Available online at [www.sciencedirect.com](http://www.sciencedirect.com/)

[Electronic Notes in Theoretical Computer Science 325 (2016) 85–110](http://dx.doi.org/10.1016/j.entcs.2016.09.033)

[www.elsevier.com/locate/entcs](http://www.elsevier.com/locate/entcs)

Giry and the Machine

Fredrik Dahlqvist[1](#_bookmark0) Vincent Danos[2](#_bookmark0) Ilias Garnier[3](#_bookmark0)

*University College London Ecole Normale Sup´erieure University of Edinburgh*

**Abstract**

We present a general method - the Machine - to analyse and characterise in finitary terms natural transfor- mations between well-known functors in the category **Pol** of Polish spaces. The method relies on a detailed analysis of the structure of **Pol** and a small set of categorical conditions on the domain and codomain functors. We apply the Machine to transformations from the Giry and positive measures functors to com- binations of the Vietoris, multiset, Giry and positive measures functors. The multiset functor is shown to be defined in **Pol** and its properties established. We also show that for some combinations of these func- tors, there cannot exist more than one natural transformation between the functors, in particular the Giry monad has no natural transformations to itself apart from the identity. Finally we show how the Dirichlet and Poisson processes can be constructed with the Machine.

*Keywords:* probability, topology, category theory, monads

# Introduction

Classical tools of probability theory are not geared towards compositionality, and especially not compositional approximation (Kozen, [[13](#_bookmark45)]). This has not prevented authors from developing powerful techniques (Chaput et al. [[5](#_bookmark36)], Kozen et al. [[14](#_bookmark46)]) based on structural approaches to probability theory (Giry, [[9](#_bookmark41)]). Here, we adopt a slightly different standpoint: we propose to tackle this tooling problem *globally*, by combining structural insights of **Pol** together with some classical tools of proba- bility theory and topology put in functorial form. The outcome is the Machine, an axiomatic reconstruction in category-theoretic terms of developments carried out in [[7](#_bookmark39)]. Thus, we get a simpler and more conceptual proof of our previous results. We also obtain a much more comprehensive picture and prove that natural transforma- tions between Giry-like functors are entirely characterised by their components on

This work was sponsored by the European Research Council (ERC) under the grant RULE (320823).

1 [f.dahlqvist@ucl.ac.uk](mailto:f.dahlqvist@ucl.ac.uk)

2 [vincent.danos@ens.fr](mailto:vincent.danos@ens.fr)

3 [igarnier@inf.ed.ac.uk](mailto:igarnier@inf.ed.ac.uk)

<http://dx.doi.org/10.1016/j.entcs.2016.09.033>

1571-0661/© 2016 The Author(s). Published by Elsevier B.V.

This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

finite spaces. For instance, the monadic data of the Giry functor are easily obtained from the finite case (which is completely elementary) and applying the Machine. But the construction is not limited to probability functors: we deal similarly with the multiset and the Vietoris (the topological powerdomain of compact subsets) func- tors. This allows one to consider transformations mixing probabilistic and ordinary non-determinism, in a way which is reminiscent of (Keimel et al., [[12](#_bookmark44)]). Another byproduct of our Machine is that we reconstruct from finitary data classical objects of probability theory and statistics, namely the Poisson and Dirichlet processes. It is worth noting that Poisson, Dirichlet (and many other similar constructions ob- tained by recombining the basic ingredients differently) are obtained as natural and continuous maps: naturality expresses the stability of the “behaviour” in a change of granularity, and as such is a fundamental property of consistency, but continuity (which to our knowledge is proved here for the first time) expresses a no less impor- tant property, namely the robustness of the behaviour in changes in “parameters”. This has potential implications in Bayesian learning.

The structure of the paper is as follows. In Sec. [3](#_bookmark2), we show that **Pol** is stratified into the subcategories **Pol***f* , **Pol***cz* **Pol***z* of finite, compact zero-dimensional and zero-dimensional Polish spaces respectively and show how these subcategories are related. In Sec. [4](#_bookmark9), the Machine is introduced: we identify a small set of categorical conditions on functors *F, G* that guarantee that any natural transformation from *F* to *G* in **Pol***f* can be extended step-by-step through the subcategories to a nat- ural transformation on **Pol**. In Sec. [5](#_bookmark20), we illustrate the workings of the Machine on natural transformations connecting the Giry and positive measure functors to combinations of the Vietoris, multiset, Giry and positive measure functors. As far as we know, the multiset functor is defined in **Pol** for the first time and its prop- erties are established. As a first application of the Machine, we develop in Sec. [6](#_bookmark23) general criteria under which there can exist at most one natural transformation from a functor *F* to the Giry functor. In particular, we show that there exists at most one natural transformation between the Vietoris, multiset, positive measure and Giry functor to the Giry functor. Lastly, we show in Sec. [7](#_bookmark29) how transformations of the type M+ *⇒* G*H* where M+ is the finite measure functor and *H* is either the multiset or the finite measure functor can be built in **Pol***f* from a single generating morphism M+(1) *→* G*H*(1) and give criteria for this transformation to be natural. In particular, we show that the Dirichlet and Poisson distributions satisfy these criteria and use the Machine to build Dirichlet and Poisson processes.

# Notations

Most of our developments take place in the category **Pol** of Polish spaces and continuous maps. **Pol** is a full subcategory of the category **Top** of topological spaces and continuous maps. **Pol** has all countable limits and all countable coproducts (Bourbaki [[4](#_bookmark37)], IX). The functor mapping any space to the measurable space having the same underlying set and the Borel *σ*-algebra and interpreting continuous maps as measurable ones will be denoted by *B* : **Pol** *→* **Meas**, where **Meas** is the category

of measurable spaces and measurable maps. A *countable codirected diagram* (*ccd* for short) is given by a countable directed partial order *I* and a contravariant functor *D* : *Iop →* **Pol** such that for all *i ≤Iop j*, *D*(*i ≤Iop j*) is surjective. We moreover assume that *ccd* s range over non-empty spaces. With that assumption, the categorical limit of a *ccd D*, which we denote by lim *D*, is always non-empty.

# The structure of Pol

**Pol** can be decomposed according to the following diagram of inclusions:

**Pol** ,˛*Icz* /**P**¸**ol** ,˛*Iz* /**P**¸**ol** ,˛*Ip* /**P**¸**ol** (1)

*f cz z*

Here, **Pol***f* is the full subcategory of finite (hence discrete) spaces, **Pol***cz* is the full subcategory of compact zero-dimensional spaces and **Pol***z* is the full subcategory of zero-dimensional spaces while *Icz, Iz* and *Ip* are the obvious inclusion functors. To this picture, we add categories of *based spaces* and *base-preserving maps*.

**Definition 3.1 (Categories of based spaces)** A *based space* is a pair (*X, F*) of *X ∈ Obj*(**Pol**) and of a countable base *F* of the topology of *X*. A *base-preserving map* from (*X, F*) to (*Y, G*) is a function *f* : *X → Y* such that *f−*1(*G*) *⊆F* (it is there- fore continuous). One easily checks that this defines a category having based spaces as objects and base-preserving maps as morphisms. We denote this category by **Pol***b*. Similarly, a *based zero-dimensional space* is a pair (*Z, F*) where *Z ∈ Obj*(**Pol***z*) and *F* is a *countable base of clopen sets which is also a boolean algebra*. We denote by **Pol***b* the category of based zero-dimensional spaces and base-preserving maps.

*z*

Of course, there exists for each such based category a (faithful, but not full!) forgetful functor, that we will denote by resp. *Uz* and *Up*. The situation is summed up in the following commutative diagram in **Cat**:

˛ *Ib*

**Pol**

*b z*

*I*

,˛ *Icz* /¸ ¸,

*b* ,

*z*

*p* /**P**¸**ol***b*

**Pol***f*

,¸

*Iz*

**Pol***cz*

*Uz Up*

z˛ ,˛ /

**Pol***z Ip* **P**¸**ol**

In the remainder of this section, we will unravel further relationships between these categories.

**Pol***f* **is a codense subcategory of Pol***cz***.** Objects of **Pol***f* are finite discrete spaces. Note that every subset of a discrete space is clopen; as a consequence, any map between two finite spaces is continuous. We will denote objects of **Pol***f* by their cardinality *m, n*. The objects of **Pol***cz* are the compact zero-dimensional (or *proﬁnite*) spaces, a prime example being the Cantor space 2N. These spaces are homeomorphic to limits of countable codirected diagrams (*ccd* s for short) taking values in **Pol***f* . This is exactly captured by the concept of codensity (see [[15](#_bookmark47)], X.6).

**Proposition 3.2 Pol***f is codense in* **Pol***cz.*

**Proof.** Let *X* be a compact zero-dimensional space, and consider the comma cat- egory *X ↓ Icz*. We denote by *DX* : (*X ↓ Icz*) *→* **Pol***f* the diagram correspond- ing to the base of this cone. It is enough to prove that for all *X ∈ Obj*(**Pol***cz*), *X ∼*= lim *DX* . Following (Mac Lane [[15](#_bookmark47)], IX.3), it is in turn enough to exhibit a di- agram *D* : *Iop →* **Pol***f* verifying *X ∼*= lim *D* and a cofinal (“initial” in [[15](#_bookmark47)]) functor *c* : *Iop →* (*X ↓ Icz*). Proposition 3.1 of [[7](#_bookmark39)] yields the existence of such a diagram *D* where *I* is the set of finite partitions of *X* taken in the boolean algebra of clopen sets of *X* (that we denote by *Clo*(*X*)), partially ordered by partition refinement and directed by partition intersection. Observe that any continuous map *f* : *X → n* induces a finite clopen partition of *X* by considering its fibres. Let us denote this partition by *X/f* . Let *c* be the functor mapping any finite partition *n ∈ Iop* seen as an object of **Pol***f* to the quotient map *qn* : *X* → *n*, and any refinement *m ≤£op n* to to the obvious map *πmn* such that *qm* = *πmn ◦ qn*. For any *f* : *X → n* the partition *X/f* is mapped to *c*(*X/f* ) : *X → X/f* , and there trivially exists a map *π* : *c*(*X/f* ) *→ f* . For any two *f, fj ∈ Obj*(*X ↓ Icz*), one can easily exhibit a partition *i ∈I* of *X* such that there exists *π* : *c*(*i*) *→ f* and *πj* : *c*(*i*) *→ fj*. *2*

**Pol***cz* **is a reflective subcategory of Pol***b* **.** Objects of **Pol***z* are *zero-dimensional* spaces, i.e. spaces whose topology admits a (countable) base of clopen sets. Discrete spaces (such as N) are always zero-dimensional. A less trivial example is the Baire space NN. The bridge between **Pol***cz* and **Pol***z* is provided by *compactiﬁying* zero- dimensional spaces, as explained in full length in ([[7](#_bookmark39)], Sec. 3). Let us recall the underpinnings of this compactification. Let *Z* be some zero-dimensional space and *F* be a countable base of clopens of *Z*. One easily verifies that the boolean algebra generated by *F*, that we denote by *Bool*(*F*), still generates the same topology and is still countable. Therefore, one can witout loss of generality assume that the base *F* of *Z* is a countable Boolean algebra of clopen sets. Let *I7* be the directed partial order of finite partitions of *Z* taken in *F* and let *D7* : *Iop →* **Pol***f* be the diagram

*z*

*op* *7*

defined by *D7* (*i ∈ I7* ) *i* on objects (seeing finite partitions of *Z* as finite discrete spaces) and *D7* (*j ≤ op i*)= *qij* where *qij* : *j → i* is the obvious quotient map.

