a projective space has a non-distributive lattice of subspaces and hence the deducted natural level of abstraction was a lattice-theoretic one, but after seven decades there is still no satisfactory abstract counterpart to the role which the tensor product plays in von Neumann’s Hilbert space formalism. Also, the world of lattices is insufficiently comprehensive to give any explicit account on probabilities, which are traditionally left implicit by relying on Gleason’s theorem [[14](#_bookmark44)] e.g. Piron’s book [[19](#_bookmark54)]. As discussed in [[2](#_bookmark37)], another shortcoming of von Neumann’s formalism is the total lack of *types* reflecting kinds e.g. *f* : H → H can be reversible dynamics, a measurement, either destructive or non-destructive, or a mixed state.

Typing and finding an appropriate abstraction of the quantum formal- ism was (re-)addressed by Abramsky and myself in [[2](#_bookmark37)] where we recast the formalism of quantum mechanics in purely category-theoretic terms. We con- sidered *strongly compact closed categories* with *biproducts* and we showed that all the Hilbert space machinery necessary for quantum mechanics arises in that setting, but now equipped with appropriate types and high-level tools for reasoning about entanglement — following the tradition of *linear logic* we will refer to the strongly compact closed structure as the *multiplicative* part of the structure and to the *biproducts* as the *additive* part of the struc- ture. The abstract counterpart to the Hilbert space tensor product is now a structural primitive from which, surprisingly, most of the required ingredi- ents for a quantum formalism can be derived [[2](#_bookmark37),[3](#_bookmark38)]. Hence we postulated the axioms of (finitary dimensional) quantum mechanics in terms of strong com- pact closure, and biproducts, and it turned out that many non-trivial results obtained within von Neumann’s formalism such as quantum teleportation, logic-gate teleportation and entanglement swapping become almost trivial in the abstract setting. Moreover, the abstract setting is far more *expressive* and is *explicitly operational* (in the compositional sense), and of course, ad- mits a lot more *axiomatic freedom*, and, last but definitely not least, turns out to still be a *quantitative* setting. But unfortunately the biproduct struc- ture comes together with redundant global phases and also with semi-additive

enrichment, [3](#_bookmark1) in layman’s terms, a vector space like calculus which excludes anything of the projective kind which is non-trivial. Biproducts as in [[2](#_bookmark37)] [4](#_bookmark2) and in particular the pairing operation of the product structure also cause a col- lapse of the classical information flow onto the superposition structure, due to which the physically and syntactically different entities ‘classical bit equipped with probability weights’ and ‘qubit’ become categorically isomorphic.

The goal of this paper is to address these problems of the additive part of the structure by reconsidering von Neumann’s initial concern which led to quantum logic, but this time not with Birkhoff but with category theory as a close friend. While in [[7](#_bookmark42)], starting from a single Hilbert *space*, one first eliminates global scalars and then aims at finding the appropriate abstraction,

i.e. diagramatically,

Hilbert space H ¸¸¸

¸¸¸¸¸

kill redundant global scalars

J

¸¸¸¸¸

¸¸¸¸¸

¸¸¸z ˛

lattice of subspaces L(H)

go abstract

abstract lattices

we will start from the whole *category* of finite dimensional Hilbert spaces and linear maps FdHilb, with strongly compact closed categories with ‘some additive structure’ as its appropriate abstraction, and then study the abstract counterpart to ‘elimination of redundant global scalars’, i.e. diagramatically,

go abstract

FdHilb ¸ ‘vectorial’ strong compact closure

¸¸¸¸¸

¸¸¸¸¸

¸¸¸¸¸

kill redundant global scalars

¸¸¸¸¸zJ˛

‘projective’ strong compact closure

Since the bulk of the required linear structure is already present in the strong compact closed structure, there is no need for commitment to the highly de- manding biproduct structure, and we expose the *necessary* and *suﬃcient* ad- ditive structure required for an effective categorical quantum formalism which

3 Following Selinger [[21](#_bookmark56),[22](#_bookmark57)] semi-additive enrichment does admit a probabilistic interpreta- tion when considering density matrices, but these only arise from an irreversible construction on the initial biproduct category cf. [[22](#_bookmark57)] and Definition [3.1](#_bookmark13) in this paper.

4 By this we mean biproducts as the type for superposition e.g. defining a qubit as *Q* I⊕I.

In [[21](#_bookmark56)] biproducts do not play this role, they encode classical control. In [[22](#_bookmark57)] there are two levels of biproducts, one which generates superposition but of which the pairing operation gets erased in more or less the same manner as we do it in the construction in Def. [3.1](#_bookmark13) of this paper — see also the paragraph ‘parallel work’ below.

includes a probability calculus, but excludes global phases. Abstract counter- parts to ‘eliminating global phases’ and ‘absence of global phases’ are intro- duced in Sections [2](#_bookmark3) and [3](#_bookmark14). In Sections [4](#_bookmark17), [7](#_bookmark33) we study the qualitative and the quantitative structural requirements on the additive component of a categor- ical quantum formalism, respectively referred to as an *ortho-structure* and an *ortho-Bornian structure*. In Section [5](#_bookmark24) we re-address the categorical semantics of [[2](#_bookmark37)] and deal with the above mentioned problem of the collapse of the clas- sical information flow onto the superposition structure in two possible ways. An important physically significant feature of dumping biproducts is that the dominant role of the scalars vanishes — cf. in the case of biproducts all finitary morphisms arise as matrices in the semiring of scalars. The resulting sole sig- nificance of a scalar is that of a probability weight e.g. there is no connection anymore with the relative phases responsible for interference phenomena. We discuss this issue briefly in Section [8](#_bookmark35).

Proofs, details, discussion and some more results.

These can be found in [[9](#_bookmark45)] which is an extended version of this paper. Additional sections includes a construction which adds abstract global phases and hence provides a (partial) converse to Def. [3.1](#_bookmark13); this yields an abstract equivalence which resembles the fundamental theorem of projective geometry relating projective spaces and vector spaces.

Other work.

The aim of this paper and the conception of the utterance ‘quantum logic’ is different from the work by Abramsky and Duncan in [[4](#_bookmark39)] and by Abramsky in [[1](#_bookmark36)]. Their aim is to find a geometric model and syntax for automated rea- soning within our categorical formalism of [[2](#_bookmark37),[3](#_bookmark38)] in the spirit of the *proof-net* calculus for linear logic, anticipating on the fact that many quantum protocols such as quantum teleportation have an underlying diagrammatic interpreta- tion in terms of the *quantum information-flow*, introduced in [[8](#_bookmark43)] and abstractly axiomatized by Abramsky and myself

Parallel work.

Selinger’s latest [[22](#_bookmark57)] and this paper — which were simultaneously and inde- pendently written — have a non-empty intersection. Our *WProj*-construction for strongly compact closed categories coincides with Selinger’s canonical em- bedding of a strongly compact closed category C in its category of *completely positive maps* CPM(C). Also in [[22](#_bookmark57)], Selinger proposes a graphical language for strong compact closure for which he proved completeness for equational reasoning — we have been using a similar language in a more informal manner

[[2](#_bookmark37),[3](#_bookmark38)] and continue(d) to do so in this paper. We also mention the independent work by Baez [[5](#_bookmark40)] which relates to the developments in [[2](#_bookmark37),[3](#_bookmark38)] and by Kauffman

[[16](#_bookmark51)] which relates to those in [[8](#_bookmark43)].

