 Electronic Notes in Theoretical Computer Science 167 (2007) 117–130 

[www.elsevier.com/locate/entcs](http://www.elsevier.com/locate/entcs)

Effective Randomness for Computable Probability Measures

# Laurent Bienvenu [1](#_bookmark0) ,[2](#_bookmark0)

*Laboratoire d’Informatique Fondamentale Universit´e de Provence*

*Marseille, France*

Wolfgang Merkle[1](#_bookmark0) ,[3](#_bookmark0)

*Institut fu¨r Informatik, Ruprecht-Karls-Universit¨at Heidelberg, Germany*

Abstract

Any notion of effective randomness that is defined with respect to arbitrary computable probability measures canonically induces an equivalence relation on such measures for which two measures are considered equivalent if their respective classes of random elements coincide. Elaborating on work of Bienvenu [[1](#_bookmark11)], we determine all the implications that hold between the equivalence relations induced by Martin-L¨of randomness, computable randomness, Schnorr randomness, and weak randomness, and the equivalence and consistency relations of probability measures, except that we do not know whether two computable probability measures need to be equivalent in case their respective concepts of weakly randomness coincide.

*Keywords:* computable probability measures, Martin-L¨of randomness, computable randomness, Schnorr randomness, weak randomness, equivalence of probability measures, consistency of probability measures.

1 We would like to thank Alexander Shen and an anonymous referee for helpful comments and suggestions

2 Email: [laurent.bienvenu@lif.univ-mrs.fr](mailto:laurent.bienvenu@lif.univ-mrs.fr)

3 Email: [merkle@math.uni-heidelberg.de](mailto:merkle@math.uni-heidelberg.de)

1571-0661 © 2007 Elsevier B.V. Open access under [CC BY-NC-ND license.](http://creativecommons.org/licenses/by-nc-nd/3.0/)

doi:10.1016/j.entcs.2006.08.010

# Introduction

Since the work of von Mises around 1920, several concepts of randomness for individual infinite sequences of zeros and ones have been proposed. The most important and most satisfactory concept known today is Martin-L¨of random- ness (introduced by Martin-Lo¨f [[5](#_bookmark15)] in 1966), but other notions have received a lot of attention, too (for a detailed and comprehensive account, see the upcom- ing monograph of Downey and Hirschfledt [[2](#_bookmark12)]). On the one hand, there are various notions of stochasticity, which are defined in terms of selection rules. On the other hand, there are notions that can be defined in terms of betting strategies such as Martin-Lo¨f randomness or computable randomness. The concepts of stochasticity and randomness are generally applied in connection with the uniform measure on Cantor spaces. Indeed, notions of stochastic- ity, which rely on the converge of frequencies as asserted in the law of large numbers, cannot be extended to arbitrary or even to arbitrary computable probability measures. In contrast to this, concepts of randomness defined in terms of betting strategies usually extend naturally to arbitrary computable probability measures. Accordingly, we will focus on standard randomness no- tions that can be defined in terms of martingales. These notions are Martin- L¨of randomness, computable randomness (also called recursive randomness), Schnorr randomness, and weak randomness (also called Kurtz randomness).

Relations between the various notions of stochasticity and randomness have been extensively studied in the context of uniform measures. In the sequel, we pursue a different approach, as we compare randomness notions with respect to their behavior when the underlying probability measure is varied. In classical probability theory, two probability measures are said to be *equivalent* if they have the same nullsets, or, in other words, if they have the same sets of measure 1, which means that they are in some sense quite similar. Since defining a notion of randomness means choosing for each computable measure *μ* a particular set of *μ*-measure 1 and calling its elements *random*, it is natural to define an equivalence relation by saying that two measures are equivalent if they have the same random elements. In this paper,l we investi- gate into the question which relations hold between the equivalence relations that are obtained from the various randomness concepts. More precisely, we ask for which pairs of randomness concepts it is the case that equivalence with respect to the first concept implies equivalence with respect to the second.

# Definitions and concepts

* 1. *The Cantor space*

In what follows, we will only deal with randomness in the Cantor space (although effective randomness can be extended to more general topologi- cal spaces, see for example [[3](#_bookmark13)]). The Cantor space, which we denote by 2*ω* is the set of infinite binary sequences. It is canonically endowed with the product topology, a basis of which is given by the open sets of the form *u.*2*ω* = {*α* ∈ 2*ω* : *u* и *α*} with *u* ∈ 2∗ (2∗ is the set of finite binary sequences, and и is the prefix relation, defined on 2*ω* ∪ 2∗). If *α* ∈ 2∗ ∪ 2*ω*, *α* T *n* denotes the finite word consisting of the first *n* bits of *α*. If X ⊆ 2*ω*, X denotes the complement of X is 2*ω*.