*£*

*F*

**Proposition 3.3 (Wallman compactification ([**[**7**](#_bookmark39)**], Prop. 3.12))** lim *D7 is a zero-dimensional compactiﬁcation of Z that we denote by ω7* (*Z*)*. We denote by η7* : *Z ‹→ ω7* (*Z*) *the canonical embedding of Z into its compactiﬁcation.*

Note that this compactification is not universal, in the sense that **Pol***cz* is not a reflective subcategory of **Pol***z* (see [[15](#_bookmark47)], IV.3 for a definition of reflective subcat- egory). However, we will show that **Pol***cz is* a reflective subcategory of **Pol***b* . In the following, recall that *Clo*(*X*) is the boolean algebra of clopen sets of a compact zero-dimensional space *X*.

*z*

**Proposition 3.4** *Let Ib be the operation that maps any compact zero-dimensional space X to the pair* (*X, Clo*(*X*)) *and which acts identically on maps between such spaces. Ib is a full and faithful functor from* **Pol***cz to* **Pol***b .*

*z*

*z z*

**Proof.** For any space *X ∈ Obj*(**Pol***cz*), its boolean algebra of clopen sets *Clo*(*X*) is countable and therefore, (*X, Clo*(*X*)) is a based zero-dimensional space. By con- tinuity, maps between such spaces are base-preserving. Functoriality, fullness and faithfulness are trivial. *2*

Our compactification naturally lives in **Pol***b* :

*z*

**Proposition 3.5** *For any* (*Z, F*) *∈ Obj*(**Pol***b* )*, the embedding η7* : (*Z, F*) *→*

*z*

*Ib*(*ω7* (*Z*)) *is base preserving.*

*z*

**Proof.** By construction of *ω7* (*Z*), any finite clopen partition of this space will induce through *η7* a finite partition of *Z* taken in *F*. Therefore, *η7* is base preserv- ing. *2*

The following proposition states the functoriality of compactification in this new setting, and the fact that **Pol***cz* is a reflective subcategory of **Pol***b* .

*z*

**Proposition 3.6 (***ω* **as a reflector)** *(i) Let f* : (*Z, F*) *→* (*Zj, Fj*) *be a base- preserving map. There exists a unique ω77′* (*f* ) : *ω7* (*Z*) *→ ω7′* (*Zj*) *such that*

*ω77′* (*f* ) *◦ η7* = *η7′ ◦ f. (ii) ω* : **Pol***b*

*z*

*→* **Pol***cz is a functor deﬁned on ob-*

*jects by ω*(*Z, F*) *ω7* (*Z*) *and on base-preserving maps f* : (*Z, F*) *→* (*Zj, Fj*) *by*

*ω*(*f* ) *ω77′* (*f* )*, and it is left adjoint to the inclusion functor Ib*

*z*

*(the unit being*

*given by η).*

**Proof.** (i) This is Prop. 3.13 and Corollary 3.14 of [[7](#_bookmark39)]. Let us sketch the argument. As *f* is base-preserving, any finite clopen partition of *Zj* taken in *Fj* will induce a unique finite clopen partition of *Z* taken in *F*. Using the notations of Prop. [3.3](#_bookmark4), we deduce that *D7′* is a sub-diagram of *D7* . Therefore, there exists a unique mediating map (that we denote *ω77′* (*f* )) from lim *D7* to lim *D7′* , i.e. from *ω7* (*Z*) to *ω7′* (*Zj*), such that *η7′ ◦f* = *ω77′* (*f* ) *◦η7* . (ii) *ω* trivially preserves identities. For all *f, fj*, the equality *W* (*fj ◦ f* )= *W* (*fj*) *◦ W* (*f* ) is a consequence of the uniqueness of factorisations in (i). According to (Mac Lane [[15](#_bookmark47)], IV.3), left adjointness of *ω* is a direct consequence of (i), as any map *f* : (*Z, F*) *→ Ib*(*X*) will factor uniquely through *η7* : (*Z, F*) *→ Ib*(*ω7* (*Z*)). *2*

*z*

*z*

This reflection is summarised in the following diagram:

*Id b*

**Pol***z*

*y*

*b*

*z c*

/¸

**Pol***cz*

z

vz

/

**P**¸**ol***b*

(2)

## Pol

*b z*

*z*

*I*

**Pol***b* **is a coreflective subcategory of Pol***b***.** The penultimate step in our struc-

*z*

tural analysis of **Pol** is to relate **Pol***b* and **Pol***b*. This is accomplished by associat-

*z*

ing zero-dimensional refinements to arbitrary spaces, in an operation called *zero- dimensionalisation*. Let us define this operation.

**Proposition 3.7 (Zero-dimensionalisation ([**[**7**](#_bookmark39)**], Prop. 3.2))** *Let X be a space with underlying set U* (*X*) *and let F be a countable base of X. The topological space z7* (*X*) (*U* (*X*)*, ⟨Bool*(*F*)*⟩*) *having as underlying set U* (*X*) *and whose topology is generated by the boolean algebra Bool*(*F*) *veriﬁes the following properties:*

* 1. *z7* (*X*) *is Polish;*
  2. *z7* (*X*) *is zero-dimensional.*
  3. *measurable sets are preserved: B*(*X*)= *B*(*z7* (*X*))*.*

In a similar fashion to compactifications, this operation is better typed as a functor from **Pol***b* to **Pol***b* . Let us make zero-dimensionalisation into a functor:

*z*

**Proposition 3.8** *Let f* : (*X, F*) *→* (*Y, G*) *be a base-preserving map in* **Pol***b. Then*

*f* : (*z7* (*X*)*, Bool*(*F*)) *→* (*zG* (*Y* )*, Bool*(*G*)) *is base-preserving in* **Pol***b . We denote by*

*z*

*z* : **Pol***b →* **Pol***b*

*z*

*the functor deﬁned by z*(*X, F*)= (*z7* (*X*)*, Bool*(*F*)) *on objects and*

*acting identically on arrows.*

**Proof.** It is sufficient to consider the case of an arbitrary finite union of literals

*L* = *Aє*1 *∪ ... ∪ Aєn ∈ Bool*(*G*), where *Ai ∈ G* and *Aєi* denotes either *Ac*

or *Ai*.

1 *n i* *i*

We have *f—*1(*L*) = *∪n f—*1(*Ai*)*єi* , since *f* is base-preserving in **Pol***b* we deduce

*i*=1

that *f—*1(*L*) *∈ Bool*(*F*). Continuity of *f* in **Pol***b*

*z*

is a direct consequence of base

preservation. The fact that *z* is a functor is now trivial. *2*

The following result now follows easily:

**Proposition 3.9 (***z* **as a coreflector)** *z is right adjoint to the inclusion functor*

*Ib, i.e.* **Pol***b is a coreflective subcategory of* **Pol***b.*

*p z*

**Proof.** Observe that for all (*X, F*) *∈ Obj*(**Pol***b*), the identity function *ϵ7* *id* : *Ibz*(*X, F*) *→* (*X, F*) is base-preserving. This indeed constitutes the counit of the coreflection: one easily verifies that for all *f* : *Ib*(*Z, F*) *→* (*X, G*) there exists a

*p*

*p*

unique *fj* : *Ib*(*Z, F*) *→ Ibz*(*X, G*) such that *f* = *ϵG ◦ fj* (and *fj* is equal to *f* as a

*p p*

function). *2*

This coreflection is summarised in the following diagram:

*Id***Pol***b*

,,

*є*

zz

(3)

**Pol***b z* /**P**¸**ol***b*  /**P**¸**ol***b*

*z b*

*I*

*p*

**Relating Pol***b* **and Pol.** For all space *X ∈ Obj*(**Pol**), let us denote the set of countable bases of *X*, partially ordered by inclusion, by *Bases*(*X*). Observe that *Bases*(*X*) is directed by taking the union of the bases and closing under finite intersections. Accordingly, if *F ⊆ G* are two countable bases of *X*, the identity function *id* : (*X, G*) *→* (*Y, F*) is trivially base-preserving. This defines a codirected diagram *BX* : *Bases*(*X*)*op →* **Pol***b* mapping any base *F* to (*X, F*) and any pair

*F ⊆ G* to the identity function. Recall that *Up* : **Pol***b →* **Pol** is the base-forgetting functor. The next definition and proposition provide a universal characterisation of Polish spaces in terms of their zero-dimensionalisation.

**Definition 3.10 (Diagram of zero-dimensionals)** We define the *diagram of zero-dimensionals of X*:

*ZX* *UpIbzBX* : *Bases*(*X*)*op →* **Pol**

*p*

that maps bases *F ∈ Bases*(*X*) to *ZX* (*F*) *z7* (*X*).

We state without proof the following result, which is a category-theoretic refor- mulation of ([[7](#_bookmark39)], Theorem 3.5):

**Proposition 3.11** *For all space X ∈ Obj*(**Pol**)*, X ∼*= colim *ZX.*

In more concrete terms, any space *X* has the final topology for the family of iden- tity functions *{id* : *z7* (*X*) *→ X}7* where *F* ranges over *Bases*(*X*). Let us conclude this section by summarising our structural decomposition of **Pol** in the following diagram:

**Pol** ,˛*Icz* /**P**¸**ol**

*c*

**Pol***b*

*⊥*

*z*

*T* **Pol***b*

(4)

*f* , *cz*¸

,

,

*b z*

*I*

*z* ¸ ¸

*Uz b Up*

*I*

*p*

,˛ /

*Iz* ˛**P** **ol***z Ip*

**P**¸**ol**

# The Machine

We will leverage the structural decomposition of **Pol** given in the previous section to characterise some “profinite” natural transformations, in the sense that their behaviour on arbitrary spaces is entirely determined by their behaviour on finite spaces. We proceed in a stepwise and modular fashion: the Machine is presented as a series of extension theorems giving sufficient conditions for a natural transformation to be uniquely extended from a subcategory to the ambient one (Theorems [4.2](#_bookmark10)-[4.11](#_bookmark18)). These results are combined in Theorem [4.12](#_bookmark19).

1. **From Pol***f* **to Pol***cz***.** One can completely characterise the subcategory of the functor category [**Pol***cz*; **Pol**] consisting of functors commuting with certain codi- rected limits in terms of [**Pol***f* ; **Pol**]. These functors are defined below.

**Definition 4.1 (Pol***f* **-continuous functors)** A functor *F* : **Pol** *→* **Pol** is **Pol***f - continuous* if for all *ccd D* : *Iop →* **Pol***f* , *F* (lim *D*) *∼*= lim *FD*.

