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**Full Length Article**

**Generalized fuzzy *b*-closed and generalized**



⋆**-fuzzy *b*-closed sets in double fuzzy topological spaces**

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The purpose of this paper is to introduce and study a new class of fuzzy sets called (*r*, *s*)- generalized fuzzy *b*-closed sets and (*r*, *s*)-generalized ⋆-fuzzy *b*-closed sets in double fuzzy topological spaces. Furthermore, the relationships between the new concepts are intro- duced and established with some interesting examples.

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# Introduction

A progressive development of fuzzy sets [[1]](#_bookmark4) has been made to discover the fuzzy analogues of the crisp sets theory. On the other hand, the idea of intuitionistic fuzzy sets was first in- troduced by Atanassov [[2]](#_bookmark5). Later on, Çoker [[3]](#_bookmark6) presented the

notion of intuitionistic fuzzy topology. Samanta and Mondal [[4]](#_bookmark7), introduced and characterized the intuitionistic gradation of openness of fuzzy sets which is a generalization of smooth topology and the topology of intuitionistic fuzzy sets. The name “intuitionistic” is discontinued in mathematics and applica- tions. Garcia and Rodabaugh [[5]](#_bookmark8) concluded that they work under the name “double”.

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In 2009, Omari and Noorani [[6]](#_bookmark9) introduced generalized *b*-closed sets (briefly, g*b*-closed) in general topology. As a gen- eralization of the results in [References 6 and 7](#_bookmark9), we introduce and study (*r*, *s*)-generalized fuzzy *b*-closed sets in double fuzzy topological spaces, then a new class of fuzzy sets between an (*r*, *s*)-fuzzy *b*-closed sets and an (*r*, *s*)-generalized fuzzy *b*-closed sets namely (*r*, *s*)-generalized ⋆-fuzzy *b*-closed sets is intro- duced and investigated. Finally, the relationships between (*r*, *s*)-generalized fuzzy *b*-closed and (*r*, *s*)-generalized ⋆-fuzzy *b*-closed sets are introduced and established with some inter- esting counter examples.

# Preliminaries

Throughout this paper, *X* will be a non-empty set, *I* = [0, 1], *I*0 = (0, 1] and *I*1 = [0, 1). A fuzzy set *h* is quasi-coincident with a fuzzy set *μ* (denoted by, *hqμ*) iff there exists *x* ∈ *X* such that

**(*x*)  **(*x*)  1 and they are not quasi-coincident otherwise (denoted by, *q* ). The family of all fuzzy sets on *X* is denoted

by *IX*. By 0 and 1, we denote the smallest and the greatest fuzzy sets on *X*. For a fuzzy set *h* ∈ *IX*, 1  ** denotes its comple- ment. All other notations are standard notations of fuzzy set

theory.

Now, we recall the following definitions which are useful in the sequel.

Definition 2.1. (see [[4]](#_bookmark7)) A double fuzzy topology (*τ*, *τ*\*) on *X* is a pair of maps *τ*, *τ*\* : *IX* → *I*, which satisfies the following properties:

(O1) **(**)  1  ** \*(** ) for each *h* ∈ *IX*.

(O2) **(**1  **2 )  **(**1 )  **(**2 ) and ** \*(**1  **2 )  ** \*(**1 )  ** \*(**2 ) for each *h*1, *h*2 ∈ *IX*.

(O3) **(*i* *i* )  *i* **(*i* ) and ** \*(*i* *i* )  *i* ** \*(*i* ) for each *hi* ∈ *IX*, *i* ∈ Γ.

The triplet (*X*, *τ*, *τ*\*) is called a double fuzzy topological space (briefly, dfts). A fuzzy set *h* is called an (*r*, *s*)-fuzzy open (briefly, (*r*, *s*)-fo) if *τ*(*h*) ≥ *r* and *τ*\*(*h*) ≤ *s*. A fuzzy set *h* is called an (*r*, *s*)- fuzzy closed (briefly, (*r*, *s*)-fc) set iff 1  ** is an (*r*, *s*)-fo set.

Theorem 2.1. (see [[8]](#_bookmark10)) Let (*X*, *τ*, *τ*\*) be a dfts. Then double fuzzy closure operator and double fuzzy interior operator of *h* ∈ *IX* are defined by

*C*,** \*(**, *r*, *s*)  {** *IX *  **, **(1  **)  *r*, ** \*(1  **)  *s*},

*I*,** \*(**, *r*, *s*)  {** *IX *  **, **(**)  *r*, ** \*(**)  *s*}.

(*r*, *s*)-generalized fuzzy open (briefly, (*r*, *s*)-gfo) iff 1  ** is (*r*, *s*)-gfc set.

Definition 2.3. (see [[11,12]](#_bookmark13)) Let (*X*, *τ*, *τ*\*) be a dfts. For each *h*, *μ* ∈ *IX* and *r* ∈ *I*0, *s* ∈ *I*1. Then, a fuzzy set *h* is said to be (*r*, *s*)- fuzzy generalized *ψρ*-closed (briefly, (*r*, *s*)-fg*ψρ*-closed) if

*C*,** \*(**, *r*, *s*)  ** such that *h* ≤ *μ* and *μ* is (*r*, *s*)-fuzzy *ρ*-open set. *h* is called (*r*, *s*)-fuzzy generalized *ψρ*-open (briefly, (*r*, *s*)-fg*ψρ*- open) iff 1  ** is (*r*, *s*)-fg*ψρ*-closed set.

# (*r*, *s*)-generalized fuzzy *b*-closed sets

In this section, we introduce and study some basic proper- ties of a new class of fuzzy sets called an (*r*, *s*)-fuzzy *b*-closed sets and an (*r*, *s*)-generalized fuzzy *b*-closed.