By Caratheodory’s extension theorem, every function *m* defined on the subsets of 2*ω* and taking its values in [0*,* 1], such that *m*(2*ω*) = 1 and for all *u* 2∗ *m*(*u.*2*ω*)= *m*(*u*0*.*2*ω*)+ *m*(*u*1*.*2*ω*), induces a unique probability measure on 2*ω*. Hence, from now on we can identify a probability measure with its restriction to the open sets of the form *u.*2*ω*, and we abbreviate *μ*(*u.*2*ω*) by *μ*(*u*). The canonical measure on 2*ω* is the Lebesgue measure *λ*, defined by *λ*(*u*) = 2−|*u*| for all *u* 2∗. Moreover, since they are the only measures we will consider, we will abbreviate “probability measure on the Cantor space” by “measure”.

∈

∈

Definition 2.1 A measure *μ* is computable if there exists a computable func- tion *f* : 2∗ × N → Q such that for all (*u, n*), |*f* (*u, n*) − *μ*(*u*)|≤ 2−*n*.

The reason why we only consider computable measures is that the notions of randomness we consider, initially defined for the the uniform measure, can be extended in a very natural way to computable measures, whereas there is no such completely natural extension in the case of non-computable measures.

In classical probability theory, there are two main relations on probability measures:

Definition 2.2 Two probability measures *μ* and *ν* are equivalent (denoted by *μ ν*) if they have the same nullsets.

~

Two probability measures *μ* and *ν* are consistent if there is no set which has measure *μ*-measure 1 and *ν*-measure 0.

* 1. *Martingales*

Following Ville [[10](#_bookmark20)], we now introduce the notion of martingale. It can be seen as describing the capital of a player who is trying to guess the bits of an

infinite binary sequence, betting money (never more than his current capital) on their values, and is rewarded in a fair way. Of course, the fairness of the game depends on the underlying measure.

Definition 2.3 Let *μ* be a measure. A *μ*-martingale is a function

*d* : 2∗ R+ + such that for all *u* 2∗: *d*(*u*)*μ*(*u*) = *d*(*u*0)*μ*(*u*0) + *d*(*u*1)*μ*(*u*1) (with the convention + *.*0 = 0). A martingale *d* is said to be normed if *d*(*ε*) = 1 (*ε* being the empty word). It is said to be computable if there exists a computable function *f* : 2∗ × N → Q such that for all (*u, n*),

∞

→ ∪ { ∞} ∈

|*f* (*u, n*) − *d*(*u*)|≤ 2−*n*.

The following lemma shows that there exists an exact correspondence be- tween measures and martingales:

Lemma 2.4 *For every measure (resp. computable measure) μ, the normed μ-martingales (resp. computable normed μ-martingales) are exactly the func- tions of the form ξ , where ξ is a measure (resp. computable measure).*

*μ*

(The proof is straightforward).

The next theorem is a well-known result on martingales, which will be of great use in the sequel.

Theorem 2.5 (Ville [[10](#_bookmark20)]) *Let μ be a measure and d a μ-martingale. For all k* ∈ R+*, μ*{*α* ∈ 2*ω* : *supn d*(*α* T *n*) ≥ *k*}≤ 1*/k.*

* 1. *Martin-L¨of randomness*

Definition 2.6 An open set V is said to be computably enum erable (c.e.) if

there exists a computably enumerable *A* ⊂ 2∗ such that V =

*u*∈*A u.*2*ω*.

A collection {V*n*}*n*∈N of c.e. open sets is said to be computable if there exists a com putable function (*n, k*) ∈ N2 '→ *un,k* ∈ 2∗ such that for all *n* ∈ N,

V*n* = *k*∈N *un,k.*2 .

*ω*

A *μ*-Martin-L¨of test is a computable collection of c.e. open sets {V*n*}*n*∈N

such that for all *n*, *μ*(V*n*) ≤ 2−*n* .

*α* ∈ 2*ω* is said to pass the *μ*-Martin-L¨of test {V*n*}*n*∈N if *α* ∈*/* *n* V*n*.

*α* ∈ 2*ω* is said to be *μ*-Martin-Lo¨f random (*μ*-ML random for short) if it passes all *μ*-Martin-L¨of tests. We denote by *μMLR* the set of *μ*-ML-random infinite sequences.

Remark 2.7 The notion of ML randomness remains the same if we define a Martin-L¨of test to be a computable collection of c.e. open sets {V*n*}*n* such that *μ*( *n*) is bounded by some computable real-valued function *f* (*n*) which is decreasing and tends to 0 as *n* tends to infinity.

V

Martin-L¨of randomness can be expressed in the framework of martingales (see [[2](#_bookmark12)]) but we will not need this.

* 1. *Computable randomness*

C. P. Schnorr proposed in [[8](#_bookmark18)] and [[9](#_bookmark19)] two weaker, but in some sense more effective, alternative notions of effective randomess. They are now called respectively computable randomness and Schnorr randomness. Computable randomness is based on the so-called unpredictability paradigm: a sequence is random if no computable strategy/martingale succeeds on it.