Subsequent work.

In [[12](#_bookmark46),[11](#_bookmark47)] we take a very different approach than the one proposed in the second part of this paper. Rather than relying on additive types for describing quantum measurements we abstract over classical data and define quantum measurements purely multiplicatively, by considering self-adjoint Eilenberg- Moore coalgebras for comonads induced by a special kind of internal comonoid. Via Selinger’s construction we were then able to build a *decoherence* morphism which takes into account the informatic irreversibility, exactly what we which to accommodate in this paper by relaxing the additive structure. Moreover, this approach extends to POVMs via an abstract variant of Naimark’s theo- rem. It remains to be seen how that approach relates to the results presented here, but it’s fair to say that the results in [[11](#_bookmark47),[12](#_bookmark46)] seem at the moment more compelling than those presented in the second part of this paper. In recent work [[10](#_bookmark48)] we also provide an axiomatic characterization of Selinger’s construc- tion, which turns out to be strongly intertwined with the preparation-state agreement axiom that we introduce in the first part of this paper.

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# Some observations on strong compact closure

Recall that a *strongly compact closed category* (SCCC) [[3](#_bookmark38)] is a symmetric monoidal category (SMC) [[18](#_bookmark53)], hence with unit I, natural isomorphisms *λA* : *A* I⊗*A* and *ρA* : *A* *A*⊗I, associativity *αA,B,C* : *A*⊗(*B* ⊗*C*) (*A*⊗*B*)⊗*C* and symmetry *σA,B* : *A* ⊗ *B* *B* ⊗ *A*, and, with

* an involution *A* '→ *A*∗ on objects called *dual*,
* a contravariant identity-on-objects monoidal involution *f* '→ *f* † on mor- phisms called *adjoint*, and,
* for each object a distinct morphism *ηA* :I → *A*∗ ⊗ *A* called *unit*,

which satisfy (1)

† ◦ (*η*† ∗ ⊗ 1*A*) ◦ (1*A* ⊗ *ηA*) ◦ *ρA* = 1*A*

*λ*

*A*

*A*

and the coherence condition *ηA*∗ = *σA*∗*,A* ◦ *ηA*, and all natural isomorphisms *χ* of the symmetric monoidal structure should satisfy *χ*−1 = *χ*†, that is, they are *unitary*. Every SCCC is also a *compact closed category* (CCC) [5](#_bookmark6) [[17](#_bookmark52)] and we recall that a CCC is a ∗-autonomous category [[6](#_bookmark41)] with a self-dual tensor

i.e. with natural isomorphisms *uA,B* : (*A* ⊗ *B*)∗ *A*∗ ⊗ *B*∗ and *u*I : I∗ I. For an SCCC we will assume that *u*I is also unitary and that *uA,B* is strict. As shown in [[3](#_bookmark38)] the adjoint of an SCCC decomposes as

*f* † = (*f* ∗)∗ = (*f*∗)∗

where both (−)∗ and (−)∗ are involutive, respectively contravariant and co- variant, and have *A* '→ *A*∗ as action on objects. We will be using two distinct unfoldings of the *name* *f* ’ :I → *A*∗⊗ *B* of a morphism *f* : *A* → *B*, either the usual definition, or, the *absorption lemma* in [[2](#_bookmark37)] (Lemma 3.7), respectively,

(2)

(1*A*∗ ⊗ *f* ) ◦ *ηA*

Defn*.*

=: *f*

’

Lemma

:= (*f* ∗

⊗ 1*B*) ◦ *ηB .*

Still following [[3](#_bookmark38)] each morphism also defines a *bipartite projector*

P*f* := *f* ’ ◦ ( *f* ’)† : *A*∗ ⊗ *B* → *A*∗ ⊗ *B.*

In any SMC C there exists a commutative monoid of *scalars*, namely C(I*,* I) the endomorphism monoid of the tensor unit [[17](#_bookmark52)]. As in [[2](#_bookmark37),[3](#_bookmark38)] we define *scalar multiplication* by setting

*s* • *f* := *λ*−1 ◦ (*s* ⊗ *f* ) ◦ *λA* : *A* → *B.*

*B*

Lemma 2.1 *Let f and g be a morphisms and s, t scalars in an SMC, then*

(*s* • *f* ) ◦ (*t* • *g*)= (*s* ◦ *t*) • (*f* ◦ *g*) and (*s* • *f* ) ⊗ (*t* • *g*)= (*s* ◦ *t*) • (*f* ⊗ *g*) *.*

Each complex number can be written as *r* · *eiθ* with *r* ∈ R and *θ* ∈ [0*,* 2*π*[ to which we respectively refer as the *amplitude* and the *phase*. Quantum theory dictates that the states of quantum systems are represented as one-dimensional subspaces of a Hilbert space, that is, (non-zero) vectors in a Hilbert space *up to a* (non-zero) *scalar multiple*. Hence when specifying operations on quantum systems we need only to express to which vector a vector is mapped *up to a* (non-zero) *scalar multiple*. Hence FdHilb is saturated with *global scalars* which are superfluous for quantum theory. If we eliminate these, then, since

5 It should be clear to the reader that in the context of this paper a compact closed category cannot be confused with a cartesian closed category.

states are also encoded as morphisms, we also eliminate the redundancy in their description. We would moreover like to eliminate these global scalars using a procedure which applies to any SCCC. But in fact we only want to eliminate *global phases*, since, as shown in [[2](#_bookmark37)], *global amplitudes* allow us to encode probability weights, and are crucially intertwined with the abstract inner-product via the abstract *Born rule*. In the case of FdHilb, if *f* = *eiθ* · *g* with *θ* ∈ [0*,* 2*π*[ for *f, g* : H1 → H2 then

*f* ⊗ *f* † = *eiθ*· *g* ⊗ (*eiθ*· *g*)† = *eiθ*· *g* ⊗ *e*−*iθ*· *g*† = *g* ⊗ *g*† *.*

The following lemma indicates that the passage *f* '→ *f* ⊗*f* † causes also abstract global phases to vanish in some sort of similar manner.

Proposition 2.2 *For f and g morphisms and s, t scalars in an SCCC,*

*s* • *f* = *t* • *g , s* ◦ *s*† = *t* ◦ *t*† = 1I =⇒ *f* ⊗ *f* † = *g* ⊗ *g*†*.*

Observing that 1−1 = 1I it actually suffices to assume existence of a scalar *x* such that *s* ◦ *s*† = *t* ◦ *t*† = *x*−1. But the real surprise is the fact that there exists a converse to Proposition [2.2](#_bookmark7). It is moreover a stronger result in the sense that it extends beyond cases where there exists an inverse to *s*◦*s*† = *t*◦*t*†. This shows that abstract removal of global phases is truly genuine and not merely generalization by analogy.