The key result is the following:

**Theorem 4.2** *Let F, G* : **Pol***cz ⇒* **Pol** *be two functors. If G is* **Pol***f -continuous, then Nat*(*F|***Pol***f , G|***Pol***f* ) *∼*= *Nat*(*F, G*)*.*

This isomorphism arises from the existence of a functor computing right Kan extension along *Icz* (see [[15](#_bookmark47)], X), denoted by Ran*Icz* in the following:

**Proposition 4.3** *The functor* Ran*Icz* : [**Pol***f* ; **Pol**] *→* [**Pol***cz*; **Pol**] *is full and faith- ful.*

**Proof.** In the following, for any *X ∈ Obj*(**Pol***cz*), *DX* : (*X ↓ Icz*) *→* **Pol***f* stands for the diagram verifying *X ∼*= lim *DX* (see proof of Prop. [3.2](#_bookmark3)). We first prove that any functor *F* : **Pol***f →* **Pol** admits a right Kan extension Ran*Icz F* along *Icz*. Following (Mac Lane [[15](#_bookmark47)], X.3, Corollary 4) it is sufficient to prove that for all *X ∈ Obj*(**Pol***cz*), the diagram *F ◦ DX* : (*X ↓ Icz*) *→* **Pol** has a limit. By a cofinality argument similar to that used in the proof of Prop. [3.2](#_bookmark3), one can show that lim *F ◦ DX ∼*= lim *F ◦ D* for a countable diagram *D* and since **Pol** is countably complete this limit exists, therefore *F* admits a right Kan extension. Let us prove that the extension is full and faithful. Since *Icz* is full and faithful, the universal arrow *ϵF* : (Ran*Icz F* )*Icz ⇒ F* is an iso. Given *F, G* : **Pol***f →* **Pol***cz* and *α* : *F ⇒ G*, there exists a unique *σ* : Ran*Icz F ⇒* Ran*Icz G* such that *α ◦ ϵF* : (Ran*Icz F* )*Icz → G* factors as *α ◦ ϵF* = *ϵG ◦ σIcz*. Therefore, Ran*Icz* defines a functor from [**Pol***f* ; **Pol**] to [**Pol***cz*; **Pol**] which is full and faithful by the bijection *Nat*(Ran*Icz F,* Ran*Icz G*) *∼*= *Nat*(*F, G*). *2*

**Proof.** [Theorem [4.2](#_bookmark10)] Prop. [4.3](#_bookmark11) and the universal property of Ran yields an isomorphism *Nat*(*F|***Pol***f , G|***Pol***f* ) *∼*= *Nat*(*F,* Ran*Icz G|***Pol***f* ). Recall that Ran*Icz G|***Pol***f* (*X*) = lim *G ◦ DX ∼*= lim *G ◦ D* where *DX* and *D* are as in the proof of Prop. [4.3](#_bookmark11). By **Pol***f* -continuity of *G*, Ran*Icz G|***Pol***f* (*X*) *∼*= *G*(lim *D*)= *G*(*X*). *2*

1. **From Pol***cz* **to Pol***b* **.** As seen in Prop. [3.6](#_bookmark5), the Wallman compactification makes **Pol***cz* into a reflective subcategory of **Pol***b* . The extension of a natural transforma- tion from **Pol***cz* to **Pol***b* can be framed componentwise as a *restriction* of the natural transformation to a space embedded into its compactification, that we construct us- ing intersections.

*z*

*z*

*z*

**Definition 4.4 (Intersections, preservation of intersections)** If *j*1 : *X ‹→ Z, j*2 : *Y ‹→ Z* are two embeddings, we define the intersection *X ∩ Y → Z* as the pullback of *j*1 and *j*2 (Eq. [5](#_bookmark12)). We say that an endofunctor *G* : **Pol** *→* **Pol** preserves intersections if the diagram in Eq. [6](#_bookmark13) is an intersection.

*p*1 /¸

*G*(*X ∩*

*G*(*p*1) /*G*¸(*X*)

(6)

*X ∩ Y*

*p*2

,˛

*X*˛z

*j*1

(5)

*G*(*p*2)

*Y* )

˛

˛z

*G*(*j*1)

*Y* /*Z*¸

*j*2

*G*(*Y* ),

*G*(*j*2)

/*G*¸(*Z*)

The following Lemma characterises the topology of intersections in **Pol**.

**Lemma 4.5** *X ∩ Y is the* **Set***-theoretic intersection of X, Y together with the sub- space topology induced by Z.*

Recall that if *f* : *X → Y* is a morphism in a category C, its *cokernel pair* (if it exists) is the pushout of *f* with itself (Mac Lane [[15](#_bookmark47)], III.3). In **Top**, there is a well-known characterisation of embeddings as limits of their coker- nel pair (see e.g. (Adamek et al. [[1](#_bookmark34)], 7.56-7.58)). In **Pol**, we have the following:

*X*,˛ *f* /*Y* ¸

˛z ˛z

**Proposition 4.6** *Let X, Y be Polish and f* : *X ‹→ Y be an f j*1

*embedding. Then (i) the pushout object Y* +*X Y is Polish, (ii)*

*the cokernel arrows j*1*, j*2 : *Y → Y* +*X Y are embeddings and*

*(iii) the intersection of j*1 *and j*2 *is homeomorphic to X.*

*Y* /*Y* ¸+ *Y*

,˛

*X*

*j*2

The following Lemma ensures that the pushout object of an embedding with range in **Pol***cz* is still compact zero-dimensional.

**Lemma 4.7** *Let f* : *X ‹→ Y be an embedding in* **Pol** *such that Y ∈ Obj*(**Pol***cz*)*. Then Y* +*X Y ∈ Obj*(**Pol***cz*)*.*

**Proof.** The proof that *Y* +*X Y* is Polish is routine. It thus remains to see that it is compact and zero-dimensional. Since finite unions of compacts are compact, the coproduct *Y* + *Y* is compact. By universality of coproducts, the cokernel maps *j*1*, j*2 : *Y → Y* +*X Y* define a unique continuous map *j*1+*j*2 : *Y* +*Y → Y* +*X Y* , which is easily seen to be surjective, and it follows that *Y* +*X Y* is the continuous image of a compact, i.e. is compact. To see that it is zero-dimensional, we use the fact that on compact Hausdorff spaces zero-dimensionality coincides with being totally disconnected. Let *x ∈ Y* +*X Y* and let *Ux* be a subset such that *x ∈ Ux*. We can assume w.l.o.g. that *x* is in the first copy of *Y* and that *Ux* is included in this copy. Since *Y* is totally disconnected, if *Ux /*= *{x}* it can be written as the union of two disjoint opens *V*1*, V*2 in the subspace topology induced by *Y* and and thus also by *Y* +*X Y* . It follows that if *Ux /*= *{x}* it cannot be connected in *Y* +*X Y* . *2*

**Theorem 4.8** *Let F, G* : **Pol***b →* **Pol** *be a pair of functors such that G preserves embeddings and intersections. Then Nat*(*F, G*) *∼*= *Nat*(*F|***Pol***cz , G|***Pol***cz* )*.*

*z*

**Proof.** In the interest of readability, we will elude the inclusion *Ib* : **Pol***cz →* **Pol***b* .

*z z*

Let *α* : *F|***Pol***cz ⇒ G|***Pol***cz* be a natural transformation. We prove that (i) for all

*X ∈ Obj*(**Pol***b* ), *αc*(*X*) : *F* (*ω*(*X*)) *→ G*(*ω*(*X*)) restricts uniquely to a morphism *αX* : *F* (*X*) *→ G*(*X*) such that *αc*(*X*) *◦* (*Fη*)*X* = (*Gη*)*X ◦αX* , and (ii) this restriction uniquely extends *α* to a natural transformation from *F* to *G*.

*z*

1. Consider, given *X ∈ Obj*(**Pol***b* ), the embedding *ηX* : *X ‹→ ω*(*X*). By Prop. [4.6](#_bookmark14), *X* is the intersection of the cokernel maps *j*1*, j*2 : *ω*(*X*) *‹→ ω*(*X*) +*X ω*(*X*). Moreover by Lemma [4.7](#_bookmark15), there exists a component *αcX*+*XcX* . By functoriality and naturality of *η*, the diagram in Fig. [1](#_bookmark17) (ignoring *αX* ) commutes. Since *G* preserves embeddings and intersections, there exists a unique mediating map *αX* : *F* (*X*) *→ G*(*X*) making the whole diagram commute.

*z*

1. Finally, we need to check that extending *α* to *F|***Pol***b*

*z*

*→ G|***Pol***b*

in this way

is natural. Let *f* : *X → Y* in **Pol***b*

*z*

and let *ηX, ηY* denote the embeddings of *X*

and *Y* in their respective zero-dimensional compactifications. The corresponding diagram is depicted in Fig. [2](#_bookmark17). The top, bottom, front, back and right-hand square commute, and it follows that (*Gη*)*Y ◦ G*(*f* ) *◦ αX* = (*Gη*)*Y ◦ αY ◦ F* (*f* ). Since *ηY* is an embedding and since *G* preserves embeddings, (*GηY* ) is an embedding and in particular is injective, and it follows that i.e. *G*(*f* ) *◦ αX* = *αY ◦ F* (*f* ) as desired.

*z*

*F* (*ω*(*X*)) *F* (*j*2) /*F*¸(*ω*(*X*)+

*ω*(*X*))

*F* (*Y* ) (*Fy*)*Y* /*F*¸(*ω*(*Y* ))

(*Fy*)*X*

*F* (*X*) *α*

,

(*Fy*)*X* /*F*¸(*ω*(*X*))

*F* (*j*1)

*X*

*F* (*X*)

,

(*Fy*)*X*/*F*¸(*ω*(*X*))

*F* (*f* )

,

*F* (*c*(*f* ))

*αc*(*Y* )

*c*(*X*)

*α*

*t*(*j* )

2

*c*(*X*)+*Xc*(*X*)

*αY*

˛(*ty*)

,

*αX*

*αX*

(*ty*)

*G*( *ω*(,*X*))

*αc*(*X*)

/*G*¸(*ω*(*X* ) +*X ω*(*X*))

*G*(*Y* ),

*t*(*f* )

*Y* /*G*¸(*ω*(*Y* ))

,

*X*

(*ty*)*X*

*t*(*j*1)

*αc*(*X*)

˛ (*ty*)*X*

*t*(*c*(*f* ))

*G*(*X*)

/*G*¸(*ω*(*X*))

*G*(*X*),

/*G*¸(*ω*(*X*))

Fig. 1. Fig. 2.

## From Pol*b*

*z*

*2*

**to Pol.** The last part of the Machine is a procedure to extend

natural transformations from **Pol***b*

*z*

to **Pol**. We have seen in Prop. [3.11](#_bookmark8) that Polish

spaces are the colimits of their “diagrams of zero-dimensionals”. We will require functors in the domain of natural transformations to commute with these colimits.

**Definition 4.9 (***Z***-cocontinuous functors)** A functor *F* : **Pol** *→* **Pol** is *Z- cocontinuous* if for all *X ∈ Obj*(**Pol**), *F* (*X*) *∼*= colim *F ZX* where *ZX* is defined in Def. [3.10](#_bookmark7).

Moreover, we will require these functors to be *Z-stable*, which means that the underlying sets of the spaces in the range of the considered functors are invariant by zero-dimensionalisation. As we will prove later, this is for instance the case of the Giry, multiset and list functors.