Definition 2.8 Let *μ* be a computable probability measure. A sequence *α* 2*ω* is *μ*-computably random if there is no computable *μ*-martingale *d* such that sup*n d*(*α* T *n*) = + . We denote by *μCR* the set of *μ*-computably random sequences.

∈

∞

Remark 2.9 The notion of computable randomness remains the same if we replace the winning condition sup*n d*(*α* T *n*)= +∞ by lim*n d*(*α* T *n*)= +∞.

* 1. *Schnorr randomness*

Schnorr randomness is even weaker than computable randomness. A sequence *α* is declared to be not Schnorr random if there exists a computable strat- egy/martingale which not only succeeds on it, but succeeds at a reasonnable pace:

Definition 2.10 An order is a function *g* : N N which is nondecreasing and unbounded. A sequence *α* is *μ*-Schnorr random if there exists no computable *μ*-martingale *d* and computable order *g* such that *d*(*α* T *n*) *g*(*n*) for infinitely many *n*. We denote by *μSR* the set of *μ*-Schnorr random sequences.

→

≥

This definition can be rephrased as follows:

Proposition 2.11 *A sequence α is μ-Schnorr random iff there exists no computable μ-martingale d and computable function f* : N → N *such that d*(*α* T *f* (*n*)) ≥ *n for inﬁnitely many n.*

* 1. *Weak randomness*

Weak randomness, introduced by S. Kurtz [[4](#_bookmark14)] (and hence also known as Kurtz randomness) is in some sense the dual of Martin-L¨of randomness. Instead of requiring a random sequence to avoid all the sets effectively of measure 0, we require it to belong to all the effective sets of measure 1:

Definition 2.12 A sequence *α* is *μ*-weakly random if it belongs to all c.e. open set C of *μ*-measure 1. We denote by *μWR* the set of *μ*-weakly random sequences.

Weak randomness can be characterized using martingales, showing in par- ticular that Schnorr randomness implies weak randomness.

Proposition 2.13 (Wang [[11](#_bookmark21)]) *A sequence α* 2*ω is μ-weakly random if there is no computable μ-martingale d and computable order g such that d*(*α* T *n*) ≥ *g*(*n*) *for all n. We denote by μWR the set of μ-weakly random sequences.*

∈

# A classification of equivalence relations

The rest of the paper will be devoted to the proof of the following classification:

Theorem 3.1 *For all computable probability measures μ and ν, the following implications hold:*

*μCR* = *νCR*

↓

*μMLR* = *νMLR μSR* = *νSR*

(

*μ* ~ *ν*

↓

*μWR* = *νWR*

↓

*μ, ν consistent*

We will show that no other implication holds between these equivalence relations, except a possible equality between classical equivalence and WR- equivalence (although we conjecture this is not the case) which we leave as an open question.

Remark 3.2 It is interesting to note that the above implications between the different equivalence relations are not at all related to the implications between the underlying notions of randomness.

The first implication (if two computable measures have the same com- putably random elements, then they have the same Martin-L¨of random elements) is proven in [[6](#_bookmark16)] (see also [[1](#_bookmark11)]). We will prove all the others.

Proposition 3.3 *Let μ and ν be two computable measures.*

1. *If μMLR* = *νMLR then μ* ~ *ν*
2. *If μSR* = *νSR then μ* ~ *ν*

To prove this, we need the following lemma:

Lemma 3.4 *Let A* = *u*1*, .., uN be a ﬁnite preﬁx-free set of words (i.e. no one is a preﬁx of another). For all computable measure μ, there exists a normed*

*μ-martingale dμ , effectively computable from A, such that for all* 1 ≤ *i* ≤ *N:*

*A*

*dμ* (*u* )= Σ*N μ*(*u* ) −1*.*

*A*

*i*

*i*=1

*i*

Proof. For all *u* 2∗, let *dμ* be the normed *μ*-martingale which tries to win as much as possible on *u* (and hence loses everything on all other words of length |*u*|). Formally:

*u*

∈

, *μ*(*w*)−1 if *w* и *u dμ*(*w*)= *μ*(*u*)−1 if *u* и *w*

*u*

,,⎨

,,,

0 otherwise

Then let *dμ* = Σ*N μ*(*u* ) −1 Σ*N μ*(*u* )*dμ* . *dμ* is a martingale as it is a

*A*

*i*=1

*i*

*i*=1

*i*

*ui*

*A*

weighted sum of martingales, and by construction is normed and satisfies the required property.

Proof. [of Proposition [3.3](#_bookmark5)] We prove (a) and (b) at the same time. Suppose that *μ* and *ν* are not equivalent, i.e. for example there exists a set X such that *μ*(X )= 0 and *ν*(X ) *>* 0. Let *q* be a rational number such that *ν*(X ) *> q >* 0. By definition of a measure: *μ*(X ) = inf {*μ*(W) : W open}. Hence, for all *k* ∈ N, there exists an open set W ⊃ X such that *μ*(W) *<* 2−2*k* (and of course *ν*(W) *> q*). Since the *u.*2*ω* are a base for the Cantor space topology, there

*wi.*2*ω* ⊂ W

exists a finite (prefix-free) set of words *w*1*, ..., wN* such that

*N*

*i*=1

and such that *ν*( *N wi.*2*ω*) ≥ *q* (and of course *μ*( *N wi.*2*ω*) *<* 2−2*k*).