I

Proposition 2.3 *For f and g morphisms in an SCCC with scalars S, f* ⊗ *f* † = *g* ⊗ *g*† =⇒ ∃*s, t* ∈ *S* : *s* • *f* = *t* • *g , s* ◦ *s*† = *t* ◦ *t*†*.*

*In particular can we set* [6](#_bookmark10)

(3) *s* := ( *f* ’)† ◦ *f* ’ and *t* := ( *g*’)† ◦ *f* ’*.*

We prove Proposition [2.3](#_bookmark8) using pictures. We represent units by triangles and their adjoints by the same triangle but depicted upside down where we take a from bottom to top reading convention. Other morphisms are depicted by square boxes. E.g. the scalar *s* := ( *f* ’)† ◦ *f* ’ is depicted as

6 By symmetry we could also set *s* := ( *f* ’)† ◦ *g*’ and *t* := ( *g*’)† ◦ *g*’.

*η*†

*η*

*f*

*f* †

Bifunctoriality means that we can move these boxes upward and downward, and naturality provides additional modes of movement e.g. scalars admit ar- bitrary movements — one could say that they are not localized in time nor in space but, in Kripke’s terms, they *provide a weight for a whole world*. Given that *f* ⊗ *f* † = *g* ⊗ *g*†, that is, in a picture,

=

*f*

*g*

*f* †

*g*†

it follows from the picture below that *s* • *f* = *t* • *g*,

=

*η*†

*f* †

*f*

*f*

*η*

*η*†

*g*†

*g*

*f*

*η*

while the picture below shows that *s* ◦ *s*† = *t* ◦ *t*†,

=

*η*†

*η*†

*η*

*η*

*f*

*f* †

*f*

*f* †

*η*†

*η*†

*g*†

*f* †

*f*

*g*

*η*

*η*

This completes the proof of Proposition [2.3](#_bookmark8).

In an SCCC one can also show that

(4)

*ψ* ◦ *ψ*† = *ρ*†

◦ (*ψ* ⊗ *ψ*†) ◦ *λA*

where we note that in the case of FdHilb the linear operator *ψ* ◦*ψ*† : *A* → *A* is the density matrix representing the pure state *ψ* :I → *A* i.e. the state usually

*A*

represented by the vector *ψ*(1) ∈ H. In other words (see [[2](#_bookmark37)]), since *ψ* and *ψ*† are respectively to be conceived as a ket |*ψ*⟩ and a bra ⟨*ψ*|, their composite *ψ* ◦ *ψ*† corresponds to the ket-bra |*ψ*⟩⟨*ψ*|. Consider now von Neumann’s formalism in FdHilb. When passing from vectors *ψ* to density matrices *ψ* ◦ *ψ*† we cancel out global phases. The global amplitudes are squared and hence provide true probability weights. This *trick* however does not extend to morphisms. Indeed, for *U* : *A* → *B* unitary we have *U* ◦ *U* † = 1*B* so we lose all its content. But eq.([4](#_bookmark11)) tells us that for states we obtain the same effect (that is eliminating global phases) by passing to *ψ* ⊗ *ψ*† instead of *ψ* ◦ *ψ*†, and this method does extend in abstract generality. Propositions [2.2](#_bookmark7) and [2.3](#_bookmark8) then tell us that the desired effect also extends in abstract generality for arbitrary morphisms.

Assignments ([3](#_bookmark9)) show that to any morphism, and also to any pair of mor- phisms we can attribute a special scalar. Recall that the Hilbert-Schmidt norm of a bounded linear map *f* : H1 → H2, if it exists, is *i*⟨*f* (*ei*) | *f* (*ei*)⟩ [[13](#_bookmark49)]. Such a map which admits a Hilbert-Schmidt norm is an Hilbert-Schmidt map. When H1 = H2 = H all Hilbert-Schmidt maps S(H) constitute a Ba- nach algebra with *i*⟨*f* (*ei*) | *g*(*ei*)⟩ as an inner-product [[13](#_bookmark49)]. Hence S(H) is itself a Hilbert space. We still have such a Hilbert space structure if H1 /= H2 (we only lose the compositional structure).

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√Σ

Definition 2.4 For each morphism *f* in an SCCC C we define its *squared Hilbert-Schmidt norm* as ||*f* || := ( *f* ’)† ◦ *f* ’ ∈ C(I*,* I).

In FdHilb we have ||*f* ||(1) = Σ*i*⟨*f* (*ei*) | *f* (*ei*)⟩H˜ for *f* : H → H˜. For

Hilbert spaces H1 and H2 , and HS(H1*,* H2) the Hilbert space of all Hilbert-

Schmidt maps *f* : H1 → H2, we have HS(H1*,* H2) H1 ⊗ H2, so it should not be a surprise that exactly this norm naturally arises in our setting.

Definition 2.5 For morphisms *f, g* : *A* → *B* in an SCCC C we define the

*Hilbert-Schmidt inner-product* as ⟨*f* | *g*⟩ := ( *f* ’)† ◦ *g*’ ∈ C(I*,* I).

Recall from [[2](#_bookmark37),[3](#_bookmark38)] that the *inner-product of states ψ, φ* :I → *A* in an SCCC is given by *ψ*† ◦ *φ* ∈ C(I*,* I). The Hilbert-Schmidt inner-product provides a genuine generalization of this inner-product for states.

Proposition 2.6 *For morphisms ψ, φ* :I → *A in an SCCC we have*

( *ψ*’)† ◦ *φ*’ = *ψ*† ◦ *φ.*

A nice application of Proposition [2.6](#_bookmark12) is the derivation of the version of the Born rule which uses the trace and density matrices. Recall that a projector in the spectral decomposition attributed to a measurement decomposes as

P= *π*† ◦ *π* and that

Prob(*ψ,* P) := *ψ*† ◦ P ◦ *ψ* = (*π* ◦ *ψ*)† ◦ (*π* ◦ *ψ*)

is the corresponding *abstract probability* of P for measuring a system in state *ψ* : I → *A*. In the density matrix version of quantum mechanics [[23](#_bookmark58)] the probability rule is Prob(*ρ,* P) := Tr(P ◦ *ρ*) where *ρ* = *ψ* ◦ *ψ*† is the density matrix corresponding to the state *ψ* and Tr assigns to a matrix (*fij*)*ij* the trace *i fii*. Now recall from [[3](#_bookmark38)] that any strongly compact closed category admits

Σ

a categorical *partial trace* in the sense of [[15](#_bookmark50)] — this follows straightforwardly

from the corresponding result for compact closed categories [[17](#_bookmark52)] — for which the corresponding (*full*) *trace* of *f* : *A* → *A* is

Tr(*f* ) := *ηA* ◦ (1*A*∗ ⊗ *f* ) ◦ *ηA* ∈ C(I*,* I) *.*

†

Hence ⟨*f* | *g*⟩ = Tr(*f* † ◦ *g*). In FdHilb this categorical trace coincides with the linear algebraic one. Passing to the Hilbert-Schmidt inner-product through Proposition [2.6](#_bookmark12), applying eq.([2](#_bookmark5)), bifunctoriality and again eq.([2](#_bookmark5)),

Prob(*ψ,* P) = *ψ*† ◦ (P ◦ *ψ*)

= *η*† ◦ (1I∗ ⊗ *ψ*†) ◦ (1I∗ ⊗ (P ◦ *ψ*)) ◦ *η*I

I

= *η*†

*A*

= *η*†

*A*

= *η*†

*A*

◦ ((*ψ*†)∗ ⊗ 1*A*) ◦ (1I∗ ⊗ (P ◦ *ψ*)) ◦ *η*I

◦ (1*A*∗ ⊗ (P ◦ *ψ*)) ◦ ((*ψ*†)∗ ⊗ 1I) ◦ *η*I

◦ (1*A*∗ ⊗ (P ◦ *ψ*)) ◦ (1*A*∗ ⊗ *ψ*†) ◦ *ηA* = Tr(P ◦ *ρ*) *.*

But in a picture all this boils down to merely moving *ψ*† around a loop:

*η*†

*η*

*ψ*†

P ◦ *ψ*

*ψ*†

# The preparation-state agreement axiom

Following Propositions [2.2](#_bookmark7) and [2.3](#_bookmark8) the following construction aims at elimi- nating global phases i.e. it tries to turns a category with *vector space flavored* objects into one with *projective space ‘with weights’ flavored* objects.

