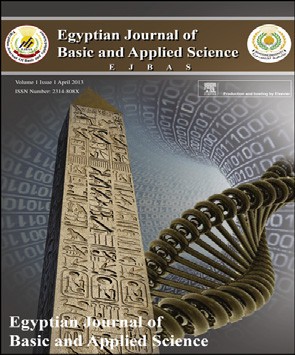
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Full Length Article

Distribution of zeros of solutions of self-adjoint fourth order differential equations



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## a r t i c l e i n f o

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## a b s t r a c t

In this paper, for self-adjoint fourth order differential equations, we establish some lower bounds on the distance between zeros of a nontrivial solution and also lower bounds on the distance between zeros of a solution and/or its derivatives. We also give new results related to boundary value problems which arise in the bending of rods. The main results will be proved by making use of some generalizations of Hardy, Opial and Wirtinger type inequalities.

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# Introduction

The oscillation and nonoscillation properties of the solutions of selfadjoint fourth order differential equations

(*p*(*t*)*x*''(*t*))'' + *q*(*t*)*x*(*t*)= 0, (1.1)

and

(*p*(*t*)*x*''(*t*))'' — *q*(*t*)*x*(*t*)= 0, (1.2)

were the subject of an extensive study in the fundamental paper of Lighton and Nehari [[25]](#_bookmark60) where the coefficients *p* and *q* are continuous positive functions. The investigation of the oscillatory behaviour of this type of equations originated with the vibrating rod problem of mathematical physics (see Ref. [[38]](#_bookmark75)). If the rod is clamped at its two endpoints *t* = a and *t* = b, it

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is well known that the deflection of the rod at time zero is an eigenfunction for the [(1.2)](#_bookmark3) with the boundary condition

*x*(a)= *x*'(a)= *x*(b)= *x*'(b)= 0. (1.3)

Later these equations and their general forms have been studied extensively by other authors, we refer the reader to the papers [[4,15,17](#_bookmark64)e[20,23,24,26,27,31,32,34](#_bookmark64)e[36]](#_bookmark64) and the book

[[33]](#_bookmark73) and the references cited therein. By a solution of [(1.1) or](#_bookmark2) [(1.2)](#_bookmark2) on the interval *J*4*I*h[a0, N), we mean a nontrivial real- valued function *x* ˛ *C*3(*J*), which has the property that

*p*(*t*)*x*''(*t*)˛*C*2(*J*) and satisfies equation [(1.1) or (1.2)](#_bookmark2) on *J*. In this

paper, we assume that [(1.1) or (1.2)](#_bookmark2) possesses such a nontrivial

solution on *I*. The nontrivial solution *x* of [(1.1) or (1.2)](#_bookmark2) is said to be oscillate or to be oscillatory, if it has arbitrarily large zeros.

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An equation of the form [(1.1) or (1.2)](#_bookmark2) is said to be disconjugate on an interval *I* if no nontrivial solution has more than three zeros on *I* counting multiplicities. If [(1.1) or (1.2)](#_bookmark2) is not oscillatory (i.e., if all solutions have only finitely many zeros), then the

*x*(a)= *x*'(a)= *x*''(a)= *x*(b)= 0,

or *x*(b)= *x*'(b)= *x*''(b)= *x*(a)= 0,

and the boundary conditions

(1.9)

equation is disconjugate on some interval [a1,N) for a1 ≥ a0 (see

Ref. [[25]](#_bookmark60)). In general, an *n*th-order differential equation

*x*(*n*)(*t*)+ *a*1(*t*)*x*(*n*—1)(*t*)+ . + *an*(*t*)*x*(*t*)= 0, (1.4)

is said to be (*k*, *n* — *k*) disconjugate on an interval *I* if no nontrivial solution has a zero of order *k* followed by a zero of order *n* — *k*. This means that, for every pair of points a, b ˛ *I*, a < b, there does not exist a nontrivial solution of [(1.4)](#_bookmark5) which satisfies

*x*(a)= *x*''(a)= *x*(b)= *x*''(b)= 0, (1.10)

which correspond to a rod hinged or supported at both ends.

We also consider the boundary conditions

*x*(a)= *x*'(a)= *x*''(b)= *x*'''(b)= 0, (1.11)

which correspond to a rod clamped at *t* = a and free at *t* = b, and the boundary conditions

*x*(*i*)(a)= 0, *i* = 0, ., *k* — 1,

*x*(*j*)(b)= 0, *j* = 0, ., *n* — *k* — 1.

(1.5)

*x*(b)= *x*'(b)= *x*''(a)= *x*'''(a)= 0, (1.12)

which correspond to a rod clamped at *t* = b and free at

The least value of b such that there exists a nontrivial so- lution which satisfies [(1.5)](#_bookmark8), is called the (*k*,*n* — *k*)-conjugate point of a.

For equation [(1.1)](#_bookmark2), disconjugacy is equivalent to (3,1)-dis- conjugacy (which, since equation [(1.1)](#_bookmark2) is selfadjoint, is also equivalent to (1,3)-disconjugacy), and for equation [(1.2)](#_bookmark3), dis- conjugacy is equivalent to (2,2)-disconjugacy (see Ref. [[25]](#_bookmark60)).

The equation [(1.2)](#_bookmark3) is said to be (2,2)-disconjugate on [a,b] if there is no nontrivial solution *x*(*t*) and *c*, *d* ˛ [a,b], *c* < *d* such

that *x*(*c*)= *x*'(*c*)= *x*(*d*)= *x*'(*d*)= 0.

Ourmotivation inthispaper comes from the oldpaper by C. de la Valle´e Poussin [[30]](#_bookmark72) and the papers [[10,12,14,28]](#_bookmark69). In Ref. [[30]](#_bookmark72) the author considered the linear *n*th-order differential equation [(1.4)](#_bookmark5)

*t* = a.

The paper is organized as follows: In Section [2](#_bookmark10), we present some inequalities of Hardy, Opial and Wirtinger types. In Section [3](#_bookmark21), we prove several results for the equations [(1.1) and](#_bookmark2) [(1.2)](#_bookmark2) subject to the above boundary conditions. In particular, the results for the equation [(1.1)](#_bookmark2) will be proved in Section [3.1](#_bookmark22) subject to the boundary conditions [(1.8)](#_bookmark6)e[(1.10)](#_bookmark6). The results

for the equation [(1.2)](#_bookmark3) will be proved in Section [3.2](#_bookmark43) subject to the boundary conditions [(1.3), (1.8) and (1.11)](#_bookmark7) when *p*(*t*) < 0. The case when [(1.12)](#_bookmark9) holds similar to the case when [(1.11)](#_bookmark7) holds and will be left to the interested reader. In Section [4](#_bookmark55),

we give some illustrative examples.

with real continuous coefficients *aj* and asserts that the equation

[(1.4)](#_bookmark5) is disconjugate onany interval sufficiently short with respect to the magnitude of the coefficients of the equation. More pre- cisely, he proved that if |*aj*(*t*)|≤ *bj* on [a,b] and the inequality

# Hardy, Opial and Wirtinger inequalities

In this section, we present the inequalities that we will need to

*n j* prove the main results. For more details, we refer the reader to

X

*bj*(b — a) < 1, (1.6)

*j*=1 *j*

holds, then [(1.4)](#_bookmark5) is disconjugate. In Ref. [[37]](#_bookmark74) it is shown that if *x* is a solution of the fourth order differential equation

*x*(4)(*t*)+ *q*(*t*)*x*(*t*)= 0, (1.7)

the books [[2,21,22]](#_bookmark61). The Hardy inequality [[21,22]](#_bookmark59) of the differ- ential form that we will need in this paper is given in the following theorem.

Theorem 2.1. [[21,22]](#_bookmark59). If *y* is absolutely continuous on (a,b) with *y*(a) = 0 or *y*(b) = 0, then the following inequality holds

' ' 0 11 0 11

which satisfies *x*(a)= *x* (a)= *x*(b)= *x* (b)= 0, then

Z

Z

Zb

3

(b — a)

≥ 192/

|*q*(*t*)|d*t*,

b *n* b

@ *q*(*t*)|*y*(*t*)| d*t*A ≤ *C*@

*n*

a

a

*m*

*r*(*t*)|*y*'(*t*)|*m*d*t*A , (2.1)

a

and if *x* satisfies *x*(a)= *x*(b)= *x*''(a)= *x*''(b)= 0, then

b

Z

(b — a) ≥ 4/ |*q*(*t*)|d*t*.

