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Induced Topologies on the Poset of Finitely Generated Saturated Sets

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**Abstract**

In [[7](#_bookmark24)], Heckmann and Keimel proved that a dcpo *P* is quasicontinuous iff the poset **Fin** *P* of nonempty finitely generated upper sets ordered by reverse inclusion is continuous. We generalize this result to general topological spaces in this paper. More precisely, for any *T*0 space (*X, τ* ) and *U ∈ τ* , we construct a topology *τF* generated by the basic open subsets *UF* = *{↑F ∈* **Fin** *X*: *F ⊆ U}*. It is shown that a *T*0 space (*X, τ* ) is a hypercontinuous lattice iff *τF* is a completely distributive lattice. In particular, we prove that if a poset *P* satisfies property DINT*op*, then *P* is quasi-hypercontinuous iff **Fin** *P* is hypercontinuous.

*Keywords:* Hypercontinuous poset, quasicontinuous domain, Scott topology, upper topology

# Introduction and Preliminaries

### Quasicontinuous domains were introduced by Gierz, Lawson and Stralka (see [[4](#_bookmark21)]) as a common generalization of both generalized continuous lattices (see [[5](#_bookmark22)]) and continuous domains (see [[6](#_bookmark23)]). It was proved that quasicontinuous domains equipped with the Scott topologies are precisely the spectra of distributive hypercontinuous

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### lattices. In [[7](#_bookmark24)], Heckmann and Keimel proved that a dcpo *P* is quasicontinuous iff the poset **Fin** *P* of nonempty finitely generated upper sets ordered by reverse inclusion is continuous. In this paper, we generalize this result to general topological spaces. Firstly, for any *T*0 space (*X, τ* ) and *U ∈ τ* , we construct a topology *τF* generated by the basic open subsets *UF* = *{↑F ∈* **Fin** *X*: *F ⊆ U}*. Then we show that a *T*0 space (*X, τ* ) is a hypercontinuous lattice iff *τF* is a completely distributive lattice. In particular, we prove that for a dcpo *P* , if the Scott topology *σ*(*P* ) is hypercontinuous or *σ*(**Fin** *P* ) is completely distributive, then *σ*(*P* )*F* =*σ*(**Fin** *P* ). Furthermore, it is proved that if a poset *P* satisfies property DINT*op*, then *P* is quasi-hypercontinuous iff **Fin** *P* is hypercontinuous.

For a poset *P* , let *P* (*<ω*) = *{F ⊆ P* : *F* is finite*}* and **Fin** *P* =*{↑F* : *F ∈ P* (*<ω*)*}*.

For all *x ∈ P* , *A ⊆ P* , let *↑x* = *{y ∈ P* : *x ≤ y}* and *↑A* = S

*a∈A*

*↑a*; *↓x* and *↓A*

### are defined dually. For a poset *P* , the topology generated by the collection of sets

*P \ ↓x* (as a subbase) is called the *upper topology* and denoted by *υ*(*P* ); the *lower topology* on *P* is dually defined and denoted by *ω*(*P* ). A subset *U* of *P* is called *Scott open* provided that *U* = *↑U* and *D ∩ U /*= *∅* for all directed sets *D ⊆ P* with

### W *D ∈ U* whenever W *D* exists. The topology formed by all the Scott open sets of

*P* is called the *Scott topology* on *P* , written as *σ*(*P* ).

### If *P* is a poset, more generally a preordered set, we introduce a preorder *≤* on the powerset of *P* , sometimes called the *Smyth preorder*, by *A ≤ B* iff *↑B ⊆ ↑A*. Throughout the paper, **Fin** *P* is always endowed with the Smyth preorder.

**Definition 1.1** ([[6,11](#_bookmark28)]) Let *P* be a poset.

### For any two elements *x* and *y* in *P* , we write *x y*, if for each directed subset

*D ⊆ P* with W *D* existing, *y ≤* W *D* implies *x ≤ d* for some *d ∈ D*. The set

*{y ∈ P* : *y x}* will be denoted *⇓x* and *{y ∈ P* : *x y}* denoted *⇑x*.