*i*=1

*i*=1

Hence, for a given *k*, one can effectively find a (prefix-free) finite set of

words *A*

*k*

1

*Nk*

= *uk, ..., uk*

such that, setting V

= *Nk*

*u .*2*ω*, we have *ν*(V

) ≥ *q*

and *μ*( *k*) 2−2*k* (it suffices to enumerate the finite sets of words until we find one which satisfies these properties, which will eventually happen by the above discussion). Then let Z = *n k>n* V*k* = {*α* : *α* ∈ V*k* for infinitely many *k*}. It is easy to see that *ν*(Z) ≥ *q* and hence that Z contains *ν*-Martin L¨of random sequences. We now show that Z ∩ *μSR* = ∅. By the above lemma,

V ≤

*k*

*i*=1

*i*

*k*

for all *k*, there exists a normed *μ*-martingale *dk* = *dμ* such that for all *uk*:

*Ak*

*i*

*dk*(*uk*) ≥ 22*k*. Next, let *d* = Σ

*i*

*k*∈N

2−*kdk*. It is a normed *μ*-martingale as it is

the weighted sum, with sum of weights equal to 1, of normed *μ*-martingales. It

is computable since for all *w*, *d*(*w*) − Σ*m* 2−*kdk*(*w*) ≤ 2−*m μ*(*w*)−1 (hence

*k*=1

one can approximate *d*(*w*) by taking *m* large enough, the error bound being

computable in *m* and tending to 0 as *m* tends to infinity). Let *f* be the function defined by *f* (*k*) = max *uk* : 1 *i Nk* . For all *α* , there are infinitely many *uk* which are prefixes of *α*. By construction of *d*, for all *k* and all *i* ≤ *Nk*, we have *d*(*uk*) ≥ 2−*kdk*(*uk*) ≥ 2*k*. Hence for infinitely many *n*:

*i*

*i*

{| | ≤ ≤ } ∈ Z

*i*

*i*

*d*(*α* T *f* (*n*)) ≥ 2*n*. This, by Proposition [2.11](#_bookmark3), asserts that *α* is not *μ*-Schnorr random. We have proven that Z ∩ *νMLR* /= ∅ and Z ∩ *μSR* = ∅, which completes the proof.

Proposition 3.5 *Let μ and ν be two computable measures. If μ ν, then*

~

*μWR* = *νWR.*

Proof. This is trivial. If a sequence *α* is, say, *ν*-weakly random and not *μ*- weakly random, then this means that there exists an effectively open set C of *μ*-measure 1 such that *α* ∈*/* C. Since *α* is *ν*-weakly random, C must have *ν*-measure less than 1. Hence, C witnesses that *μ* and *ν* are not equivalent.

Proposition 3.6 *Let μ and ν be two computable measures. If μWR* = *νWR*

*then μ and ν are consistent.*

Proof. Suppose *μ* and *ν* are not consistent, that is, there exists a set X such that *ν*(X )= 1 and *μ*(X ) = 0. We argue, similarly to the proof of Proposition [3.3](#_bookmark5), that for all *k* ∈ N, w e can effectively find a finite prefix-free set of words

*uk, ..., uk*

such that *μ*(

*u .*2*ω*) ≤ 2−*k*−2 and *ν*(

*u .*2*ω*) ≥ 1 − 2−*k*.

*i*

*i*

*Nk*

*Nk*

Then, C = *k*

*i*

1

*Nk*

*i*=1

*i*=1

*Nk i*=1

*uk.*2*ω* is an effectively open set, of *ν*-measure 1 (and hence

contains all the *ν*-weakly random sequences), and of *μ*-measure less than 1*/*2

(and hence does not contain all the *μ*-weakly random sequences, which form a set of *μ*-measure 1).

We have proven all the implications of Theorem [3.1](#_bookmark4). We now begin the more delicate task of showing that there is no other valid implication between the equivalence relations we study (with the possible exception mentioned

above).

In order to construct counter-examples, the following quantities will play a crucial role.