Definition 3.1 For each SCCC C we define a category *WProj* (C).

* The objects of *WProj* (C) are those of C.
* The Hom-sets of *WProj* (C) are

1. *WProj* (C)(*A, B*) := {*f* ⊗ *f* † | *f* ∈ C(*A, B*)}

with 1*A* ⊗ 1*A* ∈ *WProj* (C)(*A, A*) being the identities.

* Composition in *WProj* (C), for *A f*) *B g*) *C* in C, is given by (*f* ⊗ *f* †) ¯◦ (*g* ⊗ *g*†) := (*f* ◦ *g*) ⊗ (*f* ◦ *g*)†*.*

Proposition 3.2 *Let* C *be an SCCC. Then WProj* (C) *is also an SCCC.*

Proposition 3.3 *If f and g are morphisms in an SCCC then*

*f* ⊗ *f* † = *g* ⊗ *g*† ⇐⇒ *f* ⊗ *f*∗ = *g* ⊗ *g*∗ ⇐⇒ P*f* = P*g .*

While one easily verifies that *A* '→ *A*, *f* ⊗ *f* † '→ *f* yields (*WProj* ◦ *WProj* )(FdHilb) *WProj* (FdHilb) *,*

idempotence of *WProj* fails to be true for arbitrary SCCC. Hence we are mainly interested in *invariance* (up to isomorphism) under *WProj* .

Theorem 3.4 *WProj* (C) C (*canonically*) *for an SCCC* C *if and only if*

1. *f* ⊗ *f* † = *g* ⊗ *g*† =⇒ *f* = *g.*

Condition ([6](#_bookmark15)) expresses that an SCCC is a *ﬁxed point* of the *WProj* - construction — as our main example *WProj* (FdHilb) is one — and hence it guarantees absence of redundant global phases. Roughly speaking one can think of these fixed points as being the result of consecutively applying *WProj* until all global redundancies are erased. But condition ([6](#_bookmark15)) also admits a lucid interpretation in its own right which is moreover a truly compelling physical motivation to adopt condition ([6](#_bookmark15)) as an *axiom* for any categorical model of abstract quantum mechanics.

Corollary 3.5 *WProj* (C) C (*canonically*) *for an SCCC* C *if and only if*

1. P*f* = P*g* =⇒ *f* ’ = *g*’*.*

Condition ([7](#_bookmark16)) states that if two preparations P*f* and P*g* of bipartite states coincide then we have (of course) that the bipartite states *f* ’ and *g*’ which they produce coincide. And without loss of generality this fact extends to arbitrary states — recall here form [[2](#_bookmark37)] that *ψ* ◦ *ψ*† : *A* → *A* is the projector which prepares the state *ψ* :I → *A*.

Corollary 3.6 *WProj* (C) C (*canonically*) *for an SCCC* C *if and only if*

1. ∀ *ψ, φ* :I → *A, ψ* ◦ *ψ*† = *φ* ◦ *φ*† =⇒ *ψ* = *φ.*

Definition 3.7 An SCCC satisfies the *preparation-state agreement axiom* iff the equivalent conditions ([6](#_bookmark15)), ([7](#_bookmark16)) and ([8](#_bookmark18)) are satisfied.

# Ortho-structure

Besides being an SCCC FdHilb also has *biproducts* i.e. it is *semi-additive*. For an SCCC with biproducts the endomorphism monoid of the tensor unit is always an *involutive abelian semiring*, and the full subcategory of objects of type I ⊕ *...* ⊕ I is isomorphic to the category of matrices in that involu- tive abelian semiring, and conversely, each *matrix calculus* over an involutive abelian semiring provides an example of an SCCC with biproducts [[2](#_bookmark37)].

Theorem 4.1 *There exist no SCCC with biproducts which both satisﬁes the preparation-state agreement axiom and for which the endomorphism monoid* *of the tensor unit is a ring with non-trivial negatives* (*i.e.* −1= 1)*.*

So if a category with as morphisms matrices over a commutative involutive semiring *R* satisfies the preparation-state agreement axiom then *R* cannot have non-trivial negatives, with the fatal consequence that *interference phenomena* relying on cancellation of negatives cannot be modeled. Note that our key example *WProj* (FdHilb) is not isomorphic to the matrix calculus over its scalar monoid R+. Next set [*f* ] := {*g* | *f* ⊗ *f* † = *g* ⊗ *g*†}.

Theorem 4.2 *The product structure and the symmetric monoidal* − ⊕ − *structure of* FdHilb *do not carry over to WProj* (FdHilb)*. In particular, in an SCCC with biproducts f* ' ∈ [*f* ] *and g*' ∈ [*g*] *do not imply f* ' ⊕ *g*' ∈ [*f* ⊕ *g*] *and hence the operation* [*f* ] ⊕¯ [*g*] := [*f* ⊕ *g*] *is ill-deﬁned.*

Proof: While 1C ∈ [1C] and (*eiθ*◦ −) ∈ [1C] we have ⟨1C*,* (*eiθ*◦ −)⟩ /∈ [⟨1C*,* 1C⟩] and 1C ⊕ (*eiθ* ◦ −) /∈ [1C ⊕ 1C].

There is a physical argument why we do not want a product structure. A pairing operation ⟨−*,* −⟩ : C(I*,* I) × C(I*,* I) → C(I*,* I ⊕ I) would allow to deduce

the initial state *ψ* :I → I ⊕ I of a *qubit* from the probabilities

Prob(*ψ, p*†

I*,*I

* *p*I*,*I) ∈ C+(I*,* I) and Prob(*ψ, p*†
* *p*I*,*I) ∈ C+(I*,* I)

of it being subjected to the dichotomic measurement with projectors

I*,*I

†

*p*

I*,*I

* *p*I*,*I :I ⊕ I → I ⊕ I and *p*†
* *p*I*,*I :I ⊕ I → I ⊕ I

since *ψ* = ⟨Prob(*ψ, p*†

I*,*I

I*,*I

* *p*I*,*I) *,* Prob(*ψ, p*†
* *p*I*,*I)⟩. But this contradicts the

*empirical evidence* that a qubit state comprises *relative phase data* — which

I*,*I

is responsible for interference phenomena — which gets erased by a measure- ment. So the intrinsic *informatic irreversibility* [7](#_bookmark21) of quantum measurements clashes with the very nature of the concept of a categorical product.