1. *P* is called a *continuous poset* if *x* = W *⇓x* and *⇓x* is directed for all *x ∈ P* .
2. *P* is called an *algebraic poset* if *x* = W*{y ∈ P* : *y y ≤ x}* for all *x ∈ P* and

the set *{y ∈ P* : *y y ≤ x}* is directed.

**Definition 1.2** ([[5,6](#_bookmark23)]) Let *P* be a poset.

1. We define a relation *≺* on *P* by *x ≺ y ⇔ y ∈ intυ*(*P* )*↑x*.
2. *P* is called a *hypercontinuous poset* if *{u ∈ P* : *u ≺ x}* is directed and *x* =

W*{u ∈ P* : *u ≺ x}* for each *x ∈ P* . A complete lattice which is hypercontinuous

as a poset is called a *hypercontinuous lattice*.

1. *P* is called a *hyperalgebraic poset* if *{u ∈ P* : *u ≺ u ≤ x}* is directed and

*x* = W*{u ∈ P* : *u ≺ u ≤ x}* for each *x ∈ P* . A complete lattice which is

hyperalgebraic as a poset is called a *hyperalgebraic lattice*.

**Theorem 1.3** ([[1,11](#_bookmark28)]) *Let P be a poset. Then the following conditions are equiva- lent:*

1. *P is a continuous poset;*
2. *For all x ∈ U ∈ σ*(*P* )*, there exists y ∈ P such that x ∈ intσ*(*P* )*↑y ⊆ ↑y ⊆ U;*

(2) *σ*(*P* ) *is a completely distributive lattice.*

**Theorem 1.4** ([[5,6](#_bookmark23)]) *Let P be a poset. Then the following conditions are equiva- lent:*

1. *P is a hypercontinuous poset;*
2. *For all x ∈ U ∈ υ*(*P* )*, there exists y ∈ P such that x ∈ intυ*(*P* )*↑y ⊆ ↑y ⊆ U;*

(2) *υ*(*P* ) *is a completely distributive lattice.*

**Definition 1.5** ([[2,3](#_bookmark20)]) A *T*0 space (*X, τ* ) is called a *web space* if for each *x ∈ X*

and *Y ⊆ X* with *x ∈ clτ Y* , one has *x ∈ clτ* (*↓x ∩ ↓Y* ).

**Definition 1.6** ([[10](#_bookmark27)]) A poset *P* is called *meet continuous* if for any *x ∈ P* and any directed set *D*, if W *D* exists and *x ≤* W *D*, then *x ∈ clσ*(*P* )(*↓x ∩ ↓D*).

**Theorem 1.7** ([[2,3](#_bookmark20)]) *Let P be a poset. Then the following conditions are equiva- lent:*

1. *P is meet continuous;*
2. *P is a web space endowed with the Scott topology;*
3. *For any Scott open set U and any x ∈ P, ↑*(*U ∩ ↓x*) *is Scott open.*

### The proof of the following lemma is similar to that of the analogous results for dcpos in [[6](#_bookmark23)].

**Lemma 1.8** *If F is a ﬁnite set in a meet continuous poset P, then we have*

*intσ*(*P* )*↑F ⊆* S*{⇑x* : *x ∈ F}.*

# Quasicontinuous domains and quasihypercontinuous posets

**Definition 2.1** ([[4,6](#_bookmark23)]) Let *P* be a dcpo.

* 1. For all *F* , *G ⊆ P* , we say that *G* is *way below F* and write *G F* if for every directed set *D ⊆ P* , W *D ∈ ↑F* implies *d ∈ ↑G* for some *d ∈ D*.
  2. *P* is called a *quasicontinuous domain* if *{↑F ∈* **Fin** *P* : *F x}* is directed and *↑x* = *{↑F ∈* **Fin** *P* : *F x}* for each *x ∈ P* .
  3. *P* is called a *quasialgebraic domain* if *{↑F ∈* **Fin** *P* : *F F x}* is directed and *↑x* = *{↑F ∈* **Fin** *P* : *F F x}* for each *x ∈ P* .