2

a

where *q*, *r* the weighted functions, are measurable positive functions in the interval (a,b) and *m*, *n* are real parameters satisfy 0 < *n* ≤ N and 1 ≤ *m* ≤ N and the constant *C* satisfies

*C* ≤ *k*(*m*, *n*)*A*(a, b), for 1 < *m* ≤ *n*, (2.2) where *k*(*m*, *n*) := *n*1/*m*(*m*\*)1/*m*\* ,

1 0 *t*

In Ref. [[10]](#_bookmark69) the author proved that if *x* is a solution of [(1.7)](#_bookmark11) which satisfies *x*(a)= *x*(b)= *x*''(a)= *x*''(b)= 0, then

b

Z

3

*A*(a, b) := sup 0 Z

*t*

a<*t*<b @

b

1

*n*

*q*(*t*)d*t*A @ Z

a

11/*m*\*

*r*1—*m*\* (*s*)d*s*A

, if *y*(a)= 0,

(b — a) ≥ 16/ |*q*(*t*)|d*t*.

a

0 Z*t* 11 0 Zb

11/*m*\*

In this paper, we obtain new lower bounds for the spacing (b — a) subject to the following boundary conditions:

*A* a, b : sup

a<*t*<b

*n*

( ) = @

*q*(*t*)d*t*A @

a

*t*

*r*1—*m*\* (*s*)d*s*A

, if *y*(b)= 0,

*x*(a)= *x*'(a)= *x*''(a)= *x*''(b)= 0,

or *x*(b)= *x*'(b)= *x*''(b)= *x*''(a)= 0,

(1.8)

and *m*\* = *m*/(*m* — 1).

Note that the inequality [(2.1)](#_bookmark12) can be considered when

*y*(a) = *y*(b) = 0. In this case, we see that [(2.1)](#_bookmark12) is satisfied with

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0 Z*d* 1

1

*n*

Theorem 2.4. [[6]](#_bookmark66). If *y* ˛ *C*1[a,b] with *y*(a) = 0 (or *y*(b) = 0), then

*A* a, b sup

( )=

@

(*c*,*d*)3(a,b) *c*

8<0 Z*c*

*q*(*t*)d*t*A

11/*m*\* 0 Zb

11/*m*\* 9=

(2.3)

b b

|*y*(*t*)|n|*y*'(*t*)|hd*t* ≤ *N*(n, h, *s*)(b — a)n

Z

@

Z

a a

1n+h

|*y*'(*t*)|*s*d*t*A

*s*

, (2.8)

×min @

0

*r*1—*m*\* (*s*)d*s*A

, @ *r*1—*m*\* (*s*)d*s*A .

where n > 0, *s* > 1, 0 ≤ h < *s*,

: ; (*s* — h)n

n

n+h—*s*

a *d N*(n, h, *s*) := (*s* — 1)(n + h)(*I*(n, h, *s*))ns ,

(2.9)

The Opial inequalities that we will use in this paper are

given in the following theorems.

Theorem 2.2. [[2]](#_bookmark61). Assume that the functions w and f are non-

s :=

and

1

n(*s* — 1)+ (*s* — h) *s*

,

(*s* — 1)(n + h)

negative and measurable on the interval (a,b), *m*, *n* are real numbers such that m/*m* > 1, and 0 ≤ *k* ≤ *n* — 1 (*n* ≥ 1) fixed. Let *x*˛*C*(*n*—1)[a, b] be such that *x*(*n*—1)(*t*) is absolutely continuous on

*I*(n, h, *s*) :=

1

*s* h 1 —(n+h+*s*n)/*s*n

Z

( — )

1 + *s* h *t*

—

0

[1 + (h — 1)*t*]*t*1/n—1d*t*.

(a,b). If *x*(*i*)(a) = 0, for *k* ≤ *i* ≤ *n* — 1 (*n* ≥ 1), then

Note that the inequality [(2.8)](#_bookmark13) can be considered when

b b

2Z

Z

f(*t*) *x*(*k*)(*t*) *l* *x*(*n*)(*t*) *m*d*t* ≤ *K* (a,b)4

1

3(*l*+*m*)/m

w(*t*) *x*(*n*)(*t*) md*t*5 ,

*y*(a) = *y*(b) = 0. Choose *c* = (a + b)/2 and apply [(2.8)](#_bookmark13) to [a,*c*] and [*c*,b] and then add we obtain

n+h

a a Zb

n 0 Zb 1 *s*

where

2 b

Z

(2.4)

3m—*m*

|*y*(*t*)|n|*y*'(*t*)|hd*t* ≤ *N*(n, h, *s*)

a

b — a

2

@ |*y*'(*t*)|*s*d*t*A

a

, (2.10)

*m*

*K* ( ) 4

*m* m

1(a, b) :=

*l*+*m*

*l*

m *t* —*m t*

f ( )w

( )

1/(m—*m*) *P*

m

*t l*(m—1)/(m—*m*) d*t*5

( 1,*k*( ))

,

where *N*(n,h,*s*) is defined as in [(2.9)](#_bookmark14). The inequality [(2.8)](#_bookmark13) can be

((*n*—*k*—1)!)

a

Z*t*

equation [(2.8)](#_bookmark13) becomes

b 0 b

considered when h = *s* and *y*(a) = 0 (or *y*(b) = 0). In this case the

@

1n+h

*P*1,*k*(*t*) :=

a

(*t* — *s*)(*n*—*k*—1)m/(m—1)

(w(*s*))—1/(m—1)

d*s*.

(2.5)

Z |*y*(*t*)|n|*y*'(*t*)|hd*t* ≤ *L*(n, h)(b — a)n Z

a a

h

|*y*'(*t*)|hd*t*A

, (2.11)

If we replace *x*(*i*)(a) = 0 by *x*(*i*)(b) = 0, then [(2.4)](#_bookmark15) holds where *K*1

where

0 1n

h

n G h+1 + 1

is replaced by *K*2 which is given by

*m*

hn n h h n

*K*2(a,b):=

*m*

m

*l*+*m*

*l*

*L*(n, h) := n + h

n + h

@G h+1

, (2.12)

1

G A

B

C

n

h

where

((*n*— *k*— 1)!)

b

2Z

× f (*t*)w (*t*) (*P*

2,*k*

4

m —*m* 1/(m—*m*)

a

Zb

3m—*m*

*l*(m—1)/(m—*m*) 5

m

(*t*)) d*t* ,

(2.6)

and G is the Gamma function.

In the following, we present a special case of the Wirtinger type inequality proved by Agarwal et al. in Ref. [[1]](#_bookmark62).

Theorem 2.5. [[1]](#_bookmark62). For *I* = [a,b] and a positive function l ˛ *C*1(*I*) with either l'(*t*) > 0 or l'(*t*) < 0 on I, we have

*P*2,*k*(*t*) :=

*t*

(*s* — *t*)(*n*—*k*—1)m/(m—1)

(w(*s*))—1/(m—1)

d*s*.

Zb l2(*t*)

|*y*'(*t*)|2d*t* ≥

b

|l'(*t*)||*y*(*t*)|2d*t*, (2.13)

Z

1

|l'(*t*)| 4

a

a

Theorem 2.3. [[3]](#_bookmark63). Let *p*(*t*), *q*(*t*) be non-negative measurable functions on (a,b) and 0 ≤ *k* ≤ *n* — 1 (*n* ≥ 1) fixed. If *x* ˛ *Cn*—1[a,b]

is such that *x*(*i*)(a) = 0, *k* ≤ *i* ≤ *n* — 1, *x*(*n*—1) is absolutely continuous on (a,b), then

for any *y* ˛ *C*1(*I*) with *y*(a) = 0 = *y*(b).

If we put *y*(*t*)= *x*''(*t*) with *x*''(a)= 0 = *x*''(b) and *Q*(*t*)= l'(*t*),

then we have the following inequality which gives a relation between *x*'''(*t*) and *x*''(*t*) on the interval [a,b]. For *I* = [a,b], then we

Z *q*(*t*) *x*(*k*)(*t*) *x*(*k*+1)(*t*) d*t* ≤ *C*a Z

b

b

a

a

*p*(*t*) *x*(*n*)(*t*) 2 d*t*, (2.7)

have

b

Z

|*p*(*t*)||*x*'''(*t*)|2d*t* ≥

a

b

|*Q*(*t*)||*x*''(*t*)|2d*t*, (2.14)

Z

1

4

a

where

*C* := 1

a

max *q*(*t*)

Zb (b — *s*)2(*n*—*k*—1)

d*s*.

for any *x* ˛ *C*3(*I*) with *x*''(a)= 0 = *x*''(b), where *p*(*t*) and *Q*(*t*) satisfy the equation

2((*n* — *k* — 1)!)2 *t*˛[a,b]

*p*(*s*)

(*p*(*t*)(l'(*t*)))' — 2*Q*(*t*)l(*t*)= 0, (2.15)

If *x* ˛ *Cn*—1[a,b] is such that *x*(*i*)(b) = 0, *k* ≤ *i* ≤ *n* — 1, *x*(*n*—1) is

a

absolutely continuous on (a,b) then [(2.7)](#_bookmark17) holds with *C*a is replaced by *C*b where

for any function l(*t*) satisfies l'(*t*)s0.