Definition 3.1. Let (*X*, *τ*, *τ*\*) be a dfts. For each *h* ∈ *IX*, *r* ∈ *I*0 and

*s* ∈ *I*1. A fuzzy set *h* is called:

1. An (*r*, *s*)-fuzzy *b*-closed (briefly, (*r*, *s*)-fbc) if

**  (*I*,** \*(*C*,** \*(**, *r*, *s*), *r*, *s*))  (*C*,** \*(*I*,** \*(**, *r*, *s*), *r*, *s*)).

λ is called an (*r*, *s*)-fuzzy *b*-open (briefly, (*r*, *s*)-fbo) iff 1  ** is (*r*, *s*)-fbc set.

1. An (*r*, *s*)-generalized fuzzy *b*-closed (briefly, (*r*, *s*)-gfbc) if *bC*,** \*(**, *r*, *s*)  ** , *h* ≤ *μ*, *τ*(*μ*) ≥ *r* and *τ*\*(*μ*) ≤ *s*. *h* is called an (*r*, *s*)-generalized fuzzy *b*-open (briefly, (*r*, *s*)-gfbo) iff 1  ** is (*r*, *s*)-gfbc set.

Definition 3.2. Let (*X*, *τ*, *τ*\*) be a dfts. Then double fuzzy *b*-closure operator and double fuzzy *b*-interior operator of *h* ∈ *IX* are defined by

*bC*,** \*(**, *r*, *s*)  {** *IX *  ** and ** is (*r*, *s*)-fbc},

*bI*,** \*(**, *r*, *s*)  {** *IX *  ** and ** is (*r*, *s*)-fbo}.

Where *r* ∈ *I*0 and *s* ∈ *I*1 such that *r* + *s* ≤ 1.

Remark 3.1. Every (*r*, *s*)-fbc set is an (*r*, *s*)-gfbc set.

The converse of the above remark may be not true as shown by the following example.

Example 3.1. Let *X* = {*a*, *b*}. Defined *μ*, *α* and *β* by:

Where *r* ∈ *I*0 and *s* ∈ *I*1 such that *r* + *s* ≤ 1.

Definition 2.2. Let (*X*, *τ*, *τ*\*) be a dfts. For each *h* ∈ *IX*, *r* ∈ *I*0 and

*s* ∈ *I*1. A fuzzy set *h* is called:

1. An (*r*, *s*)-fuzzy semiopen (see [[9]](#_bookmark11)) (briefly, (*r*, *s*)-fso) if

**  *C*,** \*(*I*,** \*(**, *r*, *s*), *r*, *s*) . *h* is called an (*r*, *s*)-fuzzy semi closed (briefly, (*r*, *s*)-fsc) iff 1  ** is an (*r*, *s*)-fso set.

1. An (*r*, *s*)-generalized fuzzy closed (see [[10]](#_bookmark12)) (briefly, (*r*, *s*)- gfc) if *C*,** \*(**, *r*, *s*)  **, *h* ≤ *μ*, *τ*(*μ*) ≥ *r* and *τ*\*(*μ*) ≤ *s*. *h* is called an

**(*a*)  0.3,

**(*a*)  0.4,

**(*a*)  0.3,

1,

**(** )   1 ,

 2

0,

**(*b*)  0.4,

**(*b*)  0.5,

**(*b*)  0.7,

if ** {0, 1}, if **  **, otherwise.

0,

** \*(** )   1 ,

 2

1,

if ** {0, 1}, if **  **, otherwise.

Then *β* is an  1 , 1-gfbc set but not an  1 , 1-fbc set.

* 1. *bI*,** \*(1  **, *r*, *s*)  1  *bC*,** \*(**, *r*, *s*) , *bC*,** \*(1  **, *r*, *s*)  1  *bI*,** \*

2 2 2 2

(**, *r*, *s*),

Definition 3.3. Let (*X*, *τ*, *τ*\*) be a dfts, *h* ∈ *IX*, *r* ∈ *I*0 and *s* ∈ *I*1. *h*

is called an (*r*, *s*)-fuzzy *b*-*Q*-neighborhood of *xt* ∈ *Pt*(*X*) if there

* 1. *bI*,** \*(0, *r*, *s*)  0,

* 1. *bI*,** \*(**, *r*, *s*)  ** ,

*bI*,** \*(1, *r*, *s*)  1 ,

exists an (*r*, *s*)-fbo set *μ* ∈ *IX* such that *xtqμ* and *μ* ≤ *h*.

The family of all (*r*, *s*)-fuzzy *b*-*Q*-neighborhood of *xt* denoted by *b*-*Q*(*xt*, *r*, *s*).

Theorem 3.1. Let (*X*, *τ*, *τ*\*) be a dfts. Then for each *h*, *μ* ∈ *IX*, *r* ∈ *I*0

and *s* ∈ *I*1, the operator *bC*,** \* satisfies the following statements:

(C1)

(C2)

(C3)

(C4)

(C5)

(C6)

(C7)

(C8)

*bC*,**\*(0, *r*, *s*)  0 , *bC*,**\*(1, *r*, *s*)  1,

**  *bC*,**\*(**, *r*, *s*) ,

If *h* ≤ *μ*, then *bC*,**\*(**, *r*, *s*)  *bC*,**\*(**, *r*, *s*), If *h* is an (*r*, *s*)-fbc, then **  *bC*,**\*(**, *r*, *s*),