**Definition 2.2** ([[12](#_bookmark29)]) Let *P* be a poset.

1. We define a relation *≺* on 2*P* by *F ≺ G ⇔ G ⊆ intυ*(*P* )*↑F* .
2. *P* is called a *quasi*-*hypercontinuous poset* if *{↑F ∈* **Fin** *P* : *F ≺ x}* is directed and *↑x* = *{↑F ∈* **Fin** *P* : *F ≺ x}* for each *x ∈ P* .
3. *P* is called a *quasi*-*hyperalgebraic poset* if *{↑F ∈* **Fin** *P* : *F ≺ F ≺ x}* is directed and *↑x* = *{↑F ∈* **Fin** *P* : *F ≺ F ≺ x}* for each *x ∈ P* .

**Theorem 2.3** ([[4,6](#_bookmark23)]) *Let P be a dcpo. Then the following conditions are equivalent:*

1. *P is a quasicontinuous domain;*
2. *For all x ∈ U ∈ σ*(*P* )*, there exists F ∈ P* (*<ω*) *such that x ∈ intσ*(*P* )*↑F ⊆ ↑F ⊆*

*U;*

1. *σ*(*P* ) *is a hypercontinuous lattice.*

**Theorem 2.4** ([[6](#_bookmark23)]) *Let P be a dcpo. Then the following conditions are equivalent:*

1. *P is a quasialgebraic domain;*
2. *For all x ∈ U ∈ σ*(*P* )*, there exists F ∈ P* (*<ω*) *such that x ∈ intσ*(*P* )*↑F* = *↑F ⊆*

*U;*

1. *σ*(*P* ) *is a hyperalgebraic lattice.*

**Theorem 2.5** ([[12](#_bookmark29)]) *Let P be a poset. Then the following conditions are equivalent:*

1. *P is a quasi-hypercontinuous poset;*
2. *For all x ∈ U ∈ υ*(*P* )*, there exists F ∈ P* (*<ω*) *such that x ∈ intυ*(*P* )*↑F ⊆ ↑F ⊆*

*U;*

1. *υ*(*P* ) *is a hypercontinuous lattice.*

According to [[10](#_bookmark27)], a poset *P* is called a *quasicontinuous poset* (resp., *quasial- gebraic poset)* if for all *x ∈ U ∈ σ*(*P* ), there exists *F ∈ P* (*<ω*) such that *x ∈ intσ*(*P* )*↑F ⊆ ↑F ⊆ U* (resp., *x ∈ intσ*(*P* )*↑F* = *↑F ⊆ U* ).

**Theorem 2.6** *Let P be a poset. Then the following two conditions are equivalent:*

1. *P is an algebraic poset;*
2. *P is a meet continuous and quasialgebraic poset.*

**Proof.** (1) *⇒* (2): Obviously.

(2) *⇒* (1): CLAIM: Let *x ∈ P* and *F ⊆ P* be finite. If *x ∈ intσ*(*P* )*↑F* = *↑F* , then there exists *t ∈ F* with *t ∈ ↓x ∩ K*(*P* ).

Proof of Claim. Since *F* is finite, *↑F* = *↑*Min(F) where Min(F) is the set of all minimal elements in *F* . By Lemma [1.8](#_bookmark4), *x ∈ ↑*Min(F) = *↑F* = *intσ*(*P* )*↑F* = *intσ*(*P* )*↑*Min(F) *⊆ {⇑*t : t *∈* Min(F)*}*. So there exists *t ∈* Min(F) with *t x*. Since *↑*Min(F) *⊆ {⇑*t : t *∈* Min(F)*}*, there exists *s ∈* Min(F) with *s t*, hence *s ≤ t*. So *s* = *t* since *s*, *t ∈* Min(F). Thus *t ∈ ↓x ∩ K*(*P* ).