**Definition 4.10 (***Z***-stable functor)** A functor *F* : **Pol** *→* **Pol** is *Z-stable* if

*UFX* = *UFZX* (*F*) for all *F ∈ Bases*(*X*).

**Theorem 4.11** *Let F, G* : **Pol** *→* **Pol** *be a pair of functors such that F is Z- cocontinuous and Z-stable. Then Nat*(*F, G*) *∼*= *Nat*(*FUpIb, GUpIb*)*.*

*p p*

**Proof.** Let *α* : *FUpIb ⇒ GUpIb* and *X ∈ Obj*(**Pol**) be given. By *Z*-cocontinuity,

*p p*

*F* (*X*) is the colimiting object of the diagram *F ZX* = *FUpIbzBX* : *Bases*(*X*)*op →*

*p*

**Pol** (Def. [3.10](#_bookmark7)). Applying *α*, we get a natural transformation *αzBX* : *FZX ⇒*

*GZX* . Composing with the counit *ϵ* : *Ibz → Id*

*b* yields a natural transformation

*p* **Pol**

(*GUpϵ*)(*αzBX* ) : *FZX ⇒ GUpId***Pol***b BX* . Note that *GUpId***Pol***b BX* is equal to the

constant functor with value *G*(*X*). Therefore, we have constructed a cocone from

*FZX* to *G*(*X*). The situation above is summed up in the following diagram:

**P** **ol***b*  *Up* /**P**¸**ol**

*F*

*Ib*

*p*

*Bases*(*X*)*op*  *BX* /**P**¸**ol***b*  *z* /**P**¸**ol***b*

*z*

*є*

t

*Id***Pol***b*

*Ib*

*p*

*α*

z

v˛

*b*  /¸

v˛

**Po** **l**

*t*

˛**P** **ol**

*Up* **Pol**

By universality, there exists a unique map *uX* : *F* (*X*) *→ G*(*X*) such that *uX ◦* (*FUpϵBX* )*7* = (*GUpϵBX* )*7 ◦* (*αzBX* )*7* . Let us prove naturality of *{uX}X∈Obj*(**Pol**). For all *f* : *X → Y* and for all base *G* of *Y* , there exists a base *F* of *X* such that *f* : (*X, F*) *→* (*Y, G*), is base-preserving, and by functoriality, so is *z*(*f* ): *ZX* (*F*) *→ ZY* (*G*). We get the following diagram:

*FZX*

(*F*) (*αzBX* )*F* /*G*¸*Z*

*uX*

(*FUpєBX* )*F*

z

(*tUpєBX* )*F*

/¸

*X*

(*F*)

*Fz*(*f* )

*FX*

*F* (*f* )

*GX*

*t*(*f* )

/

*tz*(*f* )

¸*F Y* *uY*

*G*¸*Y* ,¸

/¸

*FZY* (*G*) (*αzBY* )*G*

*GZY* (*G*)

In the above diagram, the left and right cells commute by naturality of *ϵ* while the top and bottom cells commute by construction of the arrows *uX, uY* . Note that the arrow (*FUpϵBX* )*7* is the image through *F* of the identity function *ϵ7* = *id* : *ZX* (*F*) *→ X*. Since *F* is *Z*-stable, this arrow is surjective. We conclude that the central square commute, and we extend *α* by setting for all *X αX* = *uX* as constructed above. *2*

1. **The Machine.** Bringing the parts of the Machine together, we obtain:

**Theorem 4.12** *Let F, G* : **Pol** *→* **Pol** *be a pair of functors such that:*

* 1. *F is Z-cocontinuous and Z-stable,*
  2. *G is* **Pol***f -continuous, preserves embeddings and intersections. Then one has Nat*(*F, G*) *∼*= *Nat*(*F|***Pol***f , G|***Pol***f* )*.*

# Feeding the Machine

We now investigate the properties of some functors, with an eye on applying the Machine.

**The Giry functor.** For any space *X*, we denote by G(*X*) the space of Borel probability measures over *X*, endowed with the weak topology (Giry, [[9](#_bookmark41)]). This operation can be extended to a functor G : **Pol** *→* **Pol** which admits the Giry monad structure (*G, δ, μ*) (Giry, [[9](#_bookmark41)]). The action of G on maps *f* : *X → Y* is defined by G(*f* )(*P* ) *P ◦ f—*1. The unit is given by the Dirac delta: *δX* : *X → G*(*X*) while the multiplication is defined by averaging: *μX* : *G*2(*X*) *→ G*(*X*) *P '→ t*(*X*) *p dP* (*p*).

∫

G is a rather well-behaved functor:

**Proposition 5.1** *(i) For all* ccd *D,* G(lim *D*) *∼*= lim G *◦D; (ii)* G *is Z-cocontinuous and Z-stable; (iii)* G *preserves injections and embeddings; (iv)* G *preserves intersec- tions.*

**Proof.** (i) is the Bochner extension theorem in functorial form ([[7](#_bookmark39)], Theorem 2.5).

(ii) *Z*-cocontinuity is in ([[7](#_bookmark39)], Theorem 3.7); *Z*-stability stems from Prop. [3.7](#_bookmark6), (iii). For (iii), see e.g. ([[7](#_bookmark39)], Lemma 2.1). Now for (iv): let *j*1*, j*2 : *A, B* > *X* be two embeddings, let *p*1 : *A ∩ B → A* and *p*2 : *A ∩ B* > *B* be the corresponding embeddings and consider *μ ∈* G(*A*)*,ν ∈* G(*B*) such that G(*j*1)(*μ*) = G(*j*2)(*ν*). It follows from (Kechris [[11](#_bookmark43)], Theorem 15.1) and the fact that *p*1 is injective that whenever *U* is a Borel set of *A ∩ B*, *p*1[*U* ] is a Borel set of *A*, and similarly for *p*2. We can therefore define *λ ∈* G(*A ∩ B*) by *λ*(*U* )= *μ*(*p*1[*U* ]) = *ν*(*p*2[*U* ]). To see that the equality on the right holds, note that since *j*1 in injective *p*1[*U* ]= *j—*1(*j*1[*p*1[*U* ]]),

1

and thus

*μ*(*p*1[*U* ]) = *μ*(*j—*1(*j*1[*p*1[*U* ]])) = G(*j*1)(*μ*)(*j*1[*p*1[*U* ]]) = G(*j*2)(*ν*)(*j*1[*p*1[*U* ]])

1

= G(*j*2)(*ν*)(*j*2[*p*2[*U* ]]) = *ν*(*p*2[*U* ])

This assignment from pairs (*μ, ν*) such that G*j*1(*μ*) = G*j*2(*ν*) to *λ ∈* G(*A ∩ B*) is clearly injective, and it follows that G(*A∩ B*) *∼*= G*A∩* G*B* as sets. Since G preserves embeddings, G(*j*1 *◦ p*1)= G(*j*2 *◦ p*2) is an embedding, and it follows that G(*A ∩ B*) and G*A ∩* G*B* are in fact homeomorphic. *2*

**Example 5.2** Theorem [4.12](#_bookmark19) implies that the unit *δ* : Id *→* G of the Giry monad is entirely determined by its finite components. We do not yet know whether G2 is *Z*-cocontinuous, and thus whether the multiplication *μ* : G2 *→* G is determined by its finite components. However it follows from Theorem [4.8](#_bookmark16) that the restriction of *μ* to **Pol***z is* determined by its finite components. We conjecture that this result extends to the entire category **Pol**.

**The non-zero finite measures functors.** We will also consider functors closely related to G: we let M+ be the functor mapping any space *X* to the space of non- zero positive finite measures over *X* with the weak topology, and acting on maps similarly as G. The following is trivial (consider the normalisation of a finite non-zero measure):

**Proposition 5.3** *For all space X, we have the isomorphism* M+(*X*) *∼*= G(*X*)*×*R*>*0*.*

As a consequence, M+ verifies all the properties listed in Prop. [5.1](#_bookmark21). Note that

for all finite space *n*, M+(*n*) is also homeomorphic to R*n*

*≥*0

*\ {***0***}*.

**The multiset functor.** We consider the multiset functor B : **Pol** *→* **Pol**. It is given explicitly by

B(*X*) *Xn/Sn*

*n∈*N

where *Xn/Sn* is the quotient of *Xn* under the obvious action of *Sn* – the permuta- tion group on *n* elements – on tuples together with the quotient topology, i.e. the final topology for the quotient map *q* : *Xn* → *Xn/Sn*. See Appendix [A](#_bookmark48) for a proof that B(*X*) is Polish. Its action on maps is given by setting for any *f* : *X → Y*

and *μ ∈* B(*X*), B(*f* )(*μ*)= *y '→* Σ*x∈f−*1(*y*) *μ*(*x*). This is easily shown to be continu-

ous. Observe also that for *X* finite, B(*X*) *∼*= N*X* . The multiset functor verifies the following properties:

**Proposition 5.4** *(i)* B *is* **Pol***f -continuous; (ii)* B *preserves injections and embed- dings; (iii)* B *preserves intersections.*

**Proof.** See Appendix [B](#_bookmark50). *2*

**The Vietoris functor.** As a non-probabilistic example, we will consider the Vi- etoris functor. We recall its definition.

**Definition 5.5** We denote by V : **Pol** *→* **Pol** the functor mapping any space *X* to the space of compact subsets of *X* topologised with the Hausdorff distance, and mapping any continuous function *f* : *X → Y* to V(*f* ) *K ∈* V(*X*) *'→ f* (*K*).

See (Kechris [[11](#_bookmark43)], 4.F) for a proof that V(*X*) is indeed Polish. V has the following properties:

**Proposition 5.6** *(i)* V *is* **Pol***f -continuous; (ii)* V *preserves injections and embed- dings; (iii)* V *preserves intersections.*

**Proof.** (i) is in Appendix [B](#_bookmark50). (ii) and (iii) are in Appendix [B](#_bookmark50). *2*

**Example 5.7** An interesting example of transformation which is not natural in **Pol**, due to Michael Mislove, is provided by the *support* of a measure. Usually, the support of *p ∈* G(*X*) is defined to be the smallest closed subset of measure 1. On finite spaces, for *p ∈* G(*n*), we define supp*n*(*p*) *{x ∈ n | p*(*x*) *>* 0*}*. Let us check that this is natural: for *f* : *m → n*, we have that supp(*G*(*f* )(*p*)) = supp(*p ◦ f—*1)= *x ∈ n | f—*1(*x*) *∩* supp(*p*) */*= *∅* , i.e. supp(*G*(*f* )(*p*)) = *f* (supp(*p*)) = V(*f* )(supp(*p*)). However, this does not define a natural transformation in **Pol**: consider the sequence of measures (*pn*)*n∈*N on *X* = *{*0*,* 1*}* defined by *pn* = *n—*1 *δ*0+ 1 *δ*1. *pn* weakly converges

}

*n n*

to *δ*0 as *n → ∞* and for all *n*, supp(*pn*) = *{*0*,* 1*}* but supp(*δ*0) = *{*0*}*. Therefore,

supp is not continuous!

# Rigidity

The results presented in Sec. [4](#_bookmark9) allow to construct natural transformations from finitary specifications. In this section, we apply these results to exhibit striking *rigidity* properties of G and related functors.