Remark 1. Note that the equation [(2.15)](#_bookmark19) holds if one chooses

1

*C*b :=

max *q*(*t*)

Zb (*s* — a)2(*n*—*k*—1)

d*s*.

*p*(*t*)

= *Q*(*t*)

= 1, where in this case

2((*n* — *k* — 1)!)2 *t*˛[a,b]

a

*p*(*s*)

l(*t*) = exp,ﬃ2ﬃﬃ*t*.

52 [e gypti an j o ur nal o f b a sic and a pp l i ed sci e n c e s 1 ( 2014) 49](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) e[5](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) 9

Also, the inequality [(2.14)](#_bookmark18) holds if *p*(*t*) = *Q*(*t*). In this case the function *p*(*t*) satisfies the differential equation

(*p*(*t*)(l'(*t*)))' = 2*p*(*t*)l(*t*), (2.16)

for any function l(*t*) satisfies l(*t*)s0.

# Main results

In this section, we will prove the main results. Throughout this paper in most of the results we will assume that *p*(*t*)

b

(*p*(*t*)*x*''(*t*))''*x*''(*t*)d*t* = (*p*(*t*)*x*''(*t*))'*x*''(*t*) b

a

Z

a

b

Z

— (*p*(*t*)*x*''(*t*))'*x*'''(*t*)d*t*

a

b

Z

=— *q*(*t*)*x*''(*t*)*x*(*t*)d*t*.

a

Using the assumptions *x*''(a)= *x*''(b)= 0, we have

b b

and *q*(*t*) are positive function and in the case when *p*(*t*) < 0, we will indicate it. We will also assume throughout the paper that *p* is absolutely continuous on [a,b] and the

Z (*p*(*t*)*x*''(*t*))'*x*'''(*t*)d*t* = Z

a a

Zb

a

*Q*'(*t*)*x*''(*t*)*x*(*t*)d*t*

appropriate integrals exist. Also, we assume throughout that there exists a differentiable function *Q*(*t*) with

= *Q*(*t*)*x*''(*t*)*x*(*t*)|b —

a

Zb

*Q*(*t*)*x*''(*t*)*x*'(*t*)d*t*

*q*(*t*)= *Q*'(*t*).

### *The results for equation* [*(1.1)*](#_bookmark2)

— *Q*(*t*)*x*'''(*t*)*x*(*t*)d*t*.

a

This implies after using the assumption *x*''(a)= *x*''(b)= 0,

that

For simplicity, we introduce the following notations:

Z *p*(*t*)(*x*'''(*t*))2d*t* =— Z

b

b

a

a a

b

Z

*Q*(*t*)*x*''(*t*)*x*'(*t*)d*t* —

*Q*(*t*)*x*'''(*t*)*x*(*t*)d*t*

2 Zb *Q*2 *t* 2 Z*t*

3

9

1

*t s* 4 > Zb

F1(*Q*, *p*, *P*1,0) :=

,1 ﬃﬃ4 ( ) *P*1,0(*t*)d*t*5 , *P*1,0(*t*)=

( — ) d*s*,

>

' '' '''

2 2 *p*(*t*)

a

*p*(*s*)

a

>

3

1

2

Z *t*

>>=

— *p* (*t*)*x* (*t*)*x* (*t*)d*t*.

a

q1ﬃﬃ4

2 Zb

*Q*2(*t*) 5 1

J1(*Q*, *p*, *P*1,1) :=

2 *p*(*t*) *P*1,1(*t*)d*t*

a

, *P*1,1(*t*)=

a

)

Hence

Z

Z

Z

*p*(*s* d*s*,

>

b

*p*(*t*)|*x*'''(*t*)|2d*t* ≤

b

|*Q*(*t*)||*x*'(*t*)||*x*''(*t*)|d*t* +

b

|*Q*(*t*)||*x*(*t*)||*x*'''(*t*)|d*t*

2 31 > a a a

1

qﬃﬃ Zb ' 2 2

Z *t* > Zb

fl (*p*', *p*, *P*

14

) :=

(*p* (*t*)) *P*

(*t*)d*t*5 , *P*

(*t*) =

d*s*, >;

' ''

'''

1 1,2

2 *p*(*t*)

a

1,2

1,2

*p*(*s*)

a

(3.1)

+ |*p* (*t*)||*x* (*t*)||*x* (*t*)|d*t*.

a

(3.5)

and

1

3

2

Zb *Q*2 *t* 2

Zb *s t* 4 9>

Applying the inequality [(2.4)](#_bookmark15) on the integral

b

Z

|*Q*(*t*)||*x*(*t*)||*x*'''(*t*)|d*t*,

F2(*Q*, *p*, *P*2,0) :=

,1 ﬃﬃ4

( ) *P*2,0(*t*)d*t*5 , *P*2,0(*t*) :=

( — ) d*s*,

2 2 *p t*

( )

a

2 Zb

3

*Q* (*t*) *P* (*t*)d*t*

*t p*(*s*)

1

2 Zb

>>>

=

a

with f(*t*) = |*Q*(*t*)|, w(*t*) = *p*(*t*), *m* = 1, *k* = 0, *l* = 1, *n* = 3 and m = 2, we

(*i*)

J2(*Q*, *p*, *P*2,1) :=

4

q1ﬃﬃ

2

a

2

2

*p*(*t*)

>

, *P*2,1(*t*)=

*t*

3

1. d*s*,

*p*(*s*)

2,1

5

>

>

get (note that *x*

Zb

(a) = 0, for *i* = 0,1,2) that

2 Zb 3

4

qﬃﬃ

1

Zb (*p*'(*t*))2

4

2

5

Zb 1

|*Q*(*t*)|*x*(*t*)|*x*'''(*t*)|d*t* ≤ F1(*Q*, *p*, *P*1,0)

a a

*p*(*t*)|*x*'''(*t*)|2d*t*5, (3.6)

fl2(*p*', *p*, *P*2,2) := 1

2

a

Theorem 3.1

*p*(*t*)

*P*2,2(*t*)d*t* , *P*2,2(*t*) :=

*t*

)

*p*(*s* d*s*. >;

(3.2)

where F1(*Q*, *p*, *P*1,0) is defined as in [(3.1)](#_bookmark23). Applying the inequality [(2.4)](#_bookmark15) again on the integral

b

Z

|*Q*(*t*)||*x*'(*t*)||*x*''(*t*)|d*t*,

. Suppose that x is a nontrivial solution of [(1.1)](#_bookmark2). If

*x*(*i*)(a) = 0, for *i* = 0,1,2 and *x*''(b)= 0, then

a

with f(*t*) = *Q*(*t*), w(*t*) = *p*(*t*), *k* = 1, *n* = 2, *l* = *m* = 1 and m = 2, we see that

F1(*Q*, *p*, *P*1,0)+ 4J1(*Q*, *p*, *P*1,1)+ fl1(*p*', *p*, *P*1,2)≥ 1. (3.3)

(*i*) Zb

If instead *x*

(b) = 0, for *i* = 0,1,2 and *x*''(a)= 0, then

|*Q*(*t*)||*x*'(*t*)||*x*''(*t*)|d*t* ≤ J1(*Q*, *p*, *P*1,1)4

1. Zb 3

F2(*Q*, *p*, *P*2,0)+ 4J2(*Q*, *p*, *P*2,1)+ fl2(*p*', *p*, *P*2,2)≥ 1. (3.4) a a

*p*(*t*)|*x*''(*t*)|2d*t*5, (3.7)

Proof. We prove [(3.3)](#_bookmark25). Multiplying [(1.1)](#_bookmark2) by *x*''(*t*) and inte- grating by parts we get

where J1(*Q*, *p*, *P*1,1) is defined as in [(3.1)](#_bookmark23). Applying the Wir- tinger inequality [(2.14)](#_bookmark18) on the integral

[e g ypti an j o ur nal o f b a sic and a pp l i e d sci en c e s 1 ( 2014) 49](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) e[5](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) 9 53

ß

Z

*p*(*t*)|*x*''(*t*)|2d*t*, (3.8)

a

where *x*''(a)= 0 = *x*''(ß), we see that

Using *C*2 and *C*\*2 instead of F1(*Q*, *p*, *P*1,1) and F2(*Q*, *p*, *P*2,1) in the proof of Theorem 3.1, we obtain the following result.

Theorem 3.2. Suppose that x is a nontrivial solution of [(1.1)](#_bookmark2). If

*x*(*i*)(a) = 0, for *i* = 0,1,2 and *x*''(ß)= 0, then

2

ß ß

Z

Z

*p*(*t*)|*x*''(*t*)|2d*t* ≤ 4

*p*(*t*)|*x*'''(*t*)|2d*t*, (3.9)

Ф (*Q*, *p*, *P*

ß

1

Z

)+ max |*Q*(*t*)|

(ß — *s*) d*s* + fl (*p*', *p*, *P*

)≥ 1.