Definition 3.7 Let *μ* and *ν* be two computable measures and *k* R+ + . We set

∈ ∪{ ∞}

L*ν/μ* = {*α* ∈ 2

*k*

*ω*

: sup

*n*∈N

*ν*(*α* T *n*)

*μ*(*α* T *n*) ≥ *k*}

(with the following convention: if *μ*(*α* T *n*)= *ν*(*α* T *n*)= 0, *ν*(*α*†*n*)

*μ*(*α*†*n*)

= 1, and if

*μ*(*α* T *n*)= 0 and *ν*(*α* T *n*) *>* 0, *ν*(*α*†*n*) = +∞)

*μ*(*α*†*n*)

¿From Lemma [2.4](#_bookmark1) and Theorem [2.5](#_bookmark2), we get:

Corollary 3.8 *For all computable measures μ, ν:* L∞

*ν/μ*

∩ *μCR* = ∅*. In par-*

*ticular, this implies that* L∞ ∩ *μMLR* = ∅*. Moreover, for every k* ∈ R *:*

*ν/μ*

+

*k*

*μ*(L

*ν/μ*

*ν/μ*

) ≤ 1*/k (and hence μ*(L∞

)= 0*).*

Proof. *ν* is a *μ*-martingale (by Lemma [2.4](#_bookmark1)), and L∞ is exactly the set

*μ*

*ν/μ*

of sequences on which it succeeds. Thus, L∞ ∩ *μCR* = ∅. The fact that

*ν/μ*

*k*

*μ*(L

*ν/μ*

) ≤ 1*/k* is an immediate consequence of Theorem [2.5](#_bookmark2).

The next proposition shows how some of the equivalence relations we study

are related to the L∞ .

*μ/ν*

Proposition 3.9 *For every couple* (*μ, ν*) *of computable measures:*

1. *μ* ~ *ν iff μ*(L∞ )= *ν*(L∞

*μ/ν*

*ν/μ*

)= 0*,*

1. *μMLR* = *νMLR iff* L∞ ∩ *μMLR* = L∞

*μ/ν*

*ν/μ*

∩ *νMLR* = ∅*,*

1. *μCR* = *νCR iff* L∞ ∩ *μCR* = L*ν/μ*

*μ/ν*

∞

∩ *νCR* = ∅*.*

Proof. For (a), (b) and (c), the “only if” direction is a direct consequence of Corollary [3.8](#_bookmark6). Let us now prove the “if” directions. We will use the following

fact: for every open set C ⊆ 2*ω* and all measures *μ* and *ν*: *μ*(C ∩L*k* ) ≤ *k ν*(C)

*μ/ν*

(this is a trivial consequence of the definition of L*k* ).

*μ/ν*

* 1. Suppose *μ*(L∞ )= *ν*(L∞ ) = 0, and let X ⊆ 2*ω* such that *ν*(X ) = 0.

*μ/ν*

*ν/μ*

Let *k* ∈ N. Let C be an open set such that X ⊆ C and *ν*(C) ≤ 1*/k*2. Then:

*μ*(X ) ≤ *μ*(C)

*k*

≤ *μ*(C ∩ L*μ/ν*)+ *μ*(C ∩ L*k* )

≤ *μ*(L*μ/ν*)+ *k ν*(C)

*μ/ν*

*k*

≤ *μ*(L*μ/ν*)+ 1*/k*

*k*

This being true for all *k*, and since by assumption *μ*(L∞ ) = 0 (which is

*μ/ν*

equivalent to lim*k*→+∞ *μ*(L*k* ) = 0), this proves that *μ*(X )= 0.

*μ/ν*

* 1. Suppose L∞

*μ/ν*

∩ *μMLR* = L*ν/μ*

∩ *νMLR* = ∅. Let *α* ∈*/*

*νMLR*. If

∞

∞

*α* ∈ L

*μ/ν*

, then by assumption *α* ∈*/*

*μMLR*, and we are done. If not, then

there exists *k* ∈ R+ s.t. *α* ∈*/*

*k*

L*μ/ν*

. Let {C*n*}*n*∈N be a *ν*-Martin-Lo¨

f test s.t.

*α* ∈ *n* C*n*. Let {*un,i* : (*n, i*) ∈ N2} be an effective enumeration of finite strings such that for all *n*, C*n* is the disjoint union of the {*un,i*2*ω* : *i* ∈ N}.

*ω*

Define for all *n* V*n* =

*i*{*un,i*2 : *i* ∈ N*, μ*(*un,i*) *< k ν*(*un,i*)}. It is then

clear that {V*n*}*n*∈N is a computable family of c.e. open sets such that for all

*n*, *μ*(V*n*) ≤ *k ν*(C*n*) (hence {V*n*}*n*∈N is a *μ*-Martin-Lof test) and *α* ∈

Thus, *α* ∈*/ μMLR*.

*n* V*n*.

* 1. Suppose L∞ ∩ *μCR* = L*ν/μ* ∩ *νCR* = ∅. Let *α* ∈*/ νCR*. By Lemma

*μ/ν*

∞

[2.4](#_bookmark1), there exists a computable measure *ξ* such that sup*n*

*ξ*(*α*†*n*)

*ν*(*α*†*n*)

= +∞. If

sup*n*

≥

*ξ*(*α*†*n*)

*μ*(*α*†*n*)

= +∞, then *α* ∈*/*

*μCR*, and we are done. Otherwise, there

exists a constant *k* such that for all *n*, *ξ*(*α*†*n*)

*μ*(*α*†*n*)

≤ *k*. Hence, sup

*μ*(*α*†*n*) *n ν*(*α*†*n*)

sup

*n*

*μ*(*α*†*n*) *ξ*(*α*†*n*)

1 *ξ*(*α*†*n*) = +∞. Thus *α* ∈ L∞

which by assumption

implies *α* ∈*/ μCR*.