**Definition 6.1** A pair of functors *F, G* : C *→* D is called *rigid*, if there exists at most one natural transformation *η* : *F ⇒ G*. In particular, we will say that an functor *F* : C *→* D is **rigid** if the identity natural transformation *id* : *F ⇒ F* is the only natural transformation that exists from *F* to itself.

For each finite space *k* and functor *T* : **Pol** *→* **Pol**, there exists a canonical action of *Sk*, the permutation group over *k* elements, given by:

*α* : *Sk × T* (*k*) *→ T* (*k*)*,* (*π, x*) *'→ Tπ*(*x*)

We will call this action the *canonical action*. We will call an element *x ∈ T* (*k*) stabilised by the entire group *Sk* under the canonical action an *isotropic element*. Isotropic elements will play a crucial role in our theorem.

**Theorem 6.2 (Rigidity Theorem)** *Let H* : **Pol** *→* **Pol** *be a subfunctor of the Giry monad* G *satisfying the following conditions: (i) H*(*k*)= G(*k*) *for every ﬁnite Polish space k; (ii) H is* **Pol***f -continuous;(iii) H preserves injections. Let also T* : **Pol** *→* **Pol** *be a functor such that (iv) for each ﬁnite Polish space k there exists a dense subset Qk ⊆ T* (*k*) *with the property that if x ∈ Qk there exists a ﬁnite Polish space kj, a morphism f* : *kj → k and an isotropic element xj ∈ T* (*kj*) *such that T* (*f* )(*xj*)= *x. In these circumstances the pair* (*T, H*) *is rigid.*

We prove this theorem in steps. But let us first show some example of functors satisfying the property above.

**Example 6.3** Let us show that the Vietoris functor V satisfies the condition (iv). Note first that for every *k*, the full set *k ∈* V(*k*) is isotropic: for any *π ∈ Sk α*(*π, k*) = V*π*(*k*) = *k* since *π* is bijective. Now take *Qk* = V(*k*) (which is trivially dense) and *x* = *{x*1*,..., xn} ∈* V(*k*). Consider the full set *n ∈* V(*n*) along with the map *f* : *n → k, i '→ xi*, it is clear that V(*f* (*n*)) = *x*, and *n* is isotropic.

**Example 6.4** The Giry monad G satisfies all conditions of Theorem [6.2](#_bookmark24): it satisfies

(i) trivially, it satisfies (ii) and (iii) by Prop. [5.1](#_bookmark21). Let us show that it satisfies (iv) as well. Note first that the uniform probabilities are the isotropic elements: if 1 *,...,* 1

denotes the uniform distribution on *k* elements, then

*k*

*k*

*α* *π,* 1 *,...,* 1 = G(*π*) 1 *,...,* 1 = 1 *,...,* 1 *◦ π—*1 = 1 *,...,* 1

Consider now *Qk* = Δ*k ∩* Q*k*, the rational probabilities on *k* elements. It is clearly

*k*

*k*

*k*

*k*

*k*

*k*

*k*

*k*

dense in G(*k*). Any *x ∈ Qk*, can without loss of generality be written as *p*1 *,..., pm*

*n n*

for a common denominator *n*. Now consider the projection map defined by

⎧1 if 1 *≤ i ≤ p*1

⎪

*p* : *n → k, i '→* ⎨2 if *p*1 +1 *≤ i ≤ p*1 + *p*2

*...*

Σ

⎪⎩*k* if

It is easy to check from this definition that *p*1 *,..., pm* = G(*p*) 1 *,...,* 1 , where

*k*

Σ*k—*1 *pi* +1 *≤ i ≤*

*i*=1 *pi*

*i*=1

1 *,...,* 1 is isotropic.

*n*

*n*

*n n n n*

**Example 6.5** Let M+ : **Pol** *→* **Pol** be the finite non-zero positive measure functor. It follows easily from Prop. [5.3](#_bookmark22) that this functor satisfies condition (iv): the isotropic elements are those of the shape ((1*/k,...,* 1*/k*)*, λ*) for *λ ∈* R*>*0. A dense subset is provided by (Q*k ∩* G(*k*)) *×* R*>*0 and the same argument as in Example [6.4](#_bookmark25) shows that every element ((*p*1*/n,..., pk/n*)*, λ*) is the image of ((1*/n,...,* 1*/n*)*, λ*) by G(*p*) *× id* with *p* defined as in Example [6.4](#_bookmark25).

**Example 6.6** The multiset functor B also has the property (iv). B(*k*) has one isotropic element: the unordered list [(1*,..., k*)], and any [(*x*1*,..., xk*)] *∈* B(*k*) is the image of [(1*,..., k*)] under B(*f* ) for the map *f* : *k → k, i '→ xi* (which might very well not be injective).

Let us proceed to the proof of Theorem [6.2](#_bookmark24). The following settles the finite case:

**Lemma 6.7** *Let* (*T, H*) *be a pair of functors satisfying the conditions of Theorem* [*6.2*](#_bookmark24)*, then* (*T, H*) *is rigid on* **Pol***f .*

**Proof.** Let *ν* : *T ⇒ H* be a natural transformation. We first show that if *x ∈ T* (*k*) is isotropic then

*ν* (*x*)= 1 *,...,* 1 (7)

*k*

*k*

*k*

where 1 *,...,* 1 denotes the uniform probability distribution on *k*. Fix *i ∈*

*k*

*k*

*{*1*,..., k}*, and consider the permutations (*ij*) *∈ Sk,* 1 *≤ j ≤ k* sending *i* to *j*, *j* to *i* and leaving all other elements of *k* unchanged. We have

*νk*(*x*)(*i*)= *νk*(*T* (*ij*)(*x*))(*i*) (*x* isotropic)

= *H*(*ij*)(*νk*(*x*))(*i*) (By naturality)

= G(*ij*)(*νk*(*x*))(*i*) (*H* = G on **Pol***f* )

= *νk*(*x*)(*ij*)*—*1(*i*) (By def. of G)

= *νk*(*x*)(*j*) (By def. of (*ij*))

Since this holds for every 1 *≤ j ≤ k* we have Σ*k νk*(*x*)(*j*) = Σ*k νk*(*x*)(*i*) =

*j*=1

*j*=1

*kνk*(*x*)(*i*) = 1 and thus *νk*(*x*)(*i*) = 1 for every 1 *≤ i ≤ k*, i.e. *νk*(*x*) = 1 *,...,* 1 .

*k*

*k*

*k*

Let us now consider an arbitrary *x ∈ Qk*, by assumption there exist *f* : *kj → k* and an isotropic element *xj ∈ T* (*kj*) such that *T* (*f* )(*xj*)= *x*. It follows that

*νk*(*x*)= *νk*(*T* (*f* )(*xj*)) (By assumption on *T* )

= *H*(*f* )(*νk*(*xj*)) (By naturality)

= G(*f* )(*νk*(*xj*)) (*H* = G on **Pol***f* )

= G(*f* ) 1 *,...,*  1 (*xj* is isotropic and ([7](#_bookmark27)))

*kj*

*kj*

Clearly, the same reasoning applies to any other natural transformation *ρ* : *T ⇒ H*. We have thus shown that for each finite Polish set *k*, *νk* is unique on a dense subset *Qk* of *T* (*k*). Since *νk* is a morphism in **Pol** it is continuous, and since Polish spaces are complete, it is in fact Cauchy-continuous. It follows that the restriction of *νk* to *Qk* has a unique extension to *T* (*k*). Since the restriction of *νk* to *Qk* is unique, it follows that *νk* is also unique. *2*

Note that the entire group *Sk* was necessary to show Lemma [6.7](#_bookmark26), i.e. a weaker notion of isotropic element would not be sufficient.

**Lemma 6.8** *Let* (*T, H*) *be a pair of functors satisfying the conditions of Theorem* [*6.2*](#_bookmark24)*, then* (*T, H*) *is rigid on* **Pol***cz.*

**Proof.** Assume *ν* : *T|***Pol***f ⇒ H|***Pol***f* is given. By Lemma [6.7](#_bookmark26), *ν* is unique. Since *H*

is **Pol***f* -continuous, Theorem [4.2](#_bookmark10) applies and the proof is complete. *2*

**Lemma 6.9** *Let* (*T, H*) *be a pair of functors satisfying the conditions of Theorem* [*6.2*](#_bookmark24)*, then* (*T, H*) *is rigid on* **Pol***b .*

*z*

**Proof.** It is enough to reuse the uniqueness part of the proof of Theorem [4.8](#_bookmark16). *2*

We can finally prove Theorem [6.2](#_bookmark24).

**Proof.** (**Theorem** [6.2](#_bookmark24)) Let *α* : *T|***Pol***b ⇒ H|***Pol***b*

be given. By Lemma [6.9](#_bookmark28), *α* is the

*z z*

unique such transformation. Let *β, βj* : *T ⇒ H* be given, extending *α*. For all *X* and

*F ∈ Bases*(*X*), the identity function *id* : *z7* (*X*) *→ X* is continuous. By the rigidity

assumption, *βzF*

*j*

*zF* (*X*)

(*X*) = *β*

. Using this equation and naturality,

*βX ◦ T* (*id*)= *H*(*id*) *◦ βzF*

(*X*) = *H*(*id*) *◦ βj*

= *βj*

* *T* (*id*)

Therefore *β* = *βj*. *2*

*zF* (*X*)

*X*

**Example 6.10** We have shown earlier that G satisfies all the conditions of Theorem

[6.2](#_bookmark24). It follows that there can only exist a single natural transformation G *⇒* G, and since the identity transformation is natural, it follows that G is rigid.

**Example 6.11** Let M+ : **Pol** *→* **Pol** be the finite positive measure functor. We can check that the following transformation is natural: define *ν* : M+ *→* G at a Polish

space *X* by *νX* (*Q*) *A '→ Q*(*A*)

*Q*(*X*)

for *A* a Borel set of *X*. This is well defined since

0 *< Q*(*X*) *< ∞*. It is also natural: if *f* : *X → Y* is a map in **Pol**, then for each *Q*

in M+(*X*) and Borel set *B* of *Y* we have:

*—*1 *Q*(*f—*1(*B*))

*Q*(*f—*1(*B*))

G(*f* )(*νX* (*Q*))(*B*)= *νX* (*Q*)(*f*

(*B*)) =

*Q*(*X*) = *Q*(*f—*1(*Y* ))

= *νY* (M+(*f* )(*Q*))(*B*)

Since M+ satisfies (iv), it follows from Theorem [6.2](#_bookmark24), that the normalisation trans- formation *ν* we have just defined is the only natural transformation M+ *⇒* G.