Part of the symmetric monoidal structure can actually be retained. The problem exposed in Theorem [4.2](#_bookmark19) can be overcome if both [*f* ] and [*g*] contain a particular distinguished morphism, say respectively *f* and *g* themselves. Then we can define their monoidal sum by setting [*f* ] ⊕¯ [*g*] := [*f* ⊕ *g*]. There are important equivalence classes which have such a distinguished element:

* + *Identities* 1*A* : *A* → *B* in [1*A*] as a part of the categorical structure.
  + *Natural isos λA, ρA, σA,B, αA,B,C, u*I as part of the SCCC structure.
  + *Positive scalars s* ◦ *s*† :I → I whenever they are unique in [*s* ◦ *s*†].

Such distinguished morphisms are the only ones for which we need monoidal sums, so we are going to let them play a distinguished role within the ‘min- imally required’ additive structure which we will introduce. Indeed, much of what seems to be additive at first sight turns out to be multiplicative

e.g. while the usual Hilbert-Schmidt norm involves an explicit *summation* of inner-products parameterized over a basis, abstractly it only involves units and adjoints which are both part of the multiplicative SCCC-structure.

Definition 4.3 An *ortho-SCCC* is an SCCC (C*,* ⊗*,* I*, λ, ρ, σ, α,* (−)∗*,* (−)†*, η*) that comes with an *ortho-structure* i.e. a second monoidal structure (−⊕ −) which is total on objects but can be only partial on morphisms, more specif- ically, the symmetric monoidal category (C*,* ⊕*,* 0*, l, r, s, a*) is a subcategory of C of which the objects coincide with those of C, and, which is such that (−⊗−): C×C→ Cis a strong symmetric monoidal bifunctor of which the witnessing natural isomorphisms are unitary and with *λ, ρ, α, σ, u*I symmetric monoidal natural in all variables, and which is also such that the partial bi- functor (−⊕ −) commutes with (−)∗ and (−)† i.e. 0∗ = 0, (*A* ⊕ *B*)∗ = *A*∗ ⊕ *B*∗ for all objects and (*f* ⊕ *g*)† = *f* † ⊕ *g*† for all C-morphisms.

7 Not to be confused with the irreversibility of a projector as a linear map.

The strong symmetric monoidal bifunctor provides (by definition) *distribu- tivity* natural isomorphisms

DIST0*,l* : *A* ⊗ 0 0 DIST*l* : *A* ⊗ (*B* ⊕ *C*) (*A* ⊗ *B*) ⊕ (*A* ⊗ *C*)

DIST0*,r* :0 ⊗ *A* 0 DIST*r* : (*B* ⊕ *C*) ⊗ *A* (*B* ⊗ *A*) ⊕ (*C* ⊗ *A*) *.*

By asking that (− ⊗ −) is a strongly symmetric monoidal bifunctor with *λ, ρ, σ, α, u*I all monoidal natural isomorphisms we make sure that these dis- tributivity isomorphisms behave well with respect to the natural isomorphisms of the symmetric monoidal structure on C.

Proposition 4.4 *For each pair of objects A, B in an ortho-SCCC there exists a distinguished morphism* 0*A,B* : *A* → *B which, for all f* : *B* → *C satisﬁes*

*f* ◦ 0*A,B* = 0*B,C* ◦ *f* = 0*A,C*

*and which is explicitly deﬁned in*

*A*  ) I ⊗ *A η*0 ⊗ 1)*A* (0∗ ⊗ 0) ⊗ *A*  ) 0

0*A,B* 10

v v

*B* ( I ⊗ *B* ( (0∗ ⊗ 0) ⊗ *B* ( 0

*η*† ⊗ 1*B*

0

Hence, without assuming the universal property of a zero object we do obtain a special family of morphisms which behave similarly. We will set

0*A* := 0*A,*0, hence 0*A,B* := 0†

*B*

* 0*A* : *A* → *B*. We define *pseudo-projections* and

the *pseudo-injections* respectively as

*pA,B* := *r*† ◦ (1*A* ⊕ 0*B*): *A* ⊕ *B* → *A*

*A*

*pA,B* := *l*† ◦ (0*A* ⊕ 1*B*): *A* ⊕ *B* → *B qA,B* := (1*A* ⊕ 0† ) ◦ *rA* : *A* → *A* ⊕ *B*

*A*

*B*

*qA,B* := (0†

*A*

⊕ 1*B*) ◦ *lA* : *B* → *A* ⊕ *B*

and the *pseudo-components* of a morphism *f* : *i Ai* → *j Bj* are

*fij* := *pj* ◦ *f* ◦ *qi* : *Ai* → *Bj .*

Of course in general these do not admit any kind of matrix calculus.

Proposition 4.5 *In an ortho-SCCC we have*

†

*q*

*A,B*

*pA,B* ◦ *qA,B* = 1*A pA,B* ◦ *qA,B* = 0*A,B*

= *pA,B* = *pB,A* ◦ *sA,B pB,D* ◦ (*f* ⊕ *g*)= *f* ◦ *pA,C*

1*A* ⊕ *pB,C* = *pA*⊕*B,C* ◦ *aA,B,C pA,B* ◦ *pA*⊕*B,C* = *pA,B*⊕*C* ◦ *a*† *.*

*A,B,C*

Proposition 4.6 *The components πi* := *pi* ◦ *U* : *A* → *Ai of a unitary*

*morphism U* : *A* →  *i Ai are ‘co-normalized’ i.e. πi* ◦ *π* = 1*A and ‘co-*

†

*i*

*i*

*orthogonal’ for i* /= *j i.e. πj* ◦ *π*† = 0*A ,A . Analogously, the components*

*i i j*

*ψi* := *U* ◦ *qi* : *Ai* → *A of unitary morphism U* :

*i Ai* → *A are ‘normalized’*

*i.e. ψ*† ◦ *ψ* = 1*A*

*i*

*and ‘orthogonal’ for i* /= *j i.e. ψ*† ◦ *ψi* = 0*A ,A . Unitary maps*

*preserve normality, co-normality, orthogonality and co-orthogonality.*

*i*

*j*

*i*

*j*

Note that the partial monoidal sum on morphisms did not come with an operational significance since its only aim was to provide pseudo-projections and pseudo-injections with appropriate properties. But they do much more than this, they also provide sums.

Theorem 4.7 *An ortho-SCCC is (partially) enriched in commutative monoids*

*i.e. admits a notion of sum of morphisms, where the sum of f, g* : *A* → *B for which f* ⊕ *g exists is given by*

*η*† ⊗ 1*B*

*B* ( I ⊗ *B* (2

ˆ

*f* + *g*

(2∗⊗ 2) ⊗ *B* ( 2∗ ⊗ (*B* ⊕ *B*)

ˆ

12∗ ⊗ (*f* ⊕ *g*)

*A*  ) I ⊗ *A*  ) (2∗⊗ 2) ⊗ *A*  ) 2∗ ⊗ (*A* ⊕ *A*)

*η*2 ⊗ 1*A*

*and with the additive units as in Proposition* [*4.4*](#_bookmark22)*.*

We can put this in a slogan:

SCCC + ⊕ + distributivity =⇒ CMon-enrichment

# Categorical semantics for protocols

An ortho-SCCC provides enough structure for the description and correctness proofs of the protocols considered in [[2](#_bookmark37)]. Two approaches are possible.