1 0,1

a a

2 *t*˛[a,ß]

*p*(*s*)

a

1 1,2

where *p*(*t*) satisfies the equation [(2.16)](#_bookmark20) for any positive func- tion h(*t*). Substituting [(3.9) into (3.7)](#_bookmark27), we have

If *x*(*i*)(ß) = 0, for *i* = 0,1,2 and *x*''(a)= 0, then

ß ß

Z

Z

|*Q*(*t*)||*x*'(*t*)||*x*''(*t*)|d*t* ≤ 4F1(*Q*, *p*, *P*1 1)

*p*(*t*)|*x*'''(*t*)|2d*t*. (3.10)

Ф (*Q*, *p*, *P*

ß

1

Z

) + max |*Q*(*t*)|

(*s* — a) d*s* + fl (*p*', *p*, *P*

) ≥ 1.

2

, 2 0,2

a

a a

2 *t*˛[a,ß]

*p*(*s*)

2 2,2

Applying the inequality [(2.4)](#_bookmark15) again on the integral

ß

If the function *p*(*t*) is non-increasing on [a,ß], then we see that

Z |*p*'(*t*)||*x*''(*t*)||*x*'''(*t*)|d*t*,

2 ( — ) >

Z

a

*p*(ß)

3*p*(ß)

ß

(*t* — *s*) d*s* ≤ 1

2

*p*(*s*)

Z

(3.13)

a

ß ß a 3 9

(ß — *s*) d*s* ≤ , >

a

see that

with ф(*t*)= |*p*'(*t*)|, 9(*t*) = *p*(*t*), *k* = 2, *n* = 3, *l* = *m* = 1 and μ = 2, we

Zß

2 Zß 3

Zß (*s* — *t*)2

d*s* ≤

1 Zß

(*s* — a) d*s* ≤

2

>=

(ß — a) . >

|*p*'(*t*)||*x*''(*t*)||*x*'''(*t*)|d*t* ≤ fl (*p*', *p*, *P* )4

*p*(*t*)|*x*'''(*t*)|2d*t*5, (3.11)

*p*(*s*)

*p*(ß)

3*p*(ß) ;

1 1,2

a a

where fl1(*Q*, *p*, *P*1,2) is defined as in [(3.1)](#_bookmark23). Substituting [(3.6),](#_bookmark29) [(3.10) and (3.11) into (3.5)](#_bookmark29) and cancelling the term

ß

Z

*p*(*t*)|*x*'''(*t*)|2d*t*, we have

a

a a

Substituting [(3.13)](#_bookmark30) into Theorem 3.2, we have the following result.

3

Theorem 3.3. Assume that *p*(*t*) is a non-increasing function and *x* is a nontrivial solution of [(1.1)](#_bookmark2). If *x*(*i*)(a) = 0, for *i* = 0,1,2 and *x*''(ß)= 0, then

Ф1(*Q*, *p*, *P*1,0)+ 4F1(*Q*, *p*, *P*1,1)+ fl1(*p*', *p*, *P*1,2)≥ 1,

which is the desired inequality [(3.3)](#_bookmark25). The proof of [(3.4)](#_bookmark26) is similar by using integration by parts and the constants Ф1(*Q*, *p*,

1,1

1

1,2

2

3

1 1,0 6*p*(ß)

*t*

max

Ф (*Q*, *p*, *P* )+ (ß — a)

Z

[ ]

*t*˛ a,ß

a

*q*(*s*)d*s* + fl1(*p*', *p*, *P*1,2)≥ 1.

*P* ), F (*Q*, *p*, *P*

1,0

1

) and fl (*P*, *p*, *P*

) will be replaced by Ф (*Q*, *p*,

If instead *x*(*i*)(ß) = 0, for *i* = 0,1,2 and *x*''(a)= 0, then

*P* ), F (*Q*, *p*, *P*

) and fl (*P*, *p*, *P*

) which are defined as in [(3.2)](#_bookmark24).

2,0 2

2,1

2 2,2

3 Z *t*

The proof is complete.

6*p*(ß)

*t*˛[a,ß]

Ф (*Q*, *p*, *P*

)+ (ß — a)

max

*q*(*s*)d*s* + fl (*p*', *p*, *P*

a

)≥ 1.

Remark 2. Note that when *p*(*t*) is a constant then the third term

2

2,0

2

2,2

fl*i* for *i* = 1,2 will disappear from the results in Theorem 3.1.

In the following, we apply the inequality in Theorem 2.3 to obtain a new result by using the maximum value of |*Q*|. In this

In the proof of Theorem 3.1, we have applied the Wirtinger inequality [(2.14)](#_bookmark18) on the term [(3.8)](#_bookmark28). Applying the inequality [(2.1)](#_bookmark12) on the term [(3.8)](#_bookmark28) with *y*(*t*)= *x*''(*t*) (where *x*''(a)= *x*''(ß)= 0), we see that

case F1(*Q*, *p*, *P*1,1) and F2(*Q*, *p*, *P*2,1) will be replaced by *C*2 and *C*\*2

that we will determine below. As in the proof of Theorem 3.1, we suppose that the solution *x*(*t*) of [(1.1)](#_bookmark2) satisfies *x*'(a)= *x*''(a)=

Z *p*(*t*)|*x*''(*t*)|2d*t* ≤ D2 Z *p*(*t*)|*x*'''(*t*)|2d*t*, (3.14)

a a

ß

ß

0. Then the application of the inequality [(2.7)](#_bookmark17) with *k* = 1 and

ß

where D2 = 4*A* (a, ß), and

Z ' '' \*

2

*n* = 3 on the term

Z

1

*n*

a

Zß

|*Q*(*t*)||*x*'(*t*)||*x*''(*t*)|d*t* ≤ *C*

|*Q*(*t*)||*x* (*t*)||*x* (*t*)|d*t*, gives us

Zß

*p*(*t*)|*x*'''(*t*)|2d*t*, (3.12)

2

*A*\*(a, ß)= sup

0 *d* 1

@ *p*(*t*)d*t*A

(*c*,*d*)3(a,ß)

*c*

a a 8<0 Z*c*

1

1

11/2 0 Zß

11/2 9=

where

Z

×min @

d*s*A , @

d*s*A .

1

#### *C*2 :=

max |*Q*(*t*)|

ß

(ß — *s*)

2

d*s*.

: *p*(*s*)

a

*p*(*s*) ;

*d*

2 *t*˛[a,ß]

*p*(*s*)

a

Now, we can use the inequality [(3.14)](#_bookmark31) in the proof of

If instead *x*'(ß)= *x*''(ß)= 0, then [(3.12)](#_bookmark32) holds where *C*2 is replaced by

ß

2

Theorem 3.1 to obtain new results but in this case the constant 4 in front of the coefficient F will be replaced by D2. The details will be left to the interested reader.

*C*\*2 := max |*Q*(*t*)| Z

1

(*s* — a)

d*s*.

In the following, we will apply the Boyd inequality in

2 *t*˛[a,ß]

*p*(*s*)

a

Theorem 2.4. By applying the Schwarz inequality

54 [e gypti an j o ur nal o f b a sic and a pp l i ed sci e n c e s 1 ( 2014) 49](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) e[5](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) 9

Zß 0 Zß 1 ß 1

1

1

0

Z

2

2

If *x*(*i*)

(ß) = 0, for *i* = 0,1,2 and *x*''(a)= 0, then

|*f* (*t*)*g*(*t*)|d*t* ≤

2

2

@

a a

on the term

|*f* (*t*)| d*t*A × @

a

|*g*(*t*)| d*t*A , (3.15)

Ф (*Q*, *p*, *P*

2

2,0

8 (ß — a)0 ß

@

p

a

1

2

1

|*Q*(*t*)|2d*t*A + fl (*p*', *p*, *P*

2

)≥ 1.

2,2

ß

*p*(ß)

)+ Z

Z

|*Q*(*t*)||*x*'(*t*)||*x*''(*t*)|d*t*,

a

In the following, we apply the Opial inequality due to Bes- sack and Das [[5]](#_bookmark65) to obtain new results for [(1.1)](#_bookmark2) subject to the boundary conditions [(1.10)](#_bookmark6). This inequality is a generalization

we see that

Zß

0 Zß

110 Zß

2

2

of the classical Opial inequality [[29]](#_bookmark71) and states that if *y* is

1 absolutely continuous on [*a*, *b*] with *y*(*a*) 0, then the following

1

=

2

inequality holds

|*Q*(*t*)||*x*'(*t*)||*x*''(*t*)|d*t* ≤

@

a a

|*Q*(*t*)| d*t*A @

a

|*x*'(*t*)|2|*x*''(*t*)|2d*t*A .