# Applications

In previous work [[7](#_bookmark39)], we showed that a cornerstone of nonparametric Bayesian statis- tics, the Dirichlet process [[8](#_bookmark40),[10](#_bookmark42)], is in fact a natural transformation from M+ to G2. This result hinged on a non-axiomatic version of the Machine of Sec. [4](#_bookmark9). In order to validate our new developments we first give a short construction of the Dirichlet process in axiomatic form. The value of our general framework is then illustrated by constructing the Poisson process as a natural transformation. At the heart of these constructions are families of distributions which are stable by convolution

(mistakenly taken to be infinitely divisible in [[7](#_bookmark39)]). Common examples include: the Γ distribution, the Gaussian distribution, the Poisson distribution, etc. What exam- ples such as Dirichlet or Poisson processes have in common is that they can all be represented by natural transformations of the shape M+ *⇒* G*H* where the functor *H* can be either B or M+. Since M+ is *Z*-cocontinuous, since G and *H* are **Pol***f* - continuous, preserve injections, embeddings and intersections (see Appendix [B](#_bookmark50)) we can define a natural transformation of this type by restricting ourselves to **Pol***f* and running the Machine.

In the cases which we have mentioned above, the natural transformation in **Pol***f* can in fact be defined by a single map! The fundamental property which makes this possible is that both M+ and B turn coproducts into products. When this is the case it is sometimes possible to define *φ* : M+ *⇒* G*H* on **Pol***f* from a map *φ*1 : M+(1) *→* G*H*(1). For this we need a fundamental result which holds very generally in the category **Meas** of measurable spaces and measurable maps. We define the *product measure* natural transformation between the bifunctors *π* : G *− ×*G*− →* G(*−× −*) at each pair of measurable spaces ((*X,* Σ*X* )*,* (*Y,* Σ*Y* )) by *π*(*X,Y* )(*p, q*) *'→ p × q* where *p × q* is the product measure defined on the product *σ*-algebra (Σ*X ⊗* Σ*Y* ).

**Theorem 7.1** *The transformation π* : G *− ×*G*−→* G(*−× −*) *is natural in both its arguments.*

Let us now fix a continuous map *φ*1 : M+(1) *→* G*H*(1). For any *n* in **Pol***f* we use

the fact that *n* = *n* 1 and the fact that M+ and *H* turn coproducts into products

*i*=1

to define *φn* : M+(*n*) *→* G*H*(*n*) by

*φn*

*φn*

N*n*

z˛

M+(*n*) *∼*= M+(1)*n*  1 /(G¸*H*(1))*n*  *H*1 /G(¸*H*1)*n ∼*= G*H*(*n*)

where *n* is the *n*-fold measure product at *H*(1). The maps *φn* define the com-

*H*(1)

ponent of a transformation M+ *⇒* G*H*. But when is it natural? A simple criterion is given in the following result.

**Theorem 7.2** *A transformation φ* : M+ *→* G*H built as above is natural in* **Pol***f* iff

*the following diagrams commute:*

M+(2) *φ*2 /G*H*¸(2) M+(*n*) *φn* /G*H*¸(*n*)

M+(*e*)

G*H*(*e*)

M+(*ij*)

G*H*(*ij*)

M+(1)

/G*H*¸(1)

*φ*1

(8)

M+(*n*)

/G*H*¸(*n*)

*φn*

(9)

M+(1) *φ*1 /G*H*¸(1) M+(1) *φ*2 /G*H*¸(2)

M+(*i*1)

G*H*(*i*1)

M+(*i*2)

G*H*(*i*2)

M+(2)

/G*H*¸(2)

*φ*2

(10)

M+(2)

/G*H*¸(2)

*φ*1

(11)

*where e* :2 *→* 1 *is the obvious unique epimorphism,* (*ij*): *n → n is any permutation of two elements of n, and i*1*, i*2 :1 *→* 2 *are the two injections of* 1 *into* 2=1 + 1*.*

**Proof.** Any map *f* : *m → n* between finite sets can be written as a permutation *π* : *n → n* followed by a monotone surjection *q* : *n* → *k* followed by a monotone injection *i* : *k* > *n*. Since every permutation of *n* can be written asa composition of permutation of two elements, repeated usage of Diagram ([9](#_bookmark32)) shows that G*Hπ◦φn* = *φn ◦* M+*π*. Monotone surjections *q* : *m* → *n* can be written as a composition of maps of the shape

*id*1 + *id*1 + *...* + *e* + *id*1 + *...* + *id* : *k → k −* 1

For notational clarity let us consider the case *e* + *id*1 :3 → 2. The following square commutes:

N

*φ*2*×φ*1

M+(3) *∼*= M+(2) *×* M+(1) /G*H*¸(2) *×* G*H*(1) /G(¸*H*(2) *× H*(1)) *∼*= G*H*(3)

M+(*e*)*×id*1

G*H*(*e*)*×id*1

1 1

G(*H*(*e*)*×id*1)

M+(2) *∼*= M+(1) *×* M+(1)

*φ ×φ* /G*H*¸(1) *×* G*H*(1) N /G(¸*H*(1) *× H*(1)) *∼*= G*H*(2)

Indeed, the right-hand side square commutes by Theorem [7.1](#_bookmark30), whilst the left-hand side square commutes by assumption that Diagram [8](#_bookmark32) commutes. Monotone injec- tions are treated in a similar way. *2*

We will call a family of probability distributions *φn* : M+(*n*) *→* G*H*(*n*) *additive* if ([8](#_bookmark32)) holds, *exchangeable* if ([9](#_bookmark32)) holds, and say that it *admits zero parameters* if ([10](#_bookmark33)) and ([11](#_bookmark33)) hold.

The Γ distribution Γ1 : M+(1) *→ G*M+(1) maps any parameter *λ ∈* M+(1) to a

probability with density *x '→ xλ−*1*e−x*

Γ(*λ*)

w.r.t. Lebesgue [[3](#_bookmark35)]. The family of probability

distributions Γ*n* generated by Γ1 is clearly exchangeable; it is also additive [[7](#_bookmark39)] and one can easily adapt the definition so that it admits zero parameters. It follows from Theorem [7.2](#_bookmark31) that Γ*n* : M+(*n*) *→* GM+(*n*) is a natural transformation on **Pol***f* which extends to **Pol**. The Dirichlet process is then simply defined as *D* : M+ *⇒ G*2

(G*ν*)Γ, where *ν* : M+ *⇒* G is the normalisation natural transformation (unique, by rigidity!).

Similarly, if we define Π : M+(1) *→* GB(1) *∼*= G(N) by Π (*λ*)(*k*)= *λke−λ* , then

1 1 *k*!

it is well-known that the family Π*n* generated by Π1 (similarly to the previous case)

is additive. It is also clearly exchangeable. Finally to allow for zero parameters, we extend Π1 : M*≥*0(1) *→* G(N) by putting Π1(0) = *δ*0, the Dirac delta at 0. It is clear that for any test function *f* : N *→* R

*k*Σ=0

*f* (*k*)

*λke—λ*

= *f* (0)*e—λ*

*k*!

+

*k*=1

Σ

*f* (*k*)

*λke—λ k*!

*→*

*λ* 0

*−→ f* (0) =

*f* (*k*)*δ*0

*k*

Σ

i.e. our extension is continuous for the weak topology. This fact is the exact analogue of Proposition 4.2 in [[7](#_bookmark39)]. The family Π*n* : M*≥*0(*n*) *→* GN*n* thus defines a natural transformation in **Pol***f* by Theorem [7.2](#_bookmark31), and by applying the Machine we produce

a natural transformation on **Pol**. The processes Π*X* : M+

*≥*0

(*X*) *→* GB(*X*) (for *X*

in **Pol**) defined by this natural transformation are very well-known in probability theory, they are the (inhomogeneous) *Poisson point processes* on *X* parameterised by a measure on *X*.

# Outlook

Our results allow the compositional and finitary approximation of a class of param- eterised “stochastic” processes seen as natural transformations between probability- like functors satisfying some general axioms. It is worth noting that all the conditions on endofunctors that we require for the codomain of natural transformatins are pre- served by composition (if we strengthen **Pol***f* -continuity to commutation with all limits of *ccd* s). Indeed, we are confident that compositionality can be pushed fur- ther: following coalgebraic practice, we will investigate whether functors in e.g. the polynomial closure of Giry can be fed to the Machine. For this to happen, parts of the Machine have yet to be better understood, in particular the special role played by the requirement of *Z*-cocontinuity (commutation with diagrams of zero- dimensional refinements). For instance, we ignore whether the Vietoris functor and the multiset functors are *Z*-cocontinuous, or whether *Z*-cocontinuity is preserved by composition.

Rigidity is an unexpected mathematical outcome of our structural decomposition of **Pol**. Where the Machine allows to prove existence of natural transformations, rigidity allows to prove *unicity* and is somewhat dual to the former. We expect that the notion of isotropic element will find applications beyond the scope of these developments.

On the applications side, we are confident that many processes beside Dirichlet and Poisson can be subject to the same treatment. Poisson-Dirichlet, Cox processes and some form of Gaussian processes seem to be easy targets. In the case of Dirich- let, we already know that the Machine allows to prove an asymptotic “learning” property. The work of (Culbertson et al, [[6](#_bookmark38)]) will provide a convenient setting where we will study how topological properties of Bayesian models such as continuity relate

to asymptotic properties of Bayesian update. The finitary handle provided by the Machine might also be useful in deriving new computability or complexity results in the field of probability.

# Acknowledgment

We would like to thank the anonymous referees and Michael Mislove for their helpful remarks.

# References

1. Jiri Adamek, Horst Herrlich, and George E. Strecker. *Abstract and concrete categories : the joy of cats*. Pure and applied mathematics. Wiley, New York, 1990. A Wiley-Interscience publication.
2. Charalambos D Aliprantis and Kim C Border. *Infinite dimensional analysis*. Springer, 1999.
3. N. Balakrishnan and V.B. Nevzorov. *A Primer on Statistical Distributions*. Wiley, 2004.
4. N. Bourbaki. *Elements de math´ematique. Topologie G´en´erale.* Springer, 1971.
5. Philippe Chaput, Vincent Danos, Prakash Panangaden, and Gordon Plotkin. Approximating Markov Processes by averaging. *Journal of the ACM*, 61(1), January 2014. 45 pages.
6. Jared Culbertson and Kirk Sturtz. A categorical foundation for Bayesian probability. *Applied Categorical Structures*, pages 1–16, 2012.
7. Vincent Danos and Ilias Garnier. Dirichlet is natural. *Electronic Notes in Theoretical Computer Science*, 319:137 – 164, 2015. MFPS XXXI.
8. Thomas S Ferguson. A Bayesian analysis of some nonparametric problems. *The Annals of Statistics*, pages 209–230, 1973.
9. M. Giry. A categorical approach to probability theory. In B. Banaschewski, editor, *Categorical Aspects* *of Topology and Analysis*, number 915 in LNM, pages 68–85. Springer-Verlag, 1981.
10. Amol Kapila, Bela Frigyik, and Maya R. Gupta. Introduction to the Dirichlet distribution and related processes. Technical Report UWEETR-2010-0006, University of Washington., 2010.
11. Alexander S Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Text in Mathematics*. Springer, 1995.
12. Klaus Keimel and Gordon Plotkin. Mixed powerdomains for probability and nondeterminism. *Logical Methods in Computer Science*, 2015.
13. Dexter Kozen. Kolmogorov extension, martingale convergence, and compositionality of processes. Technical report, Computing and Information Science, Cornell University, December 2015.
14. Dexter Kozen, Kim G Larsen, Radu Mardare, and Prakash Panangaden. Stone duality for markov processes. In *LICS 2013*, pages 321–330, 2013.
15. Saunders Mac Lane. *Categories for the Working Mathematician*. Springer, 1998.