If *μ* is an (*r*, *s*)-fbo, then *μqh* iff *qbC*,**\*(**, *r*, *s*),

*bC*,**\*(*bC*,**\*(**, *r*, *s*), *r*, *s*)  *bC*,**\*(**, *r*, *s*),

*bC*,**\*(**, *r*, *s*)  *bC*,**\*(**, *r*, *s*)  *bC*,**\*(**  **, *r*, *s*) ,

*bC*,**\*(**, *r*, *s*)  *bC*,**\*(**, *r*, *s*)  *bC*,**\*(**  **, *r*, *s*) ,

* 1. If *h* is an (*r*, *s*)-fbo, then **  *bI*,** \*(**, *r*, *s*),
  2. If *h* ≤ *μ*, then *bI*,** \*(**, *r*, *s*)  *bI*,** \*(**, *r*, *s*),
  3. *bI*,** \*(*bI*,** \*(**, *r*, *s*), *r*, *s*)  *bI*,** \*(**, *r*, *s*) ,
  4. *bI*,** \*(**  **, *r*, *s*)  *bI*,** \*(**, *r*, *s*)  *bI*,** \*(**, *r*, *s*),
  5. *bI*,** \*(**  **, *r*, *s*)  *bI*,** \*(**, *r*, *s*)  *bI*,** \*(**, *r*, *s*).

Proof. It is similar to Theorem 3.1.

Theorem 3.3. Let (*X*, *τ*, *τ*\*) be a dfts. *h* ∈ *IX* is (*r*, *s*)-gfbo set, *r* ∈ *I*0

and *s* ∈ *I*1 if and only if **  *bI*,** \*(**, *r*, *s*) whenever *μ* ≤ *h*,

**(1  **)  *r* and ** \*(1  **)  *s*.

Proof. Suppose that *h* is an (*r*, *s*)-gfbo set in *IX*, and let **(1  **)  *r* and ** \*(1  **)  *s* such that *μ* ≤ *h*. By the definition, 1  ** is an (*r*, *s*)-gfbc set in *IX*. So,

Proof. (1), (2), (3), and (4) are proved easily.

* + 1. Let *q* and *μ* is an (*r*, *s*)-fbo set, then **  1  **. But we have, *μqh* iff *qbC*,** \*(**, *r*, *s*) and

*bC*,** \*(**, *r*, *s*)  *bC*,** \*(1  **, *r*, *s*)  1  **,

so *qbC*,** \*(**, *r*, *s*), which is contradiction. Then *μqh* iff

*qbC*,** \*(**, *r*, *s*) .

* + 1. Let *xt* be a fuzzy point such that *xt* § *bC*,** \*(**, *r*, *s*). Then there is an (*r*, *s*)-fuzzy *b*-Q neighborhood *μ* of *xt* such that *q* . But by (5), we have an (*r*, *s*)-fuzzy *b*-*Q*-neighborhood *μ* of *xt* such that

*qbC*,** \*(**, *r*, *s*)

Also,

*xt* § *bC*,** \*(*bC*,** \*(**, *r*, *s*), *r*, *s*).

Then

*bC*,** \*(*bC*,** \*(**, *r*, *s*), *r*, *s*)  *bC*,** \*(**, *r*, *s*).

*bC*,** \*(1  **, *r*, *s*)  1  **



Also,

1  *bI*,** \*(**, *r*, *s*)  1  **.

And then,

**  *bI*,** \*(**, *r*, *s*).

Conversely, let *μ* ≤ *h*, **(1  **)  *r* and ** \*(1  **)  *s*, *r* ∈ *I*0 and

*s* ∈ *I*1 such that **  *bI*,** \*(**, *r*, *s*). Now

1  *bI*,** \*(**, *r*, *s*)  1  **,

Thus

*bC*,** \*(1  **, *r*, *s*)  1  **.

That is, 1  ** is an (*r*, *s*)-gfbc set, then *h* is an (*r*, *s*)-gfbo set. Theorem 3.4. Let (*X*, *τ*, *τ*\*) be a dfts, *h* ∈ *IX*, *r* ∈ *I*0 and *s* ∈ *I*1. If *h*

is an (*r*, *s*)-gfbc set, then

But we have,

1. *bC*,** \*(**, *r*, *s*)  **

sets.

does not contain any non-zero (*r*, *s*)-fc

*bC*,** \*(*bC*,** \*(**, *r*, *s*), *r*, *s*)  *bC*,** \*(**, *r*, *s*).

Therefore

*bC*,** \*(*bC*,** \*(**, *r*, *s*), *r*, *s*)  *bC*,** \*(**, *r*, *s*).

(7) and (8) are obvious.

Theorem 3.2. Let (*X*, *τ*, *τ*\*) be a dfts. Then for each *h*, *μ* ∈ *IX*, *r* ∈ *I*0 and *s* ∈ *I*1, the operator *bI*,** \* satisfies the following statements:

1. *h* is an (*r*, *s*)-fbc iff *bC*,** \*(**, *r*, *s*)  ** is (*r*, *s*)-fc.
2. *μ* is (*r*, *s*)-gfbc set for each set *μ* ∈ *IX* such that

**  **  *bC*,** \*(**, *r*, *s*).

1. For each (*r*, *s*)-fo set *μ* ∈ *IX* such that *μ* ≤ *h*, *μ* is an (*r*, *s*)-gfbc relative to *h* if and only if *μ* is an (*r*, *s*)-gfbc in *IX*.
2. For each an (*r*, *s*)-fbo set *μ* ∈ *IX* such that *bC*,** \*(**, *r*, *s*)*q* iff

*q* .