S

S

Firstly, we show that *x* = W(*↓x ∩ K*(*P* )) for all *x ∈ P* . Clearly, *x* is an upper

bound of *↓x ∩ K*(*P* ). Let *y* be any upper bound of *↓x ∩ K*(*P* ) and assume *x* ¢ *y*. Then *x ∈ P\↓y ∈ σ*(*P* ). By (2), there exists *F ∈ P* (*<ω*) such that *x ∈ intσ*(*P* )*↑F* =

*↑F ⊆ P\↓y*. By Claim, there exists *t ∈ F* with *t ∈ ↓x ∩ K*(*P* ), a contradiction to

*↓x ∩ K*(*P* ) *⊆ ↓y*.

Then we show that *↓x ∩ K*(*P* ) is directed for all *x ∈ P* . On the one hand, since *P* is quasialgebraic, there exists *G ∈ P* (*<ω*) such that *x ∈ intσ*(*P* )*↑G* = *↑G ⊆ P* . By Claim, there is a *y ∈ G* with *y ∈ ↓x ∩ K*(*P* ). Thus *↓x ∩ K*(*P* ) */*= *∅*. On the other hand, let *u*, *v ∈ ↓x ∩ K*(*P* ). Then *x ∈ ↑u ∩ ↑v ∈ σ*(*P* ). By (2), there exists *H ∈ P* (*<ω*) such that *x ∈ intσ*(*P* )*↑H* = *↑H ⊆ ↑u ∩ ↑v*. By Claim, there exists *m ∈ H* with *m ∈ ↓x ∩ K*(*P* ). Whence *m ∈ ↑u ∩ ↑v*. Hence *↓x ∩ K*(*P* ) is directed.*2*

**Proposition 2.7** *Let P be a poset. Then the following two conditions are equiva-* lent:

* 1. *P is a quasi-hyperalgebraic poset;*
  2. *For all x ∈ U ∈ υ*(*P* )*, there exists F ∈ P* (*<ω*) *such that x ∈ intυ*(*P* )*↑F* = *↑F ⊆*

*U;*

* 1. *υ*(*P* ) *is a hyperalgebraic lattice.*

**Proof.** (1) *⇒* (2): For all *U ∈ υ*(*P* ) with *x ∈ U* , there exists *H ∈ P* (*<ω*) such that *x ∈ P\↓H ⊆ U* . For all *h ∈ H*, by (1), there exists *Fh ∈ P* (*<ω*) such that *x ∈ intυ*(*P* )*↑Fh* = *↑Fh ⊆ P\↓h*. Since *H ∈ P* (*<ω*) and *{↑F ∈* **Fin** *P* : *x ∈*

*intυ*(*P* )*↑F* = *↑F}* is directed, there exists *G ∈ P* (*<ω*) such that *x ∈ intυ*(*P* )*↑G* =

*↑G ⊆* *↑Fh ⊆* *P\↓h* = *P\↓H ⊆ U* .

*h∈H h∈H*

(2) *⇒* (1): Suppose *↑F*1, *↑F*2 *∈ {↑F ∈* **Fin** *P* : *x ∈ intυ*(*P* )*↑F* = *↑F}*. Then

*x ∈ intυ*(*P* )*↑F*1 *∩ intυ*(*P* )*↑F*2 *∈ υ*(*P* ). By (2), there is *F*3 *∈ P* (*<ω*) such that *x ∈ intυ*(*P* )*↑F*3 = *↑F*3 *⊆ intυ*(*P* )*↑F*1 *∩ intυ*(*P* )*↑F*2 *⊆ ↑F*1 *∩ ↑F*2. Therefore, *{↑F ∈* **Fin** *P* : *x ∈ intυ*(*P* )*↑F* = *↑F}* is directed. Clearly, *↑x ⊆ {↑F ∈* **Fin** *P* : *x ∈*

*intυ*(*P* )*↑F* = *↑F}*. If *z ∈/ ↑x*, then *x ∈ P\↓z ∈ υ*(*P* ). By (2), there is *G ∈ P* (*<ω*)

with *x ∈ intυ*(*P* )*↑G* = *↑G ⊆ P\↓z*. It follows that *z ∈/* *{↑F ∈* **Fin** *P* : *x ∈*

*intυ*(*P* )*↑F* = *↑F}*. Therefore, *↑x* = *{↑F ∈* **Fin** *P* : *x ∈ intυ*(*P* )*↑F* = *↑F}*.