# Construction of the multiset functor B

**Proposition A.1** *For X Polish, let* B(*X*) *n∈*N *Xn/Sn, where Xn/Sn is the quotient of Xn under the obvious action of Sn on tuples with the quotient topology,*

*i.e. the ﬁnal topology for the quotient map q* : *Xn* → *Xn/Sn.* B(*X*) *is Polish.*

**Proof.** We first shown that if *Q* is dense in *X*, then *Qn/Sn* is dense in *Xn/Sn*: let *U* be an open set of *Xn/Sn*, then *q—*1(*U* ) is open in *Xn* and intersects *Qn*,

i.e. there exists (*r*1*,..., rn*) *∈ Qn* with (*r*1*,..., rn*) *∈ q—*1(*U* ), but this means that *q*(*r*1*,..., rn*) *∈ U* and *q*(*r*1*,..., rn*) *∈ Qn/Sn*. To see that it is completely metrisable, let *d* be a complete metric for *X*,and consider the metric on *Xn/Sn* given by:

*dq*([*x*]*,* [*y*]) = min *dn*(*x, π*(*y*))

*π∈Sn*

where [*x*]*,* [*y*] represent the orbits of *x, y ∈ Xn* respectively, and *dn* is the product metric given by

*dn*((*x*1*,..., xn*)*,* (*y*1*,..., yn*)) =

1

*p*

Σ

*d*(*xi, yi*)*p*

*i*

(A.1)

for some 0 *< p < ∞* (any choice of *p* generates an equivalent topology on *Xn*). Note that *dn* is invariant under permutations of *Sn*, i.e. for any permutation *π ∈ Sn*, *dn*(*x, y*)= *dn*(*π*(*x*)*, π*(*y*)) since this simply amounts to re-arranging the summands in Eq. ([A.1](#_bookmark49)). It is not immediately clear that *dq* is well-defined or that it defines a metric. To see that it is well defined let *xj* be another representative of [*x*], then by definition there exists *ρ ∈ Sn* such that *ρ*(*x*)= *xj*, and it follows that

min *dn*(*xj, π*(*y*)) = min *dn*(*ρ*(*x*)*, π*(*y*)) = min *dn*(*x, ρ—*1*π*(*y*)) = min *dn*(*x, π*(*y*))

*π∈Sn*

*π∈Sn*

*π∈Sn*

*π∈Sn*

It follows that *dq* is well-defined. Let us now check that it is a metric. For any *x, y* we clearly have *dq*([*x*]*,* [*y*]) *≥* 0 and *dq*([*x*]*,* [*y*]) = 0 means that there exists *π ∈ Sn* such that *dn*(*x, π*(*y*)) = 0 i.e. *x* = *π*(*y*) since *dn* is a metric, and it follows that [*x*]= [*y*]. It is straightforward to verify symmetry condition:

*dq*([*x*]*,* [*y*]) = min *dn*(*x, π*(*y*))

*π∈Sn*

= min *dn*(*π—*1(*x*)*, y*) *dn* invariant under *π—*1

*π∈Sn*

= min *dn*(*y, π—*1(*x*)) *dn* is symmetric

*π∈Sn*

= *dq*([*y*]*,* [*x*])

Finally, we need to check the triangular inequality. Since *dn* satisfies the triangular inequality we have for any choice *π*1*, π*2 *∈ Sn* that:

*dn*(*x, π*1(*y*)) *≤ dn*(*x, π*2(*z*)) + *dn*(*π*2(*z*)*, π*1(*x*))

*≤ dn*(*x, π*2(*z*)) + *dn*(*z, π—*1*π*1(*x*)) *dn* invariant under *π—*1

2 2

and it follows that *dq*([*x*]*,* [*y*]) *≤ dq*([*x*]*,* [*z*]) + *dq*([*z*]*,* [*y*]) since going through all the combinations *π—*1*π*1 will exhaust the entire group *Sn*. The fact that (*Xn/Sn, dq*) is complete follows from the fact that (*Xn, dn*) is. Let us prove that *dq* induces the topology of *Xn/Sn*. Let us take an open set *U* in *Xn/Sn*. By definition *q—*1(*U* ) is open in *Xn*, and can therefore be written as a union of open balls (for the metric *dn*) *U* = *∪iBdn* (*xi, ϵi*). By definition *q—*1(*U* ) is invariant under permutation, so

2

*q—*1(*U* )= *π*(*q—*1(*U* )) =

*π* *Bdn* (*xi, ϵi*) =

*π* (*Bdn* (*xi, ϵi*))

*π∈Sn*

=

*i π∈Sn*

*π∈Sn*

*i*

*π∈Sn*

*i*

*π* (*Bdn* (*xi, ϵi*))

since direct images commute with unions. It follows from the fact that each

*π* is an homeomorphism that *π∈Sn π*(*Bdn* (*xi, ϵi*)) is open in *X* . Moreover,

*n*

*π∈Sn π*(*Bdn* (*xi, ϵi*)) is by construction invariant under permutation, so

*q—*1(*q*(

*π∈Sn*

*π*(*Bdn* (*xi, ϵi*)))) =

*π∈Sn*

*π*(*Bdn* (*xi, ϵi*))

and therefore each *q*( *π∈Sn*

*π∈Sn π*(*Bdn* (*xi, ϵi*)) = *Bdq* (*q*(*xi*)*, ϵi*) and that

observing that *q*(

*π*(*Bdn* (*xi, ϵi*)) is an open in *Xn/Sn*. We conclude by

*q—*1(*Bd* (*q*(*xi*)*, ϵi*)) = *q—*1(*q*(  *π*(*Bdn* (*xi, ϵi*))) =

*q*

*π*(*Bdn* (*xi, ϵi*))

*π∈Sn π∈Sn*

is open in *Xn*. Therefore, the balls *Bd* (*q*(*xi*)*, ϵi*) are open in *Xn/Sn*, and since direct

*q*

images commute with unions it is not difficult to see that *U* = *i Bdq* (*q*(*xi*)*, ϵi*) isa

union of opens from the basis generated by the metric. Since each *Xn/Sn* is Polish

and since **Pol** has countable coproducts, B(*X*) is Polish. *2*

# Properties of the functors B and V

**Proposition B.1** *The multiset functor* B *preserves injections, embeddings and in- tersections.*

**Proof.** Let *i* : *B* > *X* be a mono, i.e. an injective continuous map. Note first that B(*i*) is defined component-wise i.e. via B*n*(*i*) : *Bn/Sn* > *Xn/Sn* injecting an equivalence class of *n*-tuples of element of *B* in *Xn/Sn*. The fact that B(*i*) is injective follows from the fact that every component B*n*(*i*) is. Similarly, to show that if *i* is an embedding so is B(*i*), it is enough to show that each B*n*(*i*) is an embedding. To see that this is the case we need to show that for every open *U* of *Bn/Sn* there exists an open *V* of *Xn/Sn* such that *U* = *V ∩ Bn/Sn* and conversely that every subset of this shape is open in *Bn/Sn*. We write *pn* : *Bn* → *Bn/Sn* and *qn* : *Xn* → *Xn/Sn*. For the first direction, let *U* be open in *Bn/Sn*, it follows that *p—*1(*U* ) is open in *Bn*, and thus that there exists an open *V* of *Xn* such that *p—*1(*U* )= *Bn ∩ V* . If we

*n*

*n*

can choose *V* to be closed under permutation we are done. Every permutation is a bijective isometry and thus a homeomorphism, and thus an open map, i.e. *π*(*V* ) is open for every *π ∈ Sn*. It follows that

*V ∗* =

*π∈Sn*

*π*(*V* )

is open and closed under permutations (this procedure amounts to taking all the reflections of tuples along the diagonal). It follows that *q—*1(*qn*[*V ∗*]) = *V ∗* and *qn*(*V ∗*) is thus open in *Xn/Sn*. Moreover since *Bn ∩V* is already closed under permutations *Bn ∩ V* = *Bn ∩ V ∗*, and therefore *U* = *Bn/Sn ∩ qn*(*V ∗*). For the opposite direction, let *U* be open in *Xn/Sn* and consider *U ∩ Bn/Sn*, it is clear that

*n*

*p—*1(*U ∩ Bn/Sn*)= *p—*1(*U* ) *∩ p—*1(*Bn/Sn*)= (*q—*1(*U* ) *∩ B*) *∩ B* = *q—*1(*U* ) *∩ B*

*n n n n n*

which is open in *Bn* since *qn*(*U* ) is open in *Xn*.

For intersections, we proceed as in Proposition [5.1](#_bookmark21). Let *j*1*, j*2 : *A, B* > *X* be two embeddings, let *p*1 : *A ∩ B → A* and *p*2 : *A ∩ B* > *B* be the corresponding embeddings and consider *μ ∈* B*A, ν ∈* B*B* such that B*j*1(*μ*) = B*j*2(*ν*). We define *λ ∈* B(*A ∩ B*)

*λ*(*x*)= *μ*(*p*1(*x*)) = *ν*(*p*2(*x*))

We check that the last equality holds in exactly the same way as in the proof of Proposition [5.1](#_bookmark21), and the rest of the proof also follows identically. *2*

**Proposition B.2** *The Vietoris functor* V *preserves monomorphisms, embeddings and intersections.*

**Proof.** It is clear that V preserves injective maps. To see that it preserves embed- dings, consider an element of the basis of the topology on V(*X*), i.e. an element of the form (Kechris [[11](#_bookmark43)] I, 4.F)

*W* = *{K ∈* V(*X*) *| K ⊆ U*0&*K ∩ U*1 */*= *∅*& &*K ∩ Un /*= *∅}*

for *U*0*,..., Un* opens in *X*. It follows that

*W ∩* V(*B*)

= *{K ∈* V(*B*) *| K ⊆* (*U*0 *∩ B*)&*K ∩* (*U*1 *∩ B*) */*= *∅*& &*K ∩* (*Un ∩ B*) */*= *∅}*

which is an element of the basis of the topology of V(*B*), since elements of the shape *Ui ∩ B* are precisely the opens of *B*. Conversely therefore, starting from an element *Wj* of this shape it is clear that by removing all the intersections with *B* we get an element *W* of the basis of the topology on V(*X*) such that *W ∩* V(*B*)= *Wj*, and V thus preserves embeddings.