Proof. (1) Suppose that **(1  **)  *r* and ** \*(1  **)  *s*, *r* ∈ *I*0 and *s* ∈ *I*1 such that **  *bC*,** \*(**, *r*, *s*)  ** whenever *h* ∈ *IX* is an (*r*, *s*)- gfbc set. Since 1  ** is an (*r*, *s*)-fo set,

**  (1  **)  *bC*,** \*(**, *r*, *s*)  (1  **)

 **  (1  *bC*,** \*(**, *r*, *s*))

 **  (1  *bC*,** \*(**, *r*, *s*))  (*bC*,** \*(**, *r*, *s*)  ** )

 0

and hence **  0 which is a contradiction. Then *bC*,** \*(**, *r*, *s*)  **

does not contain any non-zero (*r*, *s*)-fc sets.

* 1. Let *h* be an (*r*, *s*)-gfbc set. So, for each *r* ∈ *I*0 and *s* ∈ *I*1 if

*h* is an (*r*, *s*)-fbc set then,

*bC*,** \*(**, *r*, *s*)  **  0

which is an (*r*, *s*)-fc set.

Conversely, suppose that *bC*,** \*(**, *r*, *s*)  ** is an (*r*, *s*)-fc set. Then by (1), *bC*,** \*(**, *r*, *s*)  ** does not contain any non-zero an (*r*, *s*)-fc set. But *bC*,** \*(**, *r*, *s*)  ** is an (*r*, *s*)-fc set, then

*bC*,** \*(**, *r*, *s*)  **  0  **  *bC*,** \*(**, *r*, *s*).

So, *h* is an (*r*, *s*)-fbc set.

* 1. Suppose that *τ*(*α*) ≥ *r* and *τ*\*(*α*) ≤ *s* where *r* ∈ *I*0 and *s* ∈ *I*1

such that *μ* ≤ *α* and let *h* be an (*r*, *s*)-gfbc set such that *h* ≤ *α*.

Then

*bC*,** \*(**, *r*, *s*)  **.

So,

*bC*,** \*(**, *r*, *s*)  *bC*,** \*(**, *r*, *s*),

Therefore

*bC*,** \*(**, *r*, *s*)  **.

So, *μ* is an (*r*, *s*)-gfbc set.

* 1. Let *h* be an (*r*, *s*)-gfbc and *τ*(*h*) ≥ *r* and *τ*\*(*h*) ≤ *s*, where *r* ∈ *I*0

and *s* ∈ *I*1. Then *bC*,** \*(**, *r*, *s*)  ** . But, *μ* ≤ *h* so,

*bC*,** \*(**, *r*, *s*)  *bC*,** \*(**, *r*, *s*)  **.

Also, since *μ* is an (*r*, *s*)-gfbc relative to *h*, then

**  *bC*,** \*(**)(**, *r*, *s*)  *bC*,** \*(**, *r*, *s*),

so

*bC*,** \*(**, *r*, *s*)  *bC*,** \*(**)(**, *r*, *s*)  **.

*bC*,** \*(**, *r*, *s*)  **  *bC*,** \*(**)(**, *r*, *s*)  *bC*,** \*(**, *r*, *s*)  **  **  **  **.

That is, *μ* is an (*r*, *s*)-gfbc relative to *h*.

* 1. Suppose *μ* is an (*r*, *s*)-fbo and *q* , *r* ∈ *I*0 and *s* ∈ *I*1. Then

**  (1  **) . Since (1  **) is an (*r*, *s*)-fbc set of *IX* and *h* is an (*r*,

*s*)-gfbc set, then

*bC*,** \*(**, *r*, *s*)*q*.

Conversely, let *μ* be an (*r*, *s*)-fbc set of *IX* such that *h* ≤ *μ*, *r* ∈ *I*0

and *s* ∈ *I*1. Then

*q*(1  **).

But

*bC*,** \*(**, *r*, *s*)*q*(1  **)  *bC*,** \*(**, *r*, *s*)  **.

Hence *h* is an (*r*, *s*)-gfbc.

Proposition 3.1. Let (*X*, *τ*, *τ*\*) be a dfts, *h* ∈ *IX*, *r* ∈ *I*0 and *s* ∈ *I*1.

1. If *h* is an (*r*, *s*)-gfbc and an (*r*, *s*)-fbo set, then *h* is an (*r*, *s*)- fbc set.
2. If *h* is an (*r*, *s*)-fo and an (*r*, *s*)-gfbc, then *h* ∧ *μ* is an (*r*, *s*)-

gfbc set whenever **  *bC*,** \*(**, *r*, *s*).

Proof. (1) Suppose *h* is an (*r*, *s*)-gfbc and an (*r*, *s*)-fbo set such that *h* ≤ *h*, *r* ∈ *I*0 and *s* ∈ *I*1. Then

*bC*,** \*(**, *r*, *s*)  **.

But we have,

**  *bC*,** \*(**, *r*, *s*).

Then,

**  *bC*,** \*(**, *r*, *s*).

Therefore, *h* is an (*r*, *s*)-fbc set.

(2) Suppose that *h* is an (*r*, *s*)-fo and an (*r*, *s*)-gfbc set, *r* ∈ *I*0

and *s* ∈ *I*1. Then

*bC*,** \*(**, *r*, *s*)  **  ** is an (*r*, *s*)-fbc set

 **  ** is an (*r*, *s*)-fbc

 **  ** is an (*r*, *s*)-gfbc.

Now, if *μ* is an (*r*, *s*)-gfbc relative to *h* and *τ*(*α*) ≥ *r* and *τ*\*(*α*) ≤ *s* where *r* ∈ *I*0 and *s* ∈ *I*1 such that *μ* ≤ *α*, then for each an (*r*, *s*)-fo set *α* ∧ *h*, **  **  **  **  ** . Hence *μ* is an (*r*, *s*)-gfbc relative to *h*,

*bC*,** \*(**, *r*, *s*)  *bC*,** \*(**)(**, *r*, *s*)  (**  ** )  **.