(3.16)

*b b*

*B*(*t*)|*y*(*t*)|*m*|*y*'(*t*)|*n*d*t* ≤ *K*1(*m*, *n*)

Z

Z

*A*(*t*)|*y*'(*t*)|*m*+*n*d*t*, (3.19)

Now, by applying the inequality [(2.11)](#_bookmark16) on the integral *a a*

ß

Z

|*x*'(*t*)|2|*x*''(*t*)|2d*t*,

where *m*, *n* are real numbers such that *mn* > 0 and *m* + *n* > 1, *A*

and *B* are nonnegative, measurable functions on (*a*,*b*) such

Z*b*

a

with n = h = 2 and *y* = *x*' (note that *x*'(a)= 0), we see that

that (*A*—1/(*m*+*n*—1)(*s*)d*s* < N, and

*a*

3

Zß 2 Zß 32

*n*

2 Z*b*

*n*+*m*

0 Z*t*

1*m*+*n*—1

*m m*+*n*

4 ß a 2

2 2 ( — )

|*x*'(*t*)| |*x*''(*t*)| d*t* ≤ p2*p*2(ß) 4

a

a

*p*(*t*)|*x*''(*t*)|2d*t*5 , (3.17)

*K*1(*m*, *n*) :=

*n*

*n* + *m*

*n*+*m*

*B m* (*t*)

*Am* (*t*)

*n*

*a*

*a*

1

*A*(*m*+*n*—1)

@

—

(*s*)d*s*A

d*t*5 .

where we assumed that *p*(*t*) is a non-increasing function (note that the inequality [(3.17)](#_bookmark36) is also valid if *x*'(ß)= 0). Substituting [(3.17)](#_bookmark36) into [(3.16)](#_bookmark34), we have

4

(3.20)

If we replace *y*(*a*) = 0 by *y*(*b*) = 0, then [(3.19)](#_bookmark35) holds where

*K*1(*m*,*n*) is replaced by

0 11

*n* 2 Z*b*

*n*+*m*

0 Z*b*

1*m*+*n*—1

*m m*+*n*

3

Zß Zß 2 Zß

*n n*+*m*

*B m* (*t*)

—1

|*Q*(*t*)||*x*'(*t*)||*x*''(*t*)|d*t* ≤ 2 (ß — a)@ |*Q*(*t*)| d*t*A

2

*p*(*t*)|*x*''(*t*)|2d*t*.

*K*2(*m*, *n*) :=

4 *n* @

*A*(*m*+*n*—1) (*s*)d*s*A

d*t*5 .

p *p*(ß)

a

a

a

*n* + *m*

*Am* (*t*)

*a*

*t*

(3.21)

Again applying the Wirtinger inequality [(2.14)](#_bookmark18) on the integral

ß

Note that the inequality [(3.19)](#_bookmark35) can be considered when *y*(*a*) = *y*(*b*) =10. In this case we will assume that there exists s ˛ (*a*,*b*) such that

Z *p*(*t*)|*x*''(*t*)|2d*t*,

a

Z*b*

—1

Zs

—1

where *x*''(a)= 0 = *x*''(ß), we have

*A*(*m*+*n*—1) (*s*)d*s* =

s *a*

*A*(*m*+*n*—1) (*s*)d*s*. (3.22)

Z *p*(*t*)|*x*''(*t*)|2d*t* ≤ 4 Z *p*(*t*)|*x*'''(*t*)|2d*t*,

ß

ß

a a

where *p*(*t*) satisfies the equation [(2.16)](#_bookmark20) for any positive func- tion h(*t*). This implies that

In this case the inequality [(3.19)](#_bookmark35) holds with a new constant

*K*(*m*,*n*) which is given from the equation

*K*(*m*, *n*)= *K*1(*m*, *n*)= *K*2(*m*, *n*),

when [(3.22)](#_bookmark37) is satisfied. In the following, we assume that there exists s ˛ (a,ß) such that

Z |*Q*(*t*)||*x*'(*t*)||*x*''(*t*)|d*t* ≤ 8 (ß — a)0 Z

@

p

*p*(ß)

ß

ß

a

a

11

|*Q*(*t*)| d*t*A

2

2 Zß

*p*(*t*)|*x*'''(*t*)|2d*t*.

a

ß s

*p*—1(*s*)d*s* =

Z

Z

s

a

*p*—1(*s*)d*s*. (3.23)

(3.18)

Using this inequality and proceeding as in the proof of Theorem 3.1, we obtain the following result.

Theorem 3.4. Assume that *p*(*t*) is a non-increasing function

and assume that

*K*\*(*p*', *p*) = *K*1(1, 1) = *K*2(1, 1), (3.24)

where

1

and *x* is a nontrivial solution of [(1.1)](#_bookmark2). If *x*(*i*)( )

0, for *i*

0,1,2

2 Zß *p*' *t*

2 Z*t* 32

a = =

1

2

and *x*''(ß)= 0, then

*K* (1, 1) := ,1ﬃﬃ4

| ( )|

a

*p*(*t*)

*p*—1(*s*)d*s*d*t*5 ,

a

0 11 2 31

Zß 2

Zß *p*' *t*

2 Zß 2

Ф (*Q*, *p*, *P*

1

1,0

p

*p*(ß)

1

1,2

2

)+ 8 (ß — a)@

|*Q*(*t*)|2d*t*A + fl (*p*', *p*, *P*

a

)≥ 1.

*K*2(1, 1) := ,1ﬃﬃ4

| ( )|

a

*p*(*t*)

*p*—1(*s*)d*s*d*t*5 .

*t*

[e g ypti an j o ur nal o f b a sic and a pp l i e d sci en c e s 1 ( 2014) 49](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) e[5](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) 9 55

Theorem 3.5. Assume that *p*(*t*) is a non-increasing function. Suppose that *x* is a nontrivial solution of [(1.1)](#_bookmark2). If *x*(a)= *x*''(a)=

with *B*(*t*)=| *p*'(*t*)| and *A*(*t*) = *p*(*t*), and *y*(*t*)= *x*''(*t*), we see that

*x*(ß)= *x*''(ß)= 0, then

ß

' ''

Z

a

'''

ß

\* ' ''' 2

Z

a

7/20 Zß

2

11/2

|*p* (*t*)||*x* (*t*)||*x* (*t*)|d*t* ≤ *K* (*p* , *p*)

*p*(*t*)|*x* (*t*)| d*t*, (3.31)

(ß — a)

( ) @

72*p* ß

a

|*q*(*t*)| d*t*A

+ *K*\*(*p*', *p*)≥ 1, (3.25)

where *K*\*(*p*', *p*) is defined as in [(3.24)](#_bookmark38).

where *K*\*(*p*', *p*) is defined as in [(3.24)](#_bookmark38). Substituting [(3.30) and](#_bookmark40)

Zß

Proof. Multiplying [(1.1)](#_bookmark2) by *x*''(*t*) and proceeding as in the

proof of Theorem 3.1 we get

2

[(3.31) into (3.26)](#_bookmark40) and cancelling the term *p*(*t*)|*x*'''(*t*)| , we

a

ß ß ß

Z Z Z

*p*(*t*)|*x*'''(*t*)| d*t* ≤ |*q*(*t*)||*x*(*t*)||*x* (*t*)|d*t* + |*p* (*t*)||*x* (*t*)||*x* (*t*)|d*t*.

2 '' ' '' '''

obtain the desired inequality [(3.25)](#_bookmark39). The proof is complete.

As a special case when *p*(*t*) = 1 in Theorem 3.5, we have the

a a a

(3.26)

following result.

Applying the inequality the Schwarz inequality [(3.15)](#_bookmark33) on the integral

2

Corollary 3.1. Let *x* is a nontrivial solution of

*x*(4)(*t*)+ *q*(*t*)*x*(*t*)= 0, *t*˛[a, ß],

Zß 0 Zß

11/20 Zß

11/2

|*q*(*t*)||*x*(*t*)||*x*''(*t*)|d*t* ≤

@

a a

|*q*(*t*)| d*t*A

@ |*x*(*t*)|2|*x*''(*t*)|2d*t*A .

a

which satisfies *x*(a)= *x*''(a)= *x*(ß)= *x*''(ß)= 0. Then

Zß

7

Applying the inequality ([[[11]](#_bookmark70), Theorem 4.5]) (note that

*x*(a) = *x*(ß) = 0)

(ß — a)

≥ 5184/

a

*q*2(*t*)d*t*. (3.32)

0 ß 11/2

Z

@ |*x*(*t*)|2|*x*''(*t*)|2d*t*A

a

(ß — a)3/2 Zß

a

≤ 12

|*x*''(*t*)|2d*t*,

Remark 3. One can also obtain new results by multiplying [(1.1)](#_bookmark2) by *p*(*t*)*x*''(*t*) and considering the case when *p*(*t*) < 0. In this case after integrating by parts, we have

we get that ß

0 Zß 11/20 Zß

11/2

Z (*p*(*t*)*x*''(*t*))''*p*(*t*)*x*''(*t*)d*t* = (*p*(*t*)*x*''(*t*))'*p*(*t*)*x*''(*t*) ß

@ |*q*(*t*)| A

2

a

0 Zß

@ |*x*(*t*)|2|*x*''(*t*)|2d*t*A

a

11/2 Zß

(3.27)

a

ß

a

Z

— (*p*'(*t*)*x*''(*t*)+ *p*(*t*)*x*'''(*t*))2d*t*

(ß—a) @

3/2

a

A '' ß

2

2

≤ 12

|*q*(*t*)| d*t*

a

|*x* (*t*)| d*t*,

a

#### =— Z

*p*(*t*)*q*(*t*)*x*''(*t*)*x*(*t*)d*t*.