* 1. Distinct types for superposition and weighted branching.

When starting from an ortho-SCCC it suffices to add classical branching

*freely* as a product structure i.e. *sum types* (*A*1*,... , An*), *pairing* ⟨*f*1*,... , fn*⟩ :

*C* → (*A*1*,... , An*) and *projections p*˜*i* : (*A*1*,... , An*) → *Ai*. This *branch-*

*ing structure* enables classical statistics and measurement outcome dependent manipulation of data i.e. *classical information flow*, while the ortho-structure provides the interface between the quantum state space and the classical world. We adapt some examples from [[2](#_bookmark37)] to the context of an ortho-SCCC with freely added products. Each unitary morphism *U* : *A* → *i*=*n Ai* defines a

*i*=1

*non-destructive measurement* ⟨P*i*⟩*i*=*n* : *A* → (*A*)*i*=*n* where P*i* := *π*† ◦ *πi* with

*i*=1 *i*=1 *i*

*πi* := *pi* ◦ *U* . While in [[2](#_bookmark37)] classical communication is encoded as distributivity

isomorphisms here we have

CC*A*←(*B,C*) := ⟨1*A* ⊗ *p*˜1*,* 1*A* ⊗ *p*˜2⟩ : *A* ⊗ (*B, C*) → (*A* ⊗ *B, A* ⊗ *C*)

which admits no inverse, reflecting the fact that in absence of the ability to erase information, distributing information is irreversible. Also, while there is a canonical map ⟨*p*I*,*I*, p*I*,*I⟩ : I⊕I → (I *,* I), namely the destructive measurement associated to the unitary morphism 1I⊕I, this map has no inverse, and hence there exists no isomorphism between a *qubit Q* I⊕I anda *weighted bit* (I*,* I). More concretely, we define a *destructive teleportation measurement* by means of a unitary morphism *T* : *Q* ⊗ *Q* → I ⊕ I ⊕ I ⊕ I which is such that there exist unitary maps *β*1*, β*2*, β*3*, β*4 : *Q* → *Q* with *βi*’ = *T* † ◦ *qi*. The destructive teleportation measurement itself is

⟨*pi* ◦ *T* ⟩*i*=4 = ⟨ *βi*’†⟩*i*=4 : *Q* ⊗ *Q* → (I*,* I*,* I*,* I) *.*

*i*=1 *i*=1

Theorem 5.1 *The theorems stated in* [[2](#_bookmark37)] *on correctness of the example pro- tocols for an SCCC with biproducts carry over to any ortho-SCCC with freely added products when using the above deﬁnitions.*

Hence it indeed suffices for the ortho-structure to be limited to assuring coherent coexistence of the pseudo-projections with the SCCC structure since for all the other qualitative uses of the biproduct structure in [[2](#_bookmark37)] we can as well use the freely added product structure which does not genuinely interact with the SCCC structure. Conclusively, we decomposed the additives in a *fundamental structural component*, namely the ortho-structure, and, a *classical branching structure*, which can be freely added as a product structure. This classical branching structure can of course be of a more sophisticated nature than the one we used here, for example one might want to capture classical mixing, but the bottom line is that it can be introduced on top of the ortho- SCCC structure and hence is not an intrinsic ingredient.

* 1. [[2](#_bookmark37)]-style semantics.

One keeps a minimal number of non-isomorphic types by distinguishing between explicit and non-explicit sums. For example, when *Q* I ⊕ I then *Q* represents a qubit i.e. the *superposition* of I and I, while I⊕I represents a pair of *probabilistic weights* attributed to two *branches* of scalar type e.g. the respec- tive probabilities of a destructive non-trivial qubit measurement. Since this semantics is discussed in detail in [[2](#_bookmark37)] we only point at the required modification when starting from an ortho-SCCC rather than from an SCCC with biprod-

ucts. The key observation is that given a unitary morphism *U* : *A* → *Ai*

*i*=*n*

*i*=1

with *qi* : *Ai* → *A* pseudo-injections we can define the corresponding non- destructive measurement as

*i*=*n*

*U* † ◦

*i*=1

*i*=*n*

*qi*

*i*=1

* *U* : *A* →

*i*=*n*

*A*

*i*=1

which in the case of biproducts coincides with ⟨*π*† ◦ *πi*⟩*i*=*n* : *A* → *i*=*n A*.

*i*

*i*=1

*i*=1

Note that *qi* ∈ Cand that it is also reasonable to assume meaningfulness of *i*=*n U* † i.e. *n* copies of the same morphism. However, branch dependent

*i*=1

operations *i*=*n fi* do require a sufficiently large additive monoidal structure. The state of the *j*th branch is obtained by applying

*i*=1

*pj* ◦− : C

*i*=*n*

*A, Ai*

*i*=1

→ C(*A, Aj*) *,*

and the absence of a pairing operation as in *WProj* (FdHilb) prevents the collapse of the classical information flow onto the superposition structure.

# What is a Born-rule?

Given a model which intends to describe quantum mechanics, including *states*

S, *measurements* M, *probability weights* W, a *unit* 1 ∈ W and an addition

− + − : W × W → W, a *Born rule* assigns to each tuple consisting of a state *ψ* ∈ S, a measurement *M* ∈ M which applies to *ψ* and an outcome *i* ∈ spec(*M* ) a probability weight *s*(*ψ, M, i*) ∈ W such that

Σ *s*(*ψ, M, i*)= 1 *.*

*i*∈spec(*M* )

If our model is both *compositional* and if states *carry a probability weight*

which can be extracted by means of a *valuation* |− |*ξ* : S → W we obtain

Σ

|*Mi* ◦ *ψ*|*ξ* = |*ψ*|*ξ*

*i*

where *Mi* ◦− stands for the *action* of *M* on the state *ψ* whenever the outcome *i* occurs in that measurement. E.g. in FdHilb we have |*ψ*|FdHilb := ⟨*ψ* | *ψ*⟩ for *ψ* ∈ H and *Mi* := P*i* : H → H for the non-destructive measurement represented by the self-adjoint operator *M* = *i ai* · P*i* : H → H, so since

Σ

|P*i* ◦ *ψ*|FdHilb = ⟨P*i* ◦ *ψ* | P*i* ◦ *ψ*⟩ = ⟨*ψ* | P*i* ◦ *ψ*⟩ by self-adjointness of P*i* and

Σ

since *i* P*i* = 1H the usual Born rule slightly generalized to the case that

|*ψ*|FdHilb /= 1 arises i.e.

|P*i* ◦ *ψ*|FdHilb = |*ψ*|FdHilb *.*

Σ

*i*

Using P*i* = *π*† ◦ *πi* with *πi* : H → G*i* and hence H

G*i* we have |P*i* ◦

*i* *i*

satisfying *pi* ◦ *U* = *πi*, where *p* *j* : *i* G*i* → G*j* are the canonical projections,

*ψ*|FdHilb = |*πi* ◦ *ψ*|FdHilb, and for *U* : H →

*i* G*i* the unique unitary map

when setting *φ* := *U* ◦*ψ* : C →

introducing the *components* of *φ* as *φi* := *pi* ◦ *φ* = *pi* ◦ *U* ◦ *ψ* = *πi* ◦ *ψ* : C → G*i*

all the above results in

*i* G*i* we obtain |*ψ*|FdHilb = |*φ*|FdHilb. When also

|*φi*|FdHilb = |*φ*|FdHilb *.*

Σ

*i*

When replacing the *squared vector norm* |− |FdHilb = ⟨− | −⟩ : (H→ C) → C which only applies to morphisms of the type C → H by the *squared Hilbert- Schmidt norm* |− |FdHilb = *i*⟨−(*ei*) | −(*ei*)⟩ : (H1 → H2) → C we obtain by an analogous calculation that

Σ

Σ

1. |*fi*|FdHilb = |*f* |FdHilb

*i*

where now *f* : H → *i* G*i* and *fi* := *pi* ◦ *f* : H → G*i* with H arbitrary. For obvious reasons eq.([9](#_bookmark25)) is our favorite incarnation of the orthodox Born rule.