≥ sup

*ξ*(*α*†*n*) *ν*(*α*†*n*)

*n*

*k ν*(*α*†*n*)

*μ/ν*

Proposition 3.10 *Let α* ∈ Δ0 ∩ *λSR. There exists a computable measure μ*

2

*such that* L∞ = ∅*,* L∞ = {*α*} *and α* ∈*/ μSR*

*μ/λ*

*λ/μ*

Proof. We will in fact construct a computable *λ*-martingale *d* such that: lim*n d*(*α* T *n*)= 0 and if *β* = *α*, *d*(*β* T *n*) will be eventually constant. We will then argue that the computable measure *μ* defined by *μ*(*u*)= *λ*(*u*)*d*(*u*) for all *u* is as desired. The construction is done by stages. At stage *s*, *d*(*u*) will be defined for all words *u* such that |*u*|≤ 3*s*.

/

Since *α* is Δ0, it is the pointwise limite of a sequence of words {*ws*}*s*∈N. We can moreover assume that lim*s*→+∞ |*ws*| = +∞, that |*ws*| ≤ 3*s* for all *s*, and that *ws* is a prefix of *α* for infinitely many *s*.

2

Let *E* = *u*12|*u*| : *u* 2∗ . Notice that every *λ*-Schnorr random sequence has only finitely many prefixes in *E*. Hence, up to changing a finite number of its bits, we can assume that *α* has no prefix in *E*. Let us now proceed to the construction of *d*.

{ ∈ }

*S* tage *s* = 0. Set *d*(0) = *d*(1) = 1*/*2.

*S* tage *s* + 1. We define *d*(*u*) for every *u* of length 3*s*+1:

* if *u* is not an extension of *ws*, set *d*(*u*)= *d*(*u* T 3*s*)
* if *u* is an extension of *ws* and is not in *E*, set *d*(*u*)=

*d*(*u*†3*s*) *s*+1

* if *u* is an extension of *ws* and is in *E*, set *d*(*u*) in such a way that the average value of {*d*(*v*): *v* T 3*s* = *u* T 3*s*} is *d*(*u* T 3*s*)

Then, for the words *u* such that 3*s < u <* 3*s*+1, set inductively (in decreasing order of length) *d*(*u*)= *d*(*u*0)+*d*(*u*1) .

2

| |

The martingale *d* is as desired: since *α* has no prefix in *E*, it follows that for infinitely many *s*, *ws* is a prefix of *α* and *α* T 3*s* is not in *E*. For all such *s*, by definition of *d*: *d*(*α* T 3*s*) 1*/s*. On the other hand, if *β* = *α*, there exists *s*0 such that if *s > s*0, *ws* is not a prefix of *β*, and thus, for all *n >* 3*s*0 , *d*(*β* T *n*)= *d*(*β* T 3*s*0 ).

≤ /

Let us now consider *μ* = *λ d*. By the above discussion, L∞ = ∅, L∞ =

*μ/λ*

*λ/μ*

{*α*}. Moreover, for infinitely many *s*, Proposition [2.11](#_bookmark3), *α* ∈*/ μSR*.

≥

*λ*(*α*†3*s*)

*μ*(*α*†3*s* ) *s*. Hence, by Lemma [2.4](#_bookmark1) and

Before we can apply the above proposition to the construction of counter examples, we need the following important theorem.

Theorem 3.11 (Nies, Stephan and Terwijn [[7](#_bookmark17)]) *Let α* ∈ 2*ω. The fol- lowing are equivalent:*

1. *α has high Turing degree,*
2. *There exists β in the Turing degree of α such that β* ∈ *λCR* \ *λM LR,*
3. *There exists β in the Turing degree of α such that β* ∈ *λSR* \ *λCR.*

We are now ready to prove:

Proposition 3.12 *(a) There exists a computable measure μ such that λ μ*

~

*and nonetheless λM LR* = *μMLR, λCR* = *μCR, λSR* = *μSR.*

/ / /

1. *There exists a computable measure μ such that λM LR* = *μMLR and*

*λCR* /= *μCR.*

1. *There exists a computable measure μ such that λCR* = *μCR and λSR* /=

*μSR.*

Proof. (a) Let *μ* be, by Proposition [3.10](#_bookmark8), a computable measure such that

∞ ∞

L*μ/λ* = ∅, L*λ/μ* = {Ω} and Ω ∈*/ μSR*. Here Ω denotes Chaitin’s omega num-

ber, which is known to be left-r.e.(i.e. the set {*q* ∈ Q : *q <* Ω} is recursively enumerable, which in particular implies that Ω is Δ0). Since *λ*({Ω})= 0, by Proposition [3.9](#_bookmark7), we have *λ* ~ *μ*. Moreover, since Ω ∈ *λM LR* ⊂ *λCR* ⊂ *λSR*, and Ω ∈*/ μSR*, it follows that *λM LR* /= *μMLR*, *λCR* /= *μCR*, *λSR* /= *μSR*.