Applying the Wirtinger-inequality, see Brnetic´ and Pecˇaric´ [[7]](#_bookmark67),

Zß 2 Zß

a

Using the assumption *x*''(a)= *x*''(ß)= 0, we have

*y*2(*t*)d*t* ≤ (ß — a) (*y*'(*t*))2d*t*, (3.28)

6

a a

ß ß

*p*2(*t*)|*x*'''(*t*)|2d*t* = —2 *p*(*t*)*p*'(*t*)*x*''(*t*)*x*'''(*t*)d*t*

Z

Z

a

for any *y* ˛ *C*1[a,ß] and *y*(a) = *y*(ß) = 0, with *y*(*t*)= *x*'' (note that *x*''(a)= 0 = *x*''(ß)) and the assumption that *p* is a non- increasing function, we have

a ß

— (*p*'(*t*))2(*x*''(*t*))2d*t*

Z

a

0 Zß 11/2 Zß

ß

' ''

Z

3/2

(ß—a) @

2

A

12

|*q*(*t*)| d*t*

a

0 Z

|*x*''(*t*)|2d*t*

a

+ *Q*1(*t*)*x*(*t*)*x* (*t*)d*t*,

a

ß

7/2

≤ (ß—a) @

72*p*(ß)

11/2 ß

|*q*(*t*)| d*t*A

2

Z

*p*(*t*)|*x*'''(*t*)|2d*t*.

(3.29)

where *Q*1(*t*) is the antiderivative of *p*(*t*)*q*(*t*). Integrating by parts the last term in the right hand side, we see that

a a

This implies that

ß ß

*Q*' (*t*)*x*''(*t*)*x*(*t*)d*t* = *Q*1(*t*)*x*(*t*)*x*(*t*)|ß —

Z

Z

*Q*1(*t*)*x*'(*t*)*x*''(*t*)d*t*

Zß (ß — a)

0 Zß

1 a

11/2 Zß a a

7/2

2

Zß

|*q*(*t*)||*x*(*t*)||*x*''(*t*)|d*t* ≤

a

72*p*(ß) @

a

|*q*(*t*)| d*t*A

*p*(*t*)|*x*'''(*t*)|2d*t*.

a

— *Q*1(*t*)*x*(*t*)*x*'''(*t*)d*t*.

a

(3.30)

Applying the Opial inequality [(3.19)](#_bookmark35) on the integral (note

Using the assumption *x*''(ß)= *x*''(a)= 0, we see that

that *x*''(a)= 0 = *x*''(ß))

ß

Z

|*p*'(*t*)||*x*''(*t*)||*x*'''(*t*)|d*t*,

ß ß

*Q*1' (*t*)*x*(*t*)*x*''(*t*)d*t* =—

Z

Z

a a

Hence we obtain

ß

*Q*1(*t*)*x*'(*t*)*x*''(*t*)d*t* —

Z

a

*Q*1(*t*)*x*(*t*)*x*'''(*t*)d*t*.

a

# 56

ß

Z

*p*2(*t*)|*x*'''(*t*)|2d*t* ≤

[e gypti an j o ur nal o f b a sic and a pp l i ed sci e n c e s 1 ( 2014) 49](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) e[5](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) 9

ß

Z

|*Q* (*t*)||*x*(*t*)||*x*'''(*t*)|d*t* Substituting [(3.36) and (3.38) into (3.35)](#_bookmark45), we have

1

a a

Z

Z

ß

Z

+ |*Q*1(*t*)||*x*'(*t*)||*x*''(*t*)|d*t*

a

ß

*p*(*t*)|*x*''(*t*)|2d*t* ≤ 2

a

ß

|*Q*(*t*)||*x*(*t*)||*x*'(*t*)|d*t*. (3.39)

a

ß

Z

+2 |*p*(*t*)*p*'(*t*)||*x*''(*t*)||*x*'''(*t*)|d*t*

a

Applying the inequality [(2.7)](#_bookmark17) on the integral

ß

Z

+ |*p*'(*t*)|2|*x*''(*t*)|2d*t*.

a

One can apply the inequalities in Section [2](#_bookmark10) to establish new results. This will be left to the interested reader.

ß

|*Q*(*t*)||*x*(*t*)||*x*'(*t*)|d*t*,

Z

a

with *p*(*t*) = *Q*(*t*), *k* = 0 and *n* = 2, we see that

Z

Z ' 01

### *The results for equation* [*(1.2)*](#_bookmark3)

ß

|*Q*(*t*)||*x*(*t*)||*x* (*t*)|d*t* ≤ @

max|*Q*(*t*)|

Zß (ß— *s*)2

1. ß

d*s*A

*p*(*t*)|*x*''(*t*)|2d*t*,

a

We begin with the boundary conditions

2 *t*˛[a,ß]

*p*(*s*)

a

a

(3.40)

*x*(a)= *x*'(a)= *x*(ß)= *x*'(ß)= 0, which correspond to a rod

clamped at each end.

where *x*(a)= *x*'(a)= 0 (or *x*(ß)= *x*'(ßß)= 0). Substituting [(3.40) into](#_bookmark42) [(3.39)](#_bookmark42) and cancelling the term *p*(*t*)|*x*''(*t*)|2d*t*, we have

Theorem 3.6. Suppose that *x* is a nontrivial solution of [(1.2)](#_bookmark3). If  a

Z

Z

Z

*x*(a)= *x*'(a)= *x*(ß)= *x*'(ß)= 0, then

2

*p*(*s*)

*t*

max

*t*˛[a,ß]

a

ß

*q*(*s*)d*s*

(ß — *s*) d*s* ≥ 1,

max

Z *t*

*t*˛[a,ß]

a

*q*(*s*)d*s*

a

(ß — *s*) d*s* 1. (3.33)

*p*(*s*)

2

≥

Zß

a

which is the desired inequality [(3.33)](#_bookmark44). The proof is complete. Note that when *p*(*t*) is nonincreasing, we see that the in-

Proof. Multiplying [(1.2)](#_bookmark3) by *x*(*t*) and integrating by parts, we have

equalities in [(3.13)](#_bookmark30) are satisfied. Using these two inequalities in Theorem 3.6 give us the following result.

ß

Z

(*p*(*t*)*x*'''(*t*))''*x*(*t*)d*t* = *x*(*t*)(*p*(*t*)*x*''(*t*

)) a —

a

' ß

ß

*x*'(*t*)(*p*(*t*)*x*''(*t*))'d*t*

Z

a

Corollary 3.2. Suppose that *x* is a nontrivial solution of [(1.2)](#_bookmark3),

*p*(*t*) is nonincreasing. If *x*(a)= *x*'(a)= *x*(ß)= *x*'(ß)= 0, then

ß

Z

= *q*(*t*)*x*2d*t*.

a

a

(3.34)

3

(ß — a)

Z*t t*˛[a,ß]

*q*(*s*)d*s* . (3.41)

Using the assumptions that *x*(a) = *x*(ß) = 0 and *Q*'(*t*)= *q*(*t*), we get that

≥ 3*p*(ß)/ max

Remark 4. Corollary 3.2 gives us a condition for (2,2)-dis- conjugacy of [(1.2)](#_bookmark3). In particular, if

ß ß

Z

Z

*x*'(*t*)(*p*(*t*)*x*''(*t*))'d*t* =—

a a

*Q*'(*t*)*x*2(*t*)d*t*. (3.35)

*t*

Z

max

*q*(*s*)d*s* < 3*p*(ß) , (3.42)

Integrating by parts the right hand side, we see that

3

*t*˛[a,ß]

a

(ß — a)

ß

Z

*Q*'(*t*)*x*2(*t*)d*t* = *Q*(*t*)*x*2(*t* ß

) a —

a

ß

1. *Q*(*t*)*x*(*t*)*x*'(*t*)d*t*.