Therefore, *μ* is an (*r*, *s*)-gfbc in *IX*.

Conversely, let *μ* be an (*r*, *s*)-gfbc set in *IX* and *τ*(*α*) ≥ *r* and *τ*\*(*α*) ≤ *s* whenever *α* ≤ *h* such that *μ* ≤ *α*, *r* ∈ *I*0 and *s* ∈ *I*1. Then for each an (*r*, *s*)-fo set *β* ∈ *IX*, *α* = *β* ∧ *h*. But we have, *μ* is an (*r*, *s*)-gfbc set in *IX* such that *μ* ≤ *β*,

# (*r*, *s*)-generalized ⋆-fuzzy *b*-closed sets

In this section, we introduce and study some properties of a new class of fuzzy sets called an (*r*, *s*)-generalized ⋆-fuzzy closed sets and an (*r*, *s*)-generalized ⋆-fuzzy *b*-closed sets

Definition 4.1. Let (*X*, *τ*, *τ*\*) be a dfts. For each *h* ∈ *IX*, *r* ∈ *I*0 and

*s* ∈ *I*1. A fuzzy set *h* is called:

* 1. An (*r*, *s*)-generalized ⋆-fuzzy closed (briefly, (*r*, *s*)-g ⋆fc) if *C*,** \*(**, *r*, *s*)  ** whenever *h* ≤ *μ* and *μ* is an (*r*, *s*)-gfo set in *IX*. *h* is called an (*r*, *s*)-generalized ⋆-fuzzy open (briefly, (*r*, *s*)-g

⋆fo) iff 1  ** is (*r*, *s*)-g ⋆fc set.

* 1. An (*r*, *s*)-generalized ⋆-fuzzy *b*-closed (briefly, (*r*, *s*)-g ⋆fbc) if *bC*,** \*(**, *r*, *s*)  ** whenever *h* ≤ *μ* and *μ* is an (*r*, *s*)-gfo set in *IX*. *h* is called an (*r*, *s*)-generalized ⋆-fuzzy *b*-open (briefly, (*r*, *s*)-g ⋆fbo) iff 1  ** is (*r*, *s*)-g ⋆fbc set.

Theorem 4.1. Let (*X*, *τ*, *τ*\*) be a dfts. *h* ∈ *IX* is an (*r*, *s*)-g ⋆fbo set if and only if **  *bI*,** \*(**, *r*, *s*) whenever *μ* is an (*r*, *s*)-gfc, *r* ∈ *I*0 and *s* ∈ *I*1.

Proof. Suppose that *h* is an (*r*, *s*)-g ⋆fbo set in *IX*, and let *μ* is an (*r*, *s*)-gfc set such that *μ* ≤ *h*, *r* ∈ *I*0 and *s* ∈ *I*1. So by the defi- nition, we have 1  ** is an (*r*, *s*)-gfo set in *IX* and 1  **  1  ** .

But 1  ** is an (*r*, *s*)-g ⋆fbc set, then *bC*,** \*(1  **, *r*, *s*)  1  ** .

But

*bC*,** \*(1  **, *r*, *s*)  1  *bI*,** \*(**, *r*, *s*)  1  **.

Therefore,

**  *bI*,** \*(**, *r*, *s*).

Conversely, suppose that **  *bI*,** \*(**, *r*, *s*) whenever *μ* ≤ *h* and

*μ* is an (*r*, *s*)-gfc set, *r* ∈ *I*0 and *s* ∈ *I*1. Now

1  *bI*,** \*(**, *r*, *s*)  1  **,

Thus

But *h* is an (*r*, *s*)-g ⋆fbc set and 1  ** is an (*r*, *s*)-gfo set, then

*bC*,** \*(**, *r*, *s*)  1  **  **  *bC*,** \*(**, *r*, *s*)  (1  *bC*,** \*(**, *r*, *s*))  0.

Therefore *bC*,** \*(**, *r*, *s*)  ** contains no non-zero (*r*, *s*)-gfc set.

* + 1. Let *h* be an (*r*, *s*)-g ⋆fbc set, *r* ∈ *I*0 and *s* ∈ *I*1. Then by (1) we have, *bC*,** \*(**, *r*, *s*)  ** contains no non-zero (*r*, *s*)-gfc set. So,

*bC*,** \*(**, *r*, *s*)  ** is an (*r*, *s*)-g ⋆fbo set.

* + 1. Let *h* be an (*r*, *s*)-g ⋆fbc set. If *h* is an (*r*, *s*)-fbc, *r* ∈ *I*0 and

*s* ∈ *I*1, then

*bC*,** \*(**, *r*, *s*)  **  0.

Conversely, let *bC*,** \*(**, *r*, *s*)  ** is an (*r*, *s*)-fbc set in *IX* and *h* is an (*r*, *s*)-g ⋆fbc, *r* ∈ *I*0 and *s* ∈ *I*1, then by (1) we have, *bC*,** \*(**, *r*, *s*)  ** contains no non-zero (*r*, *s*)-gfc set. Then,

*bC*,** \*(**, *r*, *s*)  **  0,

that is

*bC*,** \*(**, *r*, *s*)  **.

Hence *h* is an (*r*, *s*)-fbc set.

* + 1. Let *μ* be an (*r*, *s*)-gfc set and *bI*,** \*(**, *r*, *s*)  (1  ** )  **, *r* ∈ *I*0

and *s* ∈ *I*1. Hence

1  **  *bC*,** \*(1  **, *r*, *s*)  **  *bC*,** \*(1  **, *r*, *s*)  (1  ** ).