Hence expressing a Born rule requires a *scalar sum* − + − and a scalar- valued *valuation* on morphisms |− |*ξ*, and when interpreting scalars as proba- bilistic weights these respectively stand for *adding probabilities* and *extracting* the *probabilistic weight* from the morphisms representing physical processes. The Born rule itself should then express that ‘taking components of mor- phisms’, that is, physically speaking, ‘branching due to measurements’, reflects through the valuation at the level of the scalars in terms of a decomposition

over the scalar sum, diagrammatically,

C(*A, B* ⊕ *...* ⊕ *B* ) ⟨*p*1 ◦− *, . . . , pn* ◦ −⟩Set) C(*A, B* ) × *...* × C(*A, B* )

1 *n* 1 *n*

|− |*ξ*

v

|− |*ξ* × *...* ×| − |*ξ*

v

C(I*,* I) ( C(I*,* I) × *...* × C(I*,* I) *.*

− + *...* + −

Physically this means that the total probability weight is preserved when considering all branches i.e. a *conservation law*, which in particular implies that *relative phases lost in measurements carry no probabilistic weight*.

For *f* : *A* → *B* and *h* : *A* → *A* we (re-)set

||*f* || := *η*†

*A*

◦ (1*A*∗ ⊗ (*f* † ◦ *f* )) ◦ *ηA* and Tr(*h*) := *η*†

◦ (1*A*∗ ⊗ *h*) ◦ *ηA .*

The *trace* Tr only applies to *endomorphic types* and ||*f* || = Tr(*f* † ◦ *f* ). In an SCCC with biproducts the trace is *linear* i.e. Tr(*h* + *h*')= Tr(*h*)+ Tr(*h*').

*A*

Proposition 6.1 *Each SCCC with biproducts admits a Born rule, namely,*

||*f*1|| + *...* + ||*fn*|| = ||*f* || *for any morphism f* : *A* → *B*1 ⊕ *...* ⊕ *Bn.*

√

We set ||C|| for the range of || − || and |*f* | := ||*f* || if ||*f* || has a unique square-root. We take a scalar to be *positive* iff it decomposes as *x* ◦ *x*† and has at most one square-root in

C+(I*,* I) := {*s*† ◦ *s* | *s* ∈ C(I*,* I)}

and an SCCC C to be *positive valued* iff all the scalars in ||C|| are positive.

Proposition 6.2 *If* C *is a positive valued SCCC with biproducts then*

|*f*1|*WProj* (C) + *...* + |*fn*|*WProj* (C) = |*f* |*WProj* (C)

*for any f* ∈ *WProj* (C)(*A, B*1 ⊕ *...* ⊕ *Bn*) *so WProj* (C) *admits a Born rule.*

When does a non-semi-additive ortho-SCCC admit a Born rule? Does it matter that the valuation involves || − || i.e. relies on the multiplicative structure? Is || − || (cf. Prop. [6.1](#_bookmark26)) or |−| (cf. Prop. [6.2](#_bookmark27)) more canonical than the other? The following lemma shows that if C(I*,* I) ⊆ C(I*,* I) , then for any valuation which is a rational power *ν* of || − || there is a single structural axiom which stands for existence of a Born-rule, and which only relies on the SCCC-structure and on −⊕ − (hence not explicitly on *ν* nor on − + −).

Lemma 6.3 *Let* C *be an ortho-SCCC and let the maps*

|− |*ξ* : C(*A, B*) → C(I*,* I) and − +− : C(I*,* I) × C(I*,* I) → C(I*,* I) *,*

*A,B*

*be such that for all morphisms f* : *A* → *B*1 ⊕ *B*2 *we have*

1. |*f* |*ξ* = |*f*1|*ξ* + |*f*2|*ξ .*
   1. *Scalars s*1*, s*2*, s*3 *satisfy an associative rule* (*s*1 + *s*2)+ *s*3 = *s*1 + (*s*2 + *s*3) *provided there exists a morphism f* : *A* → *B*1⊕*B*2⊕*B*3 *such that si* = |*fi*|*ξ.* *Hence f* : *A* → *B*1 ⊕ *...* ⊕ *Bn satisﬁes* |*f* |*ξ* = |*f*1|*ξ* + *...* + |*fn*|*ξ. If for all scalars* |*s* • −|*ξ* = |*s*|*ξ* ◦ | − |*ξ then s, s*1*, s*2 *satisfy a distributivity rule s* ◦ (*s*1 + *s*2) = (*s* ◦ *s*1)+ (*s* ◦ *s*2) *provided there exists a morphism* *f* : *A* → *B*1 ⊕ *B*2 *and a scalar t such that si* = |*fi*|*ξ and s* = |*t*|*ξ. If*

|(−)†|*ξ* = |− |*ξ and* |0*A,B*|*ξ* = 0I*,*I *then all f, g* ∈ C *satisfy*

(11) |*f* ⊕ *g*|*ξ* = |*f* |*ξ* + |*g*|*ξ .*

* 1. *Let* C(I*,* I) ⊆ C *and assume that for all s* ∈ |C|*ξ there exists a scalar sζ*

*such that* |*sζ*|*ξ* = *s. Then for all s, t* ∈ |C|*ξ we have*

(12) *s* + *t* = |*sζ* ⊕ *tζ*|*ξ ,*

*in the presence of which eq.*([10](#_bookmark29)) *can now be equivalently rewritten as*

(13) |*f* | = |*f* |*ζ* ⊕ |*f* |*ζ .*

*ξ*  1 2

*ξ*

*ξ*

*ξ*

* 1. *If moreover* |−|*ξ* := || −||*ν and if the unique square-roots, the ν*th*-powers and* 1 th*-powers consequently required in eq.*([12](#_bookmark30)) *and eq.*([13](#_bookmark31)) *exist, then eq.*([13](#_bookmark31)) *rewrites equivalently as*

*ν*

(14) ||*f* || = Tr(||*f*1|| ⊕ ||*f*2||) *.*

By Lemma [6.3](#_bookmark28) for |− |*ξ* := || − || and |− |*ξ* := |−| we respectively have

*s* + *t* := Tr(*s* ⊕ *t*) and *s* + *t* = √Tr(*s*2 ⊕ *t*2)= |*s* ⊕ *t*|

assuming self-adjointness of *s* and *t*. More generally, *s* + *t* = (Tr(*sν* ⊕ *t ν* )) for

1

1

*ν*

|− |*ξ* := || − ||*ν* — the *ν*th-power outside the trace and the 1 th-power inside the trace do not cancel out. Different choices of |− |*ξ* yield different sums and hence different abstract integers and rationals e.g. when setting 2I := 1I + 1I we have 2I = Tr(1I⊕I) for || − || and 2I = Tr(1I⊕I) for |− | — recall here that Tr(1C⊕C) = dim(C ⊕ C) = 2 in FdHilb. The key result of Lemma [6.3](#_bookmark28) is of course eq.([14](#_bookmark32)) which we call the *ortho-Bornian axiom*. We will now decompose this ortho-Bornian axiom in two tangible components.