Z

a

then (1.2) is (2,2)-disconjugate in [a,ß]. This means that there is no nontrivial solution of [(1.2)](#_bookmark3) in [a,ß] satisfies *x*(a)= *x*'(a)= *x*(ß)= *x*'(ß)= 0.

Using the assumption *x*(a) = *x*(ß) = 0, we see that

Zß Zß

Theorem 3.7. Assume that *p*(*t*) is nonincreasing. If *x* is a nontrivial solution of [(1.2)](#_bookmark3) which satisfies *x x*' *x*

*Q*'(*t*)*x*2(*t*)d*t* = —2

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| a  Integratin  Zß | a  g by parts the left hand side  Zß  ß  *x*''(*t*))'d*t* = *p*(*t*)*x*'(*t*)*x*''(*t*)|a — | | | |
| a  Using the | assumptio | n | a  *x*'(a)= *x*'(ß)= | |
| Zß  a | Z  a | | ß  *p*(*t*)(*x* | 2  ''(*t*)) d*t*. |

*Q*(*t*)*x*(*t*)*x*'(*t*)d*t*. (3.36)

*x*'(ß)= 0, then

(a) =

(a) = (ß)=

of [(3.35)](#_bookmark45), we see that

max |*Q*(*t*)| ≥ 96*p*(ß) . (3.43)

*t*˛[a,ß]

3

(ß — a)

*x*'(*t*)(*p*(*t*)

*x*'(*t*)(*p*(*t*)*x*''(*t*))'d*t* =—

*p*(*t*)(*x*''(*t*))2d*t*. (3.37)

0, we have

(3.38)

Proof. Proceeding as in the proof of Theorem 3.7 to obtain

ß ß

Z

Z

*p*(*t*)|*x*''(*t*)|2d*t* ≤ 2 max |*Q*(*t*)| |*x*(*t*)||*x*'(*t*)|d*t*. (3.44)

*t*˛[a,ß]

a a

Applying the inequality (see ([[[8]](#_bookmark68), Inequality (5.8)])) on the

integral

[e g ypti an j o ur nal o f b a sic and a pp l i e d sci en c e s 1 ( 2014) 49](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) e[5](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) 9 57

ß

Z

|*x*(*t*)||*x*'(*t*)|d*t*,

a

Next, in the following, we establish some results which allow us to consider the case when *p*(*t*) < 0. For simplicity, we denote

we see (note that *x*(a)= *x*'(a)= 0 = *x*(ß)= *x*'(ß)= 0) that

*K*\* (|*p*'(*t*)|, *P*1 2) := 1ﬃﬃ2 Z

1

3

9

2

>

|*p*'(*t*)|2*P*1 2(*t*)d*t*5 , >

ß

Z

|*x*(*t*)||*x*'(*t*)|d*t* ≤

a

(ß — a)3 Zß

a

192*p*(ß)

1

*p*(*t*)|*x*''(*t*)|2d*t*. (3.45)

, 2

a

, 4

ß

2 Zß

,

1 >=

3

2 >

(3.48)

*K*\* (|*p*'(*t*)|, *P*2 2) := ,1ﬃﬃ4

|*p*'(*t*)|2*P*2 2(*t*)d*t*5 , >;

Substituting [(3.45) into (3.44)](#_bookmark47) and cancelling the term

ß

Z

*p*(*t*)|*x*''(*t*)|2d*t*, we have

a

3

(ß — a)

max |*Q*(*t*)|

≥ 1,

2

where

, 2

a

Z*t*  1

*P*1,1(*t*) :=

*p*

*s* d*s*, *P*1,2 *t*

,

Z *t*  1

:=

*p*

*s* d*s*,

*t*˛[a,ß]

96*p*(ß)

2( )

a

2( )

a

which is the desired inequality [(3.43)](#_bookmark46). The proof is complete.

Zß 1 Zß 1

From Theorem 3.7 and Lemma 3.1, we have the following result.

*P*2,1(*t*) := *p*2 *s* d*s*, *P*2,2 *t* :=

*t t*

)

)

(

*p*2(*s* d*s*.

Corollary 3.3. Assume that *p*(*t*) is nonincreasing. If *x* is a nontrivial solution of [(1.2)](#_bookmark3) which satisfies *x x* *x*

and D1 = 4(*A*\*(a, ß))2 where

0 11

Z

*x*'(ß)= 0, then

ß

(a) = '(a)= (ß)=

*d*

*A*\* a, ß sup

( )=

@

(*c*,*d*)3(a,ß) *c*

2

*p*2(*t*)d*t*A

(3.49)

;

Z 192*p*(ß)

1

1

:

8<0 Z*c*

11/2 0 Zß

11/29=

|*q*(*t*)|d*t* ≥ (ß — a)3. (3.46)

a

×min @

d*s*A , @

a

d*s*A ,

*d*

and D2 = 4(*A*\*\*(a, ß))2 where

*p*(*s*)

*p*(*s*)

Remark 5. The contrapositive of the result in Corollary 3.4 yields a sufficient condition for (2,2)-disconjugacy of the equation [(1.2)](#_bookmark3).

In the following, we consider the boundary conditions

*A*\*\* a, ß sup

(*c*,*d*)3(a,ß)

( ) =

0 *d* 1

@ |*p*'(*t*)|d*t*A

1

Z

2

*c*

(3.50)

80 Z*c*

11/2 0 Zß

11/29

*x*(a)= *x*'(a)= *x*''(ß)= *x*'''(ß)= 0 which correspond to a beam hinged or supported at both ends. The proof will be as in the proof of Theorem 3.6, by using these boundary conditions and gives us the following result.

:

@

×min< 1 d*s*

a

1

,

*p*(*s* A

)

@

*p*(*s*)

A

*d*

*ds* =.

;

Theorem 3.8. Suppose that *x* is a nontrivial solution of [(1.2)](#_bookmark3). If Theorem 3.9. Suppose that *x* is a nontrivial solution of [(1.2)](#_bookmark3)

*x*(a)= *x*'(a)= *x*''(ß)= *x*'''(ß)= 0, then and there exists a function *Q*1 ˛ *C*1[a,ß] such that *Q*1' = *pq*. If

*x*(*i*)(a) = 0, for *i* = 0,1,2 and *x*''(ß)= 0, then

Z

Z ( — )

*t*

max

*p*(*s*)

ß ß *s* 2

*q*(*s*)d*s*

d*s* ≥ 1.

Ф1 *Q*1, *p*2, *P*1,0 + D1F1 *Q*1, *p*2, *P*1,1 + *K*1\* (|*p*'(*t*)|, *P*1,2)+ D2 ≥ 1.

*t*˛[a,ß]

a

a

Corollary 3.4. Suppose that *x* is a nontrivial solution of [(1.2)](#_bookmark3),

*p*(*t*) is nonincreasing. If *x*(a)= *x*'(a)= *x*''(ß)= *x*'''(ß)= 0, then

If *x*(*i*)(ß) = 0, for *i* = 0,1,2 and *x*''(a)= 0, then

Ф2 *Q*1, *p*2, *P*2,0 + D1F2 *Q*1, *p*2, *P*2,1 + *K*\*2(|*p*'(*t*)|, *P*2,2)+ D2 ≥ 1.

(3.51)

(ß — a)

3

Z *t t*˛[a,ß]

*q*(*s*)d*s* . (3.47)

(3.52)

Proof. We prove [(3.51)](#_bookmark52). Multiply [(1.2)](#_bookmark3) by *p*(*t*)*x*''(*t*). In this case after integrating by parts, we have

From Corollary 3.4 and the arguments before Corollary 3.1, we have the following result.

a

≥ 3*p*(ß)/ max

Corollary 3.5. Assume that *p*(*t*) is nonincreasing. If *x* is a

nontrivial solution of [(1.2)](#_bookmark3) which satisfies *x*

*x*'

*x*''

ß

(*p*(*t*)*x*''(*t*))''*p*(*t*)*x*''(*t*)d*t* = (*p*(*t*)*x*''(*t*))'*p*(*t*)*x*''(*t*) ß

a

Z

a

Zß

(*p*'(*t*)*x*''(*t*) + *p*(*t*)*x*'''(*t*))2d*t*

*x*'''(ß)= 0, then

ß 6*p* ß

Z ( )

|*q*(*t*)|d*t* ≥ (ß — a)3.

a

(a) =

(a) =

(ß)=

—

a

ß

Z

= *p*(*t*)*q*(*t*)*x*''(*t*)*x*(*t*)d*t*.

a

Using the assumption *x*''(a)= *x*''(ß)= 0, we have

58 [e gypti an j o ur nal o f b a sic and a pp l i ed sci e n c e s 1 ( 2014) 49](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) e[5](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) 9