But (1  **) is an (*r*, *s*)-gfc and 1  ** is an (*r*, *s*)-g ⋆fbc by (1),

1.  **  0 and hence **  1.

Proposition 4.2. Let (*X*, *τ*, *τ*\*) be dfts’s. For each *h* and *μ* ∈ *IX*, *r* ∈ *I*0

and *s* ∈ *I*1.

*bC*,** \*(1  **, *r*, *s*)  1  **.

Therefore, 1  ** is an (*r*, *s*)-gfbc set and *h* is an (*r*, *s*)-gfbo set.

Proposition 4.1. Let (*X*, *τ*, *τ*\*) be dfts’s. For each *h* ∈ *IX*, *r* ∈ *I*0 and

*s* ∈ *I*1

1. If a fuzzy set *h* is an (*r*, *s*)-g ⋆fbc, then *bC*,** \*(**, *r*, *s*)  ** con- tains no non-zero (*r*, *s*)-gfc set.
2. If a fuzzy set *h* is an (*r*, *s*)-g ⋆fbc, then *bC*,** \*(**, *r*, *s*)  ** is an (*r*, *s*)-g ⋆fbo.
3. An (*r*, *s*)-g ⋆fbc set *h* is an (*r*, *s*)-fbc iff *bC*,** \*(**, *r*, *s*)  ** is an

(*r*, *s*)-fbc set.

1. If a fuzzy set *h* is an (*r*, *s*)-g ⋆fbc, then **  1, whenever *μ* is an (*r*, *s*)-gfo set and *bI*,** \*(**, *r*, *s*)  (1  ** )  **.

Proof. (1) Suppose that *h* is an (*r*, *s*)-g ⋆fbc set and *μ* is an (*r*, *s*)-gfc set of *IX*, *r* ∈ *I*0 and *s* ∈ *I*1 such that

**  *bCτ*,*τ* \*(**, *r*, *s*)

And

**  1  **.

If *h* and *μ* are (*r*, *s*)-g ⋆fbc, then *h*∧*μ* is an (*r*, *s*)-g ⋆fbc. 1.

2. If *h* is an (*r*, *s*)-g ⋆fbc and *τ*(*μ*) ≥ *r*, *τ*\*(*μ*) ≤ *s*, then *h* ∧ *μ* is an

(*r*, *s*)-g ⋆fbc.

Proof. (1) Suppose that *h* and *μ* are (*r*, *s*)-g ⋆fbc sets in *IX* such that *h* ∧ *μ* ≤ *ν* for each an (*r*, *s*)-gfo set *ν* ∈ *IX*, *r* ∈ *I*0 and *s* ∈ *I*1. Since *h* is an (*r*, *s*)-g ⋆fbc,

*bC*,** \*(**, *r*, *s*)  **

for each an (*r*, *s*)-gfo set *ν* ∈ *IX* and *h* ≤ *ν*. Also, *μ* is an (*r*, *s*)-g ⋆fbc,

*bC*,** \*(**, *r*, *s*)  **

for each an (*r*, *s*)-gfo set *ν* ∈ *IX* and *μ* ≤ *ν*. Then we have,

*bC*,** \*(**, *r*, *s*)  *bC*,** \*(**, *r*, *s*)  **,

whenever *h* ∧ *μ* ≤ *ν*, Therefore, *h* ∧ *μ* is an (*r*, *s*)-g ⋆fbc.

(2) Since every an (*r*, *s*)-fc set is an (*r*, *s*)-g ⋆fbc and from

1. we get the proof.

Proposition 4.3. Let (*X*, *τ*, *τ*\*) be dfts’s. For each *h* and *μ* ∈ *IX*, *r* ∈ *I*0

and *s* ∈ *I*1.

If *h* is both an (*r*, *s*)-gfo and an (*r*, *s*)-g ⋆fbc, then *h* is an1(.*r*,

*s*)-fbc set.

2. If *h* is an (*r*, *s*)-g ⋆fbc and **  **  *bC*,** \*(**, *r*, *s*), then *μ* is an (*r*, *s*)-g ⋆fbc.

Proof. (1) Suppose that *h* is an (*r*, *s*)-gfo and an (*r*, *s*)-g ⋆fbc in

*IX* such that *bC*,** \*(**, *r*, *s*)  ** , *r* ∈ *I*0 and *s* ∈ *I*1. But

**  *bC*,** \*(**, *r*, *s*).

Thus

(**  *bC*,** \*(**, *r*, *s*))  (1  *bC*,** \*(**, *r*, *s*))  **  (1  *bC*,** \*(**, *r*, *s*)).

 **  (1  *bC*,** \*(**, *r*, *s*))  **  (1  *bC*,** \*(**, *r*, *s*)).

Since *h* is an (*r*, *s*)-g ⋆fbc, then

*bC*,** \*(**, *r*, *s*)  **  (1  **).

Also,

Therefore

**  **  *bC* (**, *r*, *s*)  *bC*

(**, *r*, *s*).

**  *bC*,** \*(**, *r*, *s*).

Hence *h* is an (*r*, *s*)-fbc set.

** ,** \*

Thus

** ,** \*

1. Suppose that *h* is an (*r*, *s*)-g ⋆fbc and *ν* is an (*r*, *s*)-gfo set in *IX* such that *μ* ≤ *ν* for each *μ* ∈ *IX*, *r* ∈ *I*0 and *s* ∈ *I*1. So *h* ≤ *ν*. But we have, *h* is an (*r*, *s*)-g ⋆fbc, then

*bC*,** \*(**, *r*, *s*)  **.