√

*ν*

# Ortho-Bornian structure

An endomorphism *h* : *A* → *A* is called *positive* iff it decomposes as *h* = *f* † ◦ *f* . If *A* = *A*1 ⊕ *...* ⊕ *An* then we call *h*11 ⊕ *...* ⊕ *hnn* : *A* → *A* the *pseudo-diagonal* of *h* with respect to that decomposition of *A*. The collection of all positive morphisms of an SCCC C will be denoted by C+. The ortho-Bornian axiom is now equivalent to validity of

Tr(*h*) = Tr(Tr(*h*11) ⊕ Tr(*h*22)) for all positive morphisms

*h* := *f* † ◦ *f* : *A*1 ⊕ *A*2 → *A*1 ⊕ *A*2 ∈ C+ *.*

Proposition 7.1 *In an ortho-SCCC for h, h*' ⊆ C∩ C+ *we have*

Tr(*h* + *h*')= Tr(*h* ⊕ *h*')

*with respect to the sum deﬁned in Theorem* [*4.7*](#_bookmark23)*. Hence on positive scalars this sum is the one corresponding to the valuation* || − || (*cf. Lemma* [*6.3*](#_bookmark28) ) *i.e.*

*s* + *t* := Tr(*s* ⊕ *t*) *.*

Proof: The proof of the first claim proceeds by graphical calculus. The second claim follows by the fact that on scalars Tr(*s*)= *s*.

Definition 7.2 Let C be an ortho-SCCC with C+ ⊆ C. Its trace satisfies the *diagonal axiom* iff for all *h* : *A*1 ⊕ *A*2 → *A*1 ⊕ *A*2 ∈ C+ we have

Tr(*h*)= Tr(*h*11 + *h*22)

and it is *linear* iff for all *h, h*' ∈ C+ we have

Tr(*h*)+ Tr(*h*')= Tr(*h* + *h*') *.*

Both the diagonal axiom and linearity are stable under the *WProj* -construction.

Theorem 7.3 *For an ortho-SCCC* C *with* C+ ⊆ CTFAE:

* *The trace of* C *satisﬁes the ortho-Bornian axiom.*
* *The trace of* C *is linear and satisﬁes the diagonal axiom.*

The (full) trace Tr(−)= *ηA* ◦ (1*A*∗ ⊗ −) ◦ *ηA* which we have been using so far is a specialization (set *B* = *C* := I) of the categorical partial trace

†

*C*

*A*

(15)

Tr(−)= *λ*†

* (*η*†

⊗ 1*C*) ◦ (1*A*∗ ⊗ −) ◦ (*ηA* ⊗ 1*B*) ◦ *λB*

which exists as primitive data in so-called *traced monoidal categories* intro- duced in [[15](#_bookmark50)], and of which compact closed categories are a special case. As also shown in [[3](#_bookmark38)] the required equation for strong compact closure, that is, eq.([1](#_bookmark4)), is equivalent to the *yanking axiom* for the partial trace i.e.

Tr(*σA,A*)= 1*A .*

This allows us to end with a conclusive definition in which an ortho-Bornian category arises from three assumptions on the canonical categorical trace — the definition below is not a self-contained definition but relies on the rest of the paper in order to be understood.

Definition 7.4 An *ortho-Bornian* SCCC is a category C which comes with a special object I of which the endomorphisms are called *scalars*, with *tensors A* ⊗ *B* and *f* ⊗ *g* of objects and morphisms, with *duals A*∗ of objects, with *adjoints f* † : *B* → *A* of morphisms *f* : *A* → *B*, with a special morphism *ηA* :I → *A*∗ ⊗ *A* called *unit* for each object, with *monoidal sums A* ⊕ *B* and *f* ⊕ *g* of arbitrary objects and of those morphisms which are included in a subcategory C, all of these pieces of data being subject to conditions which establish harmonious coexistence (incl. Def. [4.3](#_bookmark20)), furthermore Cincludes all zero and all positive morphisms, and, the canonical trace Tr(−) on C which is build from units and their adjoints as in eq.([15](#_bookmark34))

1a. satisfies the *yanking axiom* as part of the SCCC-structure,

1b. satisfies the *diagonal axiom* as part of the ortho-Bornian structure,

1c. is *linear* also as part of the ortho-Bornian structure. This category is moreover *projective with weights* iff it

2. satisfies the *preparation-state agreement axiom*.

# Weight and relative phase as distinct entities

Passing from a category such as *WProj* (FdHilb) — or any other one obtained by applying the *WProj* -construction to an SCCC with biproducts — to a *genuine* ortho-Bornian SCCC involves separating the entities which play the role of probabilistic weight and of relative phase i.e. the extra chunk of state space one gains by considering superpositions of two underlying state spaces. In *WProj* (FdHilb) these two entities respectively are

R+ := {*c*¯ · *c* | *c* ∈ C} and {*c* · (*c*1*, c*2) | *c* ∈ C0} (*c*1*, c*2) ∈ (C × C)0,

where C0 := C\{0} and (C×C)0 := C×C\{(0*,* 0)}, hence both are constructed starting from C, the scalar monoid of FdHilb. Writing these down when using more conceptual categorical machinery we get

{*s*† ◦ *s* | *s* ∈ S} and {*s* • ⟨*s*1*, s*2⟩| *s* ∈ S} *s*1*, s*2 ∈ S,

where S := FdHilb(C*,* C). The crucial ingredient which enables us to do this is the pairing operation of the biproduct structure which allows to express the morphisms *f* ∈ FdHilb(C*,* C ⊕ C) in terms of those in FdHilb(C*,* C) as *f* := ⟨*s*1*, s*2⟩. But when the ortho-structure of a weighted projective ortho- Bornian SCCC is not inherited from a biproduct structure we do not have such a connection. Denoting the scalar monoid as W := C(I*,* I) — where every member is now to be interpreted a probability weight — the new player is the set X implicitly defined within C(I*,* I ⊕ I) = W × X, that is, the *qubit states* stripped off from any information concerning probabilistic weight. While these two entities do not share a common parent anymore they do interact in an important manner via the *measurement statistics*

C(I*,* I ⊕ I) × C(I*,* I ⊕ I) → W :: (*ψ, φ*) '→ *φ*† ◦ *ψ*

where C(I*,* I ⊕ I) × C(I*,* I ⊕ I) W2 × X2, and in which the crucial component relating probabilistic weight and relative phase is of type X2 → W.

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