For intersections, let *j*1*, j*2 : *A, B* > *X* be two embeddings, let *p*1 : *A ∩ B → A* and *p*2 : *A∩B* > *B* be the corresponding embeddings and consider *KA ∈* V*A, KB ∈* V*B* such that V*j*1(*KA*)= V*j*2(*KB*), i.e. such that *j*1[*KA*]= *j*2[*KB*]. This means that *K* = *KA* = *KB* is a subset of *A∩B*. To see that it is compact in *A∩B*, let *i Ui ⊇ K*

be an open cover: for each *i* either *Ui* is of the form *p—*1(*Vi*) for some *Vi* open in *A*, or it is of the form *p—*1(*Vi*) for some *Vi* open in *B*. In the latter case, since *j*2 is an embedding, there exists *Wi* open in *C* such that *Ui* = *p—*1(*j—*1(*Wi*)), but then *Ui* = *p—*1(*j—*1(*Wi*)), which means that we can assume without loss of generality that

1

2

2

1 1

for each *i* the element *Ui* of the cover is of the form *p—*1(*Vi*) for some *Vi* open in *A*. It is easy to see that *Vi* is an open cover of *K* in *A*, from which we can extract a finite sub-cover, whose inverse image under *p*1 will be an finite sub-cover of *K* in *A ∩ B*. It follows that V*A ∩* V*B* V(*A ∩ B*) as sets, and since V preserves embeddings, they are also homeomorphic. *2*

1

**Proposition B.3** B *is* **Pol***f -continuous.*

**Proof.** Let *Xi,i ∈ I* be a ccd of **Pol***f* objects. We show lim B*Xi* = B(lim *Xi*). For this we need to show that the unique continuous map *u* : B(lim *Xi*) *→* lim B*Xi* is a homeomorphism. To show this will show that it is bijective and open. We start by defining an inverse *φ* : lim B*Xi* > B(lim *Xi*). Since the set underlying the limits are computed in **Set**, showing that *φ* exists and is an inverse as a function will be enough to prove that *u* is bijective. We can assume w.l.o.g. that the morphisms between the finite Polish spaces *Xi* are surjective.

Given a ‘thread’ (*μi*)*i∈I ∈* lim B*Xi* we need to define a finitely supported multiset on the threads (*xi*)*i∈I ∈* lim *Xi*. For the thread (*μi*) consider the projective system of supports (*supp*(*μi*))*i∈I* together with the obvious restrictions *fij* T *supp*(*μi*) of the connecting maps *fij* : *Xi → Xj* which are also surjective. We claim that lim *supp*(*μi*) is finite and forms the support of the multiset *φ*((*μi*)*i∈I* ) on lim *Xi* . We make the following observation:

* 1. Each support is finite
  2. The size of the support cannot increase by following the connecting arrows, since they are surjective.
  3. The total mass *k* of *μi,i ∈ I* is constant throughout the thread because B

applied to a connecting morphism preserves the total mass of a multiset.

* 1. The cardinality of the set *supp*(*μi*) is bounded by *k* since we cannot assign a weight less than one to any element in the support.
  2. There exists an *i ∈ I* after which the cardinality of *supp*(*μi*) remains constant,

i.e. such that *|supp*(*μk*)*|* = *|supp*(*μj*)*|* for each *j > k*. If this wasn’t the case it would contradict the previous points.

Thus let *k* be such that *|supp*(*μk*)*|* = *|supp*(*μj*)*|* for each *j > k*, we claim that *pk* : lim *supp*(*μi*) *→ supp*(*μk*) is a bijection. It is surjective since the connecting morphisms in the diagram are surjective. If (*xi*)*i∈I,* (*yi*)*i∈I* are two threads such that *pk*(*xi*) = *pk*(*yi*) then *xk* = *yk*. Now take any *kj ∈ I*, by co-directedness there exists *j > k, kj* and by assumption on *k*, *supp*(*μk*) and *supp*(*μj*) have the same cardinality, i.e. the connecting morphism *pjk* is bijective. There therefore exists a unique *xj ∈ supp*(*μj*) such that *pjk*(*xj*)= *xk* = *yk*, and it follows that both thread must go through the same element at *kj* too, for any *kj*, which shows that *pk* is

injective. We define *φ*((*μi*)*i∈I* ) as the multiset on lim *Xi* defined by:

(*xi*)*i∈I*

*'→* 0 if (*xi*)*i∈I ∈/* lim *supp*(*μi*)

*μk*(*xk*) else (where *k* is defined as above)

We need to show that the definition is independent of the choice of *k*. Consider another index *kj* such that *|supp*(*μk′* )*|* = *|supp*(*μj*)*|* for each *j > kj*. Again by co-directedness there exists *j > k, kj*. We now calculate:

*μk*(*xk*)= *μk*(*fjk*(*xj*)) = B*fjk*(*μj*)(*fjk*(*xj*)) = *μj*(*xj*)= B*fjk′* (*μj*)(*fjk′* (*xj*))

= *μk′* (*xk′* )

Let us now show that *φ* thus defined is a left and right inverse to *u*. Given a multiset *μ ∈* B(lim *Xi*) on threads of lim *Xi*, *u*(*μ*) is the thread of multisets *νi* on *Xi* defined by *νi*(*x*)= *μ*[*p—*1(*{x}*)], i.e. the mass given by *μ* to the set of threads going through *x ∈ Xi*. This family forms a thread since for every *fij* : *Xi → Xj* and *y ∈ Xi*,

*i*

*νj*(*y*)= *μ*[*p—*1(*{y}*)] = *μ*[*p—*1(*f—*1(*y*)] = *νi*(*f—*1(*y*)) = B*fij*(*νi*)(*y*)

*j i ij ij*

For *μ ∈* B(lim *Xi*), let *u*(*μ*) = (*νi*)*i∈I* . The support *supp*(*νi*) is given by the set *Yi ⊆ Xi* of points traversed by a thread in the support of *μ*, and it is therefore not hard to see that lim *supp*(*νi*) with the multiplicities defined by *φ* is precisely *μ*, i.e. *φ ◦ u* = *id*B(lim *Xi*). Conversely *u ◦ φ* = *id*lim *Xi* by universality of lim *Xi*.

Finally, we show that the unique *u* : B(lim *Xi*) *→* lim B*Xi* is a homeomorphism. We already know that it is continuous and bijective, so it remains to be shown that it is open. For this we must look at the topologies on B(lim *Xi*) and lim B*Xi*. In the former *U* is an open exactly when *q—*1(*U* ) is open in (lim *Xi*)*n* for each *n* where *qn* : (lim *Xi*)*n* → (lim *Xi*)*n/Sn*. Any subset *U* of B(lim *Xi*) can be written as an union of sets *Un* in (lim *Xi*)*n/Sn*, so it is sufficient to show that *u* maps opens

*n*

of (lim *Xi*)*n/Sn* (corresponding to sets of multisets of total mass *n*) to opens in lim B*Xi*. It is not hard to check that *U* is open in (lim *Xi*)*n/Sn* iff there exists *V*

open in (lim *Xi*)*n* such that *qn*[*V ∗*] = *U* where *V ∗* = *π∈S π*[*V* ]. To check that

*n*

*u*(*U* ) is open it is therefore enough to check that *u ◦ qn ◦ π*[*V* ] is open for any open

*V* in (lim *Xi*)*n* and any *π ∈ Sn*; and since *π* is a homeomorphism this really means checking that *u◦qn*[*V* ] is open when *V* is. By the definition of the product topology and of the topology on lim *Xi* it is enough to check that *u ◦ qn*[*Y j*] is open for *Y j*

*k k*

the set of *n*-tuples of threads of lim *Xi* whose *jth* component goes through *Yk ⊆ Xk*.

The morphism *qn* collapses such an *n*-tuple to a multiset on threads and *qn*[*Y j*] is the set of multisets of total mass *n* which assigns mass at least one to threads going through *Yk*.

*k*

To check that *u* sends these open sets to open sets we need to describe the topology on the codomain. Fortunately is it much simpler. Since each *Xi* is finite, *Xn/Sn* is finite, and must therefore have the discrete topology. Since B(*Xi*) =

*i*

*n Xn/Sn* is given the final topology for all the injections its topology must also

*i*

be discrete. The topology on lim B(*Xi*) is thus generated by the opens of the shape

*p—*1(*Ui*) where *Ui* is any subset of B(*Xi*) and *pi* is the canonical projection.

*i*

We can now check that *u* is open. Let us denote *qn*[*Y j*]= *Y n* the set of multisets

*k k*

of total mass *n* which assigns mass at least one to threads going through *Yk*. It

gets mapped to a set B(*pk*(*Y k*)) of multisets on *Xk*, which in turns defines *u*(*Y n*)=

*n k*

*p—*1(B(*pk*(*Y k*))) which is indeed open. *2*

*k n*

**Proposition B.4** *Let* (*Xi*)*i∈I be a* ccd *of compact spaces. Then* V(lim *Xi*) *∼*= lim V*Xi*

**Proof.** Let (*Xi*)*i∈I* be a *ccd* of compact Polish space; we must show that V lim *Xi* = lim V*Xi*. Let us first show that there exists a bijection between these sets. We write *pi* : lim *Xi → Xi* for the canonical projections. We know that there exists a unique continuous map *u* : V lim *Xi →* lim V*Xi*; it takes a compact *K* of lim *Xi* and maps it to the thread (*pi*[*K*])*i∈I* of lim V*Xi* (since the continuous image of a compact is compact). Let *K, Kj* be two compacts of lim *Xi* such that *pi*[*K*]= *pi*[*Kj*] for every *i ∈ I*, for every thread (*xi*) in lim *Xi* it is clear that (*xi*) *∈ K* iff *pi*(*xi*) *∈ pi*[*K*] iff *pi*(*xi*) *∈ pi*[*Kj*] iff (*xi*) *∈ Kj* and thus *u* is injective.

We now define an inverse map *φ* : lim V*Xi →* V lim *Xi* as follows. For each thread of compacts (*Ki*)*i∈I* in lim V*Xi*, since each *Xi* is Hausdorff, each *Ki* is closed and thus *p—*1(*Ki*) is a closed subset of lim *Xi*. We define

*i*

*φ*((*Ki*)*i∈I* )= *p—*1(*Ki*)

*i*

*i*

To see that this is well-defined, we need to show that *φ*((*Ki*)*i∈I* ) is compact. Since each *p—*1(*Ki*) is closed, their intersection *φ*((*Ki*)*i∈I* ) is closed. We also know that since each *Xi* is compact lim *Xi* is a closed subspace of the product *Xi* which is compact by Tychonoff’s theorem. It follows that lim *Xi* is compact, and since V sends compacts to compacts (Kechris Theorem 4.26), V lim *Xi* is compact. Finally since *φ*((*Ki*)*i∈I* ) is closed in a compact it is itself compact.

*i*

To see that *φ* is a left inverse of *u*, start with *K ∈* V lim *Xi*, *u*(*K*)= (*pi*[*K*])*i∈I* and

*φ*(*u*(*K*)) = *p—*1(*pi*[*K*])

*i*

*i*

Let (*xi*) be a thread in *K*, then clearly *pi*((*xi*)) = *xi ∈ pi*[*K*] for all *i*, and thus

(*xi*) *∈ φ*(*u*(*K*)). Conversely, let (*xi*) bea thread in *φ*(*u*(*K*)) then by definition of *φ*, *pi*((*xi*)) *∈ pi*[*K*] for every *i*, i.e. *xi ∈ pi*[*K*] for every *i*, i.e. (*xi*) *∈ K*, and it follows that *φ ◦ u* = *id*V lim *Xi* . Conversely, *φ* is a right inverse since *u ◦ φ* = *id*lim V*Xi* by universality of *u*. We have thus established that *u* is bijective.

Finally, since *u* : V lim *Xi →* lim V*Xi* is a continuous bijection with a compact domain and a Hausdorff codomain, it is a homeomorphism, which concludes the proof. *2*