*bC*,** \*(**, *r*, *s*)  *bC*,** \*(**, *r*, *s*)  **  (1  *bC*,** \*(**, *r*, *s*)).

Therefore *bC*,** \*(**, *r*, *s*)  ** , but *bC*,** \*(**, *r*, *s*) is not contained in (1  *bC*,** \*(**, *r*, *s*)). That is, *μ* is an (*r*, *s*)-g ⋆fbc relative to *X*.

Now

*bC*,** \*(**, *r*, *s*)  *bC*,** \*(*bC*,** \*(**, *r*, *s*), *r*, *s*)  *bC*,** \*(**, *r*, *s*)  **.

Therefore *μ* is an (*r*, *s*)-g ⋆fbc set.

Theorem 4.2. Let (*X*, **1, **1\* ) and (*Y*, **2, **2\* ) be dfts’s. If **  1*Y*  1*X*

such that *h* is an (*r*, *s*)-g ⋆fbc in *IX*, *r* ∈ *I*0 and *s* ∈ *I*1, then *h* is an (*r*, *s*)-g ⋆fbc relative to *Y*.

Proof. Suppose that (*X*, **1, **1\* ) and (*Y*, **2, **2\* ) are dfts’s such that

**  1*Y*  1*X* , *r* ∈ *I*0, *s* ∈ *I*1 and *h* is an (*r*, *s*)-g ⋆fbc in *IX*. Now, let

**  1*Y*  ** such that *μ* is an (*r*, *s*)-gfo set in *IX*. But we have, *h* is an (*r*, *s*)-g ⋆fbc in *IX*,

**  **  *bC*,** \*(**, *r*, *s*)  **.

So that

1*Y*  *bC*,** \*(**, *r*, *s*)  1*Y*  **.

Hence *h* is an (*r*, *s*)-g ⋆fbc relative to *Y*.

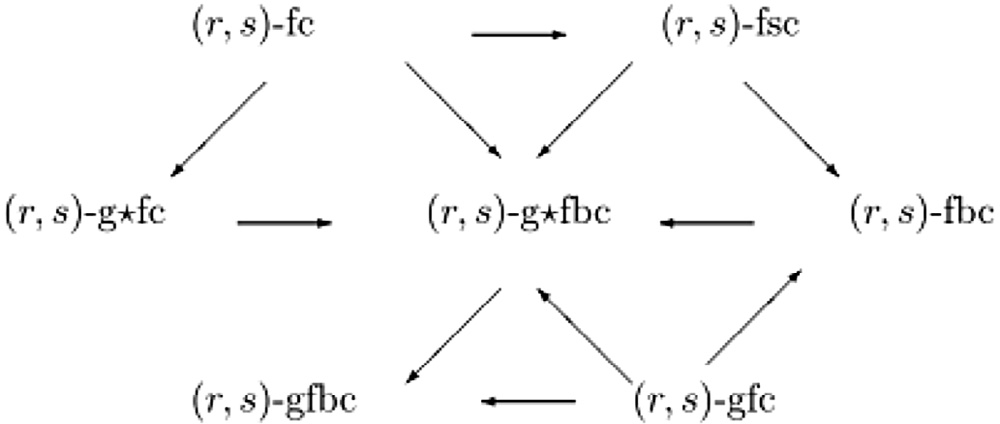
Theorem 4.3. Let (*X*, **1, **1\* ) be adfts. For each *h* and *μ* ∈ *IX*, *r* ∈ *I*0 and *s* ∈ *I*1 with *μ* ≤ *h*. If *μ* is an (*r*, *s*)-g ⋆fbc relative to *h* such that *h* is both an (*r*, *s*)-gfo and (*r*, *s*)-g ⋆fbc of *IX*, then *μ* is an (*r*,

*s*)-g ⋆fbc relative to *X*.

Proof. Suppose that *μ* is an (*r*, *s*)-g ⋆fbc and *τ*(*ν*) ≥ *r* and *τ*\*(*ν*) ≤ *s* such that *μ* ≤ *ν*, *r* ∈ *I*0, *s* ∈ *I*1. But we have, **  **  1, therefore *μ* ≤ *h* and *μ* ≤ *ν*. So

# Interrelations

The following implication illustrates the relationships between different fuzzy sets:



None of these implications is reversible where *A* → *B* represents *A* implies *B*, as shown by the following examples. But at this stage we do not have information re- garding the relationship between an (*r*, *s*)-gfbc and (*r*, *s*)-g ⋆fc sets.

Example 5.1. (1) Let *X*  {*a*, *b*, *c*} and let *μ* and *α* are fuzzy sets defined by:

**(*a*)  1.0, **(*b*)  0.5, **(*c*)  0.0,

**(*a*)  0.0, **(*b*)  0.4, **(*c*)  1.0.

Define (*τ*, *τ*\*) on *X* as follows:

**  **  **.

Also we have, *μ* is an (*r*, *s*)-g ⋆fbc relative to *h*,

1,

**(** )   1 ,

 2

0,

if ** {0, 1}, if **  **, otherwise.

0,

** \*(** )   1 ,

 2

1,

if ** {0, 1}, if **  **, otherwise.

**  *bC*,** \*(**, *r*, *s*)  **  **  **  *bC*,** \*(**, *r*, *s*)  **.

Then *α* is an  1 , 1-gfbc set, but not an  1 , 1-g ⋆fbc set.