ß ß

Z

Z

*p*2(*t*)|*x*'''(*t*)|2d*t* ≤ 2

a a

ß

|*p*(*t*)*p*'(*t*)||*x*''(*t*)||*x*'''(*t*)|d*t* +

Z

a

(*p*'(*t*))2(*x*''(*t*))2d*t*

term [(3.8)](#_bookmark28) with *y*(*t*)= *x*''(*t*) (where *x*''(a)= *x*''(ß)= 0), we see that

Zß

Zß Zß

+ *Q*1' (*t*) |*x*(*t*)||*x*''(*t*)|d*t*.

a

*p*2(*t*)|*x*''(*t*)|2d*t* ≤ D1

a a

*p*2(*t*)|*x*'''(*t*)|2d*t*, (3.56)

Integrating by parts the last term in the right hand side, we see that

where D1 = 4(*A*\*(a, ß))2 and *A*\*(a,ß) is defined as in [(3.49)](#_bookmark49). Substituting [(3.56) into (3.55)](#_bookmark56), we have

Zß

Zß Zß Zß

*Q*' (*t*) |*x*''(*t*)||*x*(*t*)|d*t* = |*Q*1(*t*)||*x*''(*t*)||*x*(*t*)||ß —

|*Q*1(*t*)||*x*'(*t*)||*x*''(*t*)|d*t*

' ''

2 2 ''' 2

1. a

a a a

Zß

|*Q*1(*t*)||*x* (*t*)||*x* (*t*)|d*t* ≤ D1F1 *Q*1, *p* , *P*1,1

*p* (*t*)|*x* (*t*)| d*t*.

a

(3.57)

— |*Q*1(*t*)||*x*(*t*)||*x*'''(*t*)|d*t*.

a

Using the assumption *x*''(ß)= *x*''(a)= 0, we see that

Applying the inequality [(2.4)](#_bookmark15) on the integral

ß

Z

|*p*(*t*)*p*'(*t*)||*x*''(*t*)||*x*'''(*t*)|d*t*

Zß Zß a

*Q*1' (*t*) |*x*''(*t*)||*x*(*t*)|d*t* =— |*Q*1(*t*)||*x*'(*t*)||*x*''(*t*)|d*t*

Z

a

a

with ф(*t*)= |.*p*(*t*)*p*'(*t*)|., 9(*t*) = *p*2(*t*), *m* = 1, *k* = 2, *l* = 1, *n* = 3 and

μ = 2, we get (note that *x*(*i*)(a) = 0, for *i* = 0,1,2) that

ß

— |*Q*1(*t*)||*x*(*t*)||*x*'''(*t*)|d*t*.

a

ß ß

2

Z

Z

|*p*(*t*)*p*'(*t*)||*x*''(*t*)||*x*'''(*t*)|d*t* ≤ *K*\* (|*p*'(*t*)|,*P*1 2) *p*2(*t*)|*x*'''(*t*)| d*t*, (3.58)

1 ,

Hence we obtain a a

ß ß

Z

Z

1. ''' 2

ß

'''

Z

' ''

where *K*\*1(|*p*'(*t*)|,*P*1,2) is defined as in [(3.48)](#_bookmark48). Applying the

Zß

*p* (*t*)|*x* (*t*)| d*t* ≤

a a

|*Q*1(*t*)||*x*(*t*)||*x* (*t*)|d*t* +

a

ß

|*Q*1(*t*)||*x* (*t*)||*x* (*t*)|d*t*

2 '' 2 ''

inequality [(2.1)](#_bookmark12) on the term |*p*'(*t*)| |*x* (*t*)| d*t* with *y*(*t*)= *x* (*t*)

a

+2 Z |*p*(*t*)*p*'(*t*)||*x*''(*t*)||*x*'''(*t*)|d*t*

(where *x*''(a)= *x*''(ß)= 0), we see that

a ß

Z

+ |*p*'(*t*)|2|*x*''(*t*)|2d*t*.

a

ß ß

|*p*'(*t*)|2|*x*''(*t*)|2d*t* ≤ D2

Z

Z

a a

*p*2(*t*)|*x*'''(*t*)|2d*t*, (3.59)

Applying the inequality [(2.4)](#_bookmark15) on the integral

(3.53)

where D2 = 4(*A*\*\*(a, ß))2 and *A*\*\*(a, ß) is defined as in [(3.50)](#_bookmark51).

Substituting [(3.54), (3.57) and (3.58) and (3.59) into (3.53)](#_bookmark54) and

ß

ß

Z

|*Q*1(*t*)||*x*(*t*)||*x*'''(*t*)|d*t*,

cancelling the term Z *p*2(*t*)|*x*'''(*t*)|2d*t*, we have

a

a

with ф(*t*) = |*Q*1(*t*)|, 9(*t*) = *p*2(*t*), *m* = 1, *k* = 0, *l* = 1, *n* = 3 and μ = 2, we get (note that *x*(*i*)(a) = 0, for *i* = 0,1,2) that

Ф1 *Q*1, *p*2, *P*1,0 + D1F1 *Q*1, *p*2, *P*1,1 + 2*K*\*1(|*p*'(*t*)|, *P*1,2)+ D2 ≥ 1,

whichisthedesiredinequality[(3.51)](#_bookmark52). Theproofof[(3.52)](#_bookmark53) issimilar to [(3.51)](#_bookmark52) by using the integration by parts and the constants

ß

Z

|*Q* (*t*)||*x*(*t*)||*x*'''(*t*)|d*t* ≤ Ф *Q* , *p*2, *P*

2 Zß

*p*2(*t*)|*x*'''(*t*)|2d*t*3,

Ф1 *Q*1, *p*2, *P*1,0 , F1 *Q*1, *p*2, *P*1,1 , *K*\*1(|*p*'(*t*)|, *P*1,2),

1 1 1

a

1,0

a

4

(3.54)

are replaced by

Ф2 *Q*1, *p*2, *P*2,0 , F2 *Q*1, *p*2, *P*2,1 , *K*\*2(|*p*'(*t*)|, *P*2,2),

where Ф1(*Q*1, *p*2, *P*1,0) is defined as in [(3.1)](#_bookmark23) and *Q* is replaced by

5

*Q*1. Applying the inequality [(2.4)](#_bookmark15) again on the integral

ß

Z

|*Q*1(*t*)||*x*'(*t*)||*x*''(*t*)|d*t*,

a

with ф(*t*) = *Q*1(*t*), 9(*t*) = *p*2(*t*), *k* = 1, *n* = 2, *l* = *m* = 1 and μ = 2, we see that

which are defined as in [(3.1) and (3.48)](#_bookmark23). The proof is complete.

# Examples

The following examples illustrate the results.

Example 1. Consider the equation

ß

Z

|*Q* (*t*)||*x*'(*t*)||*x*''(*t*)|d*t* ≤ F *Q* , *p*2, *P*

Zß

*p*2(*t*)|*x*''(*t*)|2d*t*, (3.55)

*x*(4)(*t*)+ h|cos(a*t*)|*x*(*t*)= 0, 0 ≤ *t* ≤ p, (4.1)

1 1 1

a

1,1

a

where h and a are positive constants. If *x* is a solution of [(4.1)](#_bookmark57)

where F1(*Q*1, *p*2, *P*1,1) is defined as in [(3.2)](#_bookmark24) and *Q* is replaced by

*Q*1 and *p* is replaced by *p*2. Applying the inequality [(2.1)](#_bookmark12) on the

with *x*(0)= *x*''(0)= *x*(p)= *x*''(p)= 0, we see from Corollary 3.1 that

[e g ypti an j o ur nal o f b a sic and a pp l i e d sci en c e s 1 ( 2014) 49](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) e[5](http://dx.doi.org/10.1016/j.ejbas.2013.09.001) 9 59

h Zp

h 1

5184

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2 [1 + cos(2a*t*)]d*t* = 2 p + 2a sin(2ap) ≥ p7 = 1.7164.

0

provided that h ≥ 1. Then the condition [(3.32)](#_bookmark41) reads

hp h

2 + 4a ≥ 1.7164. (4.2)

for any h ≥ 1 and a > 0.

Example 2. Consider the equation

*x*''''(*t*)— h *x*(*t*)= 0, a ≤ *t* ≤ ß, (4.3)

*t*4

where h is a positive constant and *x* is a solution of [(4.3)](#_bookmark58) which satisfies *x*(0)= *x*'(0)= *x*(ß)= *x*'(ß)= 0. Then the condition [(3.46)](#_bookmark50) implies that

[disfocality and disconjugacy of a differential equation. Pac](http://refhub.elsevier.com/S2314-808X(13)00002-X/sref14) [J Math 1979;81:379](http://refhub.elsevier.com/S2314-808X(13)00002-X/sref14)e[97](http://refhub.elsevier.com/S2314-808X(13)00002-X/sref14).

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which gives us that

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This implies that [(4.3)](#_bookmark58) is disconjugate on [a,ß] if

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