### (2) *⇔* (3): This follows from Lemma 3.3 of [[13](#_bookmark30)]. *2*

It is similar to the proof of Theorem [2.6](#_bookmark7), we have the following

**Theorem 2.8** *Let P be a poset. Then the following two conditions are equivalent:*

1. *P is a hyperalgebraic poset;*
2. *P is a meet continuous and quasi-hyperalgebraic poset.*

# Induced topologies on the poset of finitely generated saturated sets

**Definition 3.1** ([[2](#_bookmark19)]) Let (*X, τ* ) be a *T*0 space.

* 1. (*X, τ* ) is called a *c*-*space* if for all *x ∈ U ∈ τ* , there exist *y ∈ X* and *V ∈ τ* such that *x ∈ V ⊆ ↑y ⊆ U* .
  2. (*X, τ* ) is called a *locally hypercompact space* if for all *x ∈ U ∈ τ* , there exists

*F ∈ X*(*<ω*) and *V ∈ τ* such that *x ∈ V ⊆ ↑F ⊆ U* .

**Theorem 3.2** ([[1](#_bookmark18)]) *Let* (*X, τ* ) *be a T*0 *space. The following conditions are equiva- lent:*

1. *X is a c-space;*
2. *τ is a completely distributive lattice.*

**Theorem 3.3** ([[2,8](#_bookmark25)]) *Let* (*X, τ* ) *be a T*0 *space. The following conditions are equiv- alent:*

1. *X is locally hypercompact;*
2. *τ is a hypercontinuous lattice.*

Let (*X, τ* ) be a *T*0 space. For all *U ∈ τ* , let *UF* = *{↑F ∈* **Fin** *X*: *F ⊆ U}*. The topology generated by the basic open subsets *UF* is denoted by *τF* .

### It is easy to get the following

**Proposition 3.4** *Let* (*X, τ* ) *be a T*0 *space.*

1. *∅F* = *∅, XF* = **Fin** *X.*
2. *For all U, V ∈ τ,* (*U ∩ V* )*F* = *UF ∩ VF .*

**Theorem 3.5** *Let* (*X, τ* ) *be a T*0 *space. Then the following two conditions are* equivalent:

1. *τ is a hypercontinuous lattice;*
2. *τF is a completely distributive lattice.*

**Proof.** (1) *⇒* (2): For any *↑G ∈ U* = S (*Ui*)*F* , there exists *i ∈ I* such that

*i∈I*

*↑G ⊆ Ui*. By Theorem [3.3](#_bookmark9), for each *g ∈ G*, there exists *Fg ∈ X*(*<ω*) such that

*g ∈ intτ ↑Fg ⊆ ↑Fg ⊆ Ui*. Let *F* = S *Fg* and *V* = *intτ ↑F* . Obviously, *F* is finite.

*g∈G*

Thus *↑G ∈ VF ⊆ ↑***Fin***X* (*↑F* ) *⊆* (*Ui*)*F ⊆ U* . Thus *τF* is completely distributive by

### Theorem [3.2](#_bookmark8).

(2) *⇒* (1): Let *U ∈ τ* with *x ∈ U* . Then *↑x ∈ UF* . By (2), there exists *↑F ∈* **Fin** *P* such that *↑x ∈ intτF ↑***Fin***X* (*↑F* ) *⊆ ↑***Fin***X* (*↑F* ) *⊆ UF* . Thus there exists *V ∈ τ* such that *↑x ∈ VF ⊆ ↑***Fin***X* (*↑F* ) *⊆ UF* . Hence *x ∈ V ⊆ ↑F ⊆ U* . Therefore, *τ* is hypercontinuous by Theorem [3.3](#_bookmark9). *2*