2 2 2 2

1. Take *X* = {*a*, *b*} in (1) and define *μ*, *α* and *β* by: (8) Let *X* = {*a*, *b*} and let *μ* and *α* as fuzzy sets defined by:

**(*a*)  0.6,

**(*b*)  0.6,

**(*a*)  0.9,

**(*b*)  0.4,

**(*a*)  0.3,

**(*b*)  0.2,

**(*a*)  0.1,

**(*b*)  0.8.

**(*a*)  0.4,

**(*b*)  0.5.

Define (*τ*, *τ*\*) on *X* by:

Then *β* is an  1 , 1-g ⋆fbc set, but not an  1 , 1-fbc set.

1,

if ** {0, 1},

0,

if ** {0, 1},

2 2 2 2

**(** )   1 ,

if **  **,

** \*(** )   1 ,

if **  **,

1. Let *X*  {*a*, *b*, *c*}. Define *μ*, *ν* and *γ* by:

 2

0,

otherwise.

 2

1,

otherwise.

**(*a*)  1.0, **(*b*)  0.5, **(*c*)  0.3,

Then *μ* is an  1 , 1-g ⋆fc set, but not an  1 , 1-fc set.

**(*a*)  1.0, **(*b*)  0.6, **(*c*)  0.0,

1. 2 2 2

** (*a*)  0.0,

** (*b*)  0.6,

** (*c*)  0.0.

# Acknowledgments

Define (*τ*, *τ*\*) as in (1). Then *ν* is an  1 , 1-g ⋆fbc set but not an  1 , 1-fc set and not an  1 , 1-gfc. And *γ* is an  1 , 1-g ⋆fbc set, but not an  1 , 1-fsc set.

2

2

2

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2

2

1. Take (3) and defined *μ* and *ν* by:

**(*a*)  1.0, **(*b*)  1.0, **(*c*)  0.6,

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**(*a*)  0.3,

**(*b*)  0.5,

**(*c*)  0.5.

R E F E R E N C E S

Define (*τ*, *τ*\*) as in (1). Then *ν* is an  1 , 1-g ⋆fbc set, but not

[1] [Zadeh LA. Fuzzy sets. Inf Control 1965;8:338–53.](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0010)

2 2

an  1 , 1-g ⋆fc set.

2 2 [2] [Atanassov K. New operators defined over the](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0015)

1. See Example 3.1. Clearly *β* is an  1 , 1-gfbc set, but not an  1 , 1-gfc set.

2 2

[intuitionistic fuzzy sets. Fuzzy Set Syst 1993;61:](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0015) [131–42.](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0015)

2 2

1. Let *X* = {*a*, *b*}. Define *μ*, *ν* and *γ* as follows:
2. [Çoker D. An introduction to intuitionistic fuzzy topological](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0020) [spaces. Fuzzy Set Syst 1997;88:81–9.](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0020)

**(*a*)  0.7,

**(*a*)  0.3,

** (*a*)  0.4,

**(*b*)  0.6,

**(*b*)  0.2,

** (*b*)  0.5.

1. [Samanta SK, Mondal TK. On intuitionistic gradation of](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0025) [openness. Fuzzy Set Syst 2002;131:323–36.](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0025)
2. [Gutiérrez Garcia J, Rodabaugh SE. Order-theoretic,](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0030) [topological, categorical redundancies of interval-valued sets,](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0030) [grey sets, vague sets, interval-valued; intuitionistic sets,](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0030) [intuitionistic fuzzy sets and topologies. Fuzzy Set Syst](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0030) [2005;156:445–84.](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0030)
3. [Omari AA, Noorani MSM. On generalized *b*-closed sets. Bull](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0035)

Define (*τ*, *τ*\*) as in (1). Then *ν* is an  1 , 1-fbc set but not an

2 2

 1 , 1 -fsc set, also not an  1 , 1-gfc.

2 2 2 2

1. Let *X*  {*a*, *b*, *c*} and let *μ* and *α* as fuzzy sets defined by:

**(*a*)  0.9, **(*b*)  0.8, **(*c*)  0.3,

**(*a*)  0.1, **(*b*)  0.8, **(*c*)  0.3.

Define (*τ*, *τ*\*) on *X* by:

[Malays Math Sci Soc 2009;32:19–30.](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0035)

1. [Mohammed FM, Noorani MSM, Ghareeb A. Several types of](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0040) [totally continuous functions in double fuzzy topological](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0040) [spaces. Adv Fuzzy Syst 2014;2014:Article ID 361398, 9](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0040) [pages.](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0040)
2. [Lee EP, Im YB. Mated fuzzy topological spaces. J Fuzzy Log](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0045) [Int Sys 2001;11:161–5.](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0045)
3. [Kim YC, Abbas SE. Several types of fuzzy regular spaces.](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0050) [Indian J Pure Appl Math 2004;35:481–500.](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0050)
4. [Abbas SE, El-Sanousy E. Several types of double fuzzy](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0055) [semiclosed sets. J Fuzzy Math 2012;20:89–102.](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0055)
5. [Mohammed FM, Noorani MSM, Ghareeb A. Generalized *ψρ*-](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0060)

1, if ** {0, 1},



**(** )  0.6, if **  **,

0, otherwise.

0, if ** {0, 1},

** \*(** )  0.3, if **  **,

1, otherwise.



[operations on double fuzzy topological spaces. AIP Conf Proc](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0060) [2013;1571:943–8.](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0060)

1. [Mohammed FM, Noorani MSM, Ghareeb A. Generalized](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0065) [*ψρ*-closed sets and generalized *ψρ*-open sets in double](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0065) [fuzzy topological spaces. AIP Conf Proc 2014;1602:](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0065)

Then *α* is an (0.6, 0.3)-fsc set, but not an (0.6, 0.3)-fc set.

[909–17.](http://refhub.elsevier.com/S2314-808X(15)00063-9/sr0065)