2

* 1. By Theorem [3.11](#_bookmark9), let *α* be a Δ0 sequence such that *α* ∈ *λCR* \ *λM LR*.

2

By Proposition [3.10](#_bookmark8), there exists a computable measure *μ* such that L∞ = ∅,

*μ/λ*

L*λ/μ* = {*α*} and *α* ∈*/ μSR*. By Proposition [3.9](#_bookmark7), we have *λM LR* = *μMLR*

∞

(since *α* ∈*/ λM LR*) and *λCR* /= *μCR* (since *α* ∈ *λCR* \ *μCR*).

* 1. By Theorem [3.11](#_bookmark9), let *α* be a Δ0 sequence such that *α* ∈ *λSR* \ *λCR*.

2

By Proposition [3.10](#_bookmark8), there exists a computable measure *μ* such that L∞ = ∅,

*μ/λ*

L*λ/μ* = {*α*} and *α* ∈*/ μSR*. By Proposition [3.9](#_bookmark7), we have *λCR* = *μCR* (since

∞

*α* ∈*/ λCR*) and *λSR* /= *μSR* (since *α* ∈ *λSR* \ *μSR*).

Proposition 3.13 *There exists a computable measure μ such that λSR* =

*μSR, λCR* /= *μCR and λM LR* /= *μMLR*

Once again, we need a preliminary lemma.

Lemma 3.14 *Let μ and ν be two computable measures and α* ∈ 2*ω. If α* ∈

*νSR* \ *μSR, then there exists a computable order g such that ν*(*α*†*n*) ≥ *g*(*n*)

*μ*(*α*†*n*)

*inﬁnitely often.*

Proof. Let *α* ∈ *νSR* \ *μSR*. By Lemma [2.4](#_bookmark1), there exists a computable

measure *ξ* and a computable order *g* such that *ξ*(*α*†*n*) ≥ *g*(*n*) for infinitely

*μ*(*α*†*n*)

√

many *n*. Since *α* ∈ *νSR* and since *g* is a computable order, for almost all *n*,

*ξ*(*α*†*n*) ≤ √*g*(*n*). Hence, for infinitely many *n*: *ν*(*α*†*n*) = *ξ*(*α*†*n*) *ν*(*α*†*n*) ≥ √*g*(*n*) =

*ν*(*α*†*n*)

*μ*(*α*†*n*)

*μ*(*α*†*n*) *ξ*(*α*†*n*)

*g*(*n*)

√*g*(*n*).

Proof. [of Proposition [3.13](#_bookmark10)] We will construct, in a very similar way as for

Proposition [3.10](#_bookmark8) a computable measure *μ* such that L∞ = ∅, L∞ = {Ω},

*μ/λ*

*λ/μ*

but this time we want Ω to be *μ*-Schnorr random. By the above lemma,

it will be sufficient to ensure that *λ*(Ω†*n*)

*μ*(Ω†*n*)

tends to infinity slower than any

computable order. Hence, we will again construct a *λ*-martingale *d* such that

lim*n d*(Ω T *n*) = 0 and if *β* = Ω, *d*(*β* T *n*), ensuring that *d*(Ω T *n*) decreases very slowly.

/

Ω being left-r.e., let {*ws*}*s*∈N be a sequence of words such that Ω is the pointwise limit of this sequence, lim*s*→+∞ |*ws*| = +∞, |*ws*| ≤ 3*s* for all *s*, *ws* is a prefix of *α* for infinitely many *s*, and if *ws* is a prefix of Ω, *ws* и *wt* for all *t > s*. We also assume (up to modifying Ω for a finite number of bits) that Ω has no prefix in *E* (defined in the proof of Proposition [3.10](#_bookmark8)).

Let *F* : N → N such that *F* (0) = 0, and for all *s >* 0, if *ws* и Ω, *F* (*s* + 1) = |*ws*|, and otherwise *F* (*s* + 1) = *F* (*s*). By definition of the *ws*, for all *s*, the initial segment of *ws* which coincides with Ω has at least length *F* (*s*). *F* tends to infinity slower than any computable order: suppose this is

not the case, i.e. there exists a computable order *g* such that *g*(*s*) ≤ *F* (*s*) for infinitely many *s*. Then, for infinitely many *s*, looking at *ws*, it is possible to guess the first *g*(*s*) bits of Ω. From this remark, it is routine to construct a *λ*-martingale which asserts that Ω is not Schnorr random, a contradiction.

We construct a *λ*-martingale *d* such that for all *s*, *d*(Ω T 3*s*) = *F* (*s*)−1, and if *β* /= Ω, *d*(*β* T *n*) is eventually constant:

*S* tage *s* = 0. Set *d*(0) = *d*(1) = 1*/*2.

*S* tage *s* + 1. We define *d*(*u*) for every *u* of length 3*s*+1:

* if *u* is not an extension of *ws*, set *d*(*u*)= *d*(*u* T 3*s*)
* if *u* is an extension of *ws* and is not in *E*, set *d*(*u*)= 1*/*|*ws*|
* if *u* is an extension of *ws* and is in *E*, set *d*(*u*) in such a way that the average value of {*d*(*v*): *v* T 3*s* = *u* T 3*s*} is *d*(*u* T 3*s*)

Then, for the words *u* such that 3*s < u <* 3*s*+1, set inductively (in decreasing order of length) *d*(*u*)= *d*(*u*0)+*d*(*u*1) .

2

| |

Finally, set *μ* = *d λ*. It remains to show that *μ* is as desired. First, we have

∞

L

*μ/λ*

= ∅. We also see that L∞

= {Ω}. Indeed, for all *s*, *d*(Ω T 3*s*)= *F* (*s*)−1,

and hence *λ*(Ω†3*s* ) = *F* (*s*), which by definition of *F* implies sup

*λ/μ*

*μ*(Ω†3*s* )

= +∞.

*λ*(Ω†*n*) *n μ*(Ω†*n*)

Moreover, if *n <* 3*s*, by construction of *d*, *d*(Ω T *n*) ≥ *d*(Ω T 3*s*) and hence

*λ*(Ω†*n*)

*μ*(Ω†*n*) ≤ *F* (log3(*n*)) for all *n*. And of course, *F* (log3(*n*)) = *o*(*g*(*n*)) for all computable order *g*. Finally, if *β* /= Ω, as we saw in the proof of Proposition

[3.10](#_bookmark8), sup

*λ*(*β*†*n*) *<* +∞.

*n μ*(*β*†*n*)

To complete the proof, notice that by Proposition [3.9](#_bookmark7), we have *λM LR* /=

*μMLR* and *μCR* /= *μCR* (since Ω ∈ *λM LR* and Ω ∈*/ μCR*). However, we

have *λSR* = *μSR*. Indeed by the previous lemma, *μSR* \ *λSR* = ∅ since

∞

L

*μ/λ*

= ∅, and *λSR* \ *μSR* = ∅ since L∞

= {Ω} and *λ*(Ω†*n*)

= *o*(*g*(*n*)) for

every computable order *g*.

*λ/μ*

*μ*(Ω†*n*)

Proposition 3.15 *There exist a computable measure μ such that μ and λ are consistent and λWR* /= *μWR*

Proof. Let *δ* be the measure such that *δ*({0*ω*}) = 1 (which is clearly com- putable). Set *μ* = *δ/*2+ *λ/*2. *λ* and *μ* are consistent: let X ⊆ 2*ω*. If 0*ω* ∈ X , then *μ*(X ) = 1*/*2+ *λ*(X ) and if 0*ω* ∈*/* X , *μ*(X ) = *λ*(X ). In both cases, it is impossible to have *μ*(X )=0 and *λ*(X ) = 1, or vice-versa. On the other hand, 0*ω* ∈ *μWR* and 0*ω* ∈*/ λWR*.

# References

1. Bienvenu, L., *Constructive equivalence relations on computable probability measures*, International Computer Science Symposium in Russia, Lecture Notes in Computer Science 3967 (2006), 92–103.
2. Downey, R., and D. Hirschfeldt, “Algorithmic Randomness and Complexity”, Manuscript, 2003.
3. Gacs, P., “Lecture Notes on Descriptional Complexity and Randomness”, Boston University, 1988.
4. Kurtz, S., “Randomness and genericity in the degrees of unsolvability”, PhD thesis, University of Illinois at Urbana, 1981.
5. Martin-Lo¨f, P., *The deﬁnition of random sequences*, Information and Control, 9(6) (1966), 602–619.
6. Muchnik, An. A., A.L. Semenov, V.A. Uspensky, *Mathematical metaphysics of randomness*, Theoretical Computer Science 207(2) (1998), 263–317.
7. Nies, A., F. Stephan, S. Terwijn, *Randomness, relativization and Turing degrees*, Journal of Symbolic Logic 70(2) (2005), 515–535.
8. Schnorr, C. P., *A uniﬁed approach to the deﬁnition of random sequences*, Math. Systems Theory

5 (1971), 246–258.

1. Schnorr, C. P., “Zuf¨alligkeit und Wahrscheinlichkeit”, Lecture Notes in Mathematics 218

(1971), Springer-Verlag.

1. Ville, J., “Etude critique de la notion de collectif”, Gauthiers-Villars, Paris, 1939.
2. Wang, Y., “Randomness and complexity”, Doktoral dissertation, Mathematische Fakult¨at, Universit¨at Heidelberg, 1996.