### Similarly, we have the following

**Theorem 3.6** *Let* (*X, τ* ) *be a T*0 *space. Then the following two conditions are* equivalent:

1. *τ is a hyperalgebraic lattice;*
2. *τF is a completely distributive and algebraic lattice.*

**Lemma 3.7** *Let P be a poset. Then* W

*↑Fd exists in* **Fin** *P iff*

*↑Fd ∈* **Fin** *P*

*d∈D*

*for all {↑Fd* : *d ∈ D}⊆* **Fin** *P. In that case*

*d∈D*

*↑Fd* =

W

*d∈D*

*↑Fd.*

*d∈D*

**Proof.** Obviously,

*↑Fd ∈* **Fin** *P* implies that W

*↑Fd* exists in **Fin** *P* .

*d∈D*

Conversely, let *↑F ∈* **Fin** *P* with

W

*d∈D*

*d∈D*

*↑Fd* = *↑F* . Then for all *d ∈ D*, *↑Fd ≤ ↑F* ,

i.e., *↑F ⊆ ↑Fd*. Thus *↑F ⊆*

*d∈D*

### *↑Fd*. On the other hand, if

*d∈D*

*↑Fd* ¢ *↑F* , then

### there exists *x ∈*

*d∈D*

*↑Fd* but *x ∈/ ↑F* . Thus *↑x* is an upper bound of *{↑Fd* : *d ∈ D}*

in **Fin** *P* and *↑F* ¢ *↑x*, a contradiction. *2*

**Proposition 3.8** *For any poset P,* **Fin** *P is a meet continuous poset.*

**Proof.** For all *↑F ∈* **Fin** *P* and *U ∈ σ*(**Fin** *P* ), we show that *↑***Fin***P* (*↓***Fin**W*P* (*↑F* ) *∩*

*U* ) *∈ σ*(**Fin** *P* ). For all directed sets *{↑Fd* : *d ∈ D} ⊆* **Fin** *P* with *↑Fd ∈*

*d∈D*W

*↑***Fin***P* (*↓***Fin***P* (*↑F* ) *∩U* ), there exists *↑G ∈U* with *↑G ≤ ↑F* such that *↑G ≤ ↑Fd*.

### By Lemma [3.7](#_bookmark11), we have

*d∈D*

*↑Fd ⊆ ↑G*. Thus *↑G* = *↑G∪*

*d∈D*

*↑Fd* =

*d∈D*

*d∈D*

(*↑G∪↑Fd*) *∈*

*U* . Hence there exists *d ∈ D* such that *↑G ∪ ↑Fd ∈ U* . Thus *↑G ∪ ↑Fd ≤ ↑Fd*, *↑G*,

which implies *↑G ∪ ↑Fd ∈ U ∩ ↓***Fin***P* (*↑F* ). Whence *↑Fd ∈ ↑***Fin***P* (*↓***Fin***P* (*↑F* ) *∩ U* ). Hence **Fin** *P* is meet continuous by Theorem [1.7](#_bookmark3). *2*

**Lemma 3.9** *Let P be a dcpo. Then σ*(*P* )*F ⊆ σ*(**Fin** *P* )*.*

**Proof.** For all *U ∈ σ*(*P* ), we show *UF* = *{↑F ∈* **Fin** *P* : *↑F ⊆ U} ∈ σ*(**Fin** *P* ). Obviously, *UF* = *↑***Fin***P UF* . For all directed sets *{↑Fd* : *d ∈ D} ⊆* **Fin** *P* with

W *↑Fd ∈ UF* , by Lemma [3.7](#_bookmark11), we have *↑Fd* = *↑H ∈* **Fin** *P* and *↑H ⊆ U* .

*d∈D d∈D*

### By Rudin’s Lemma [6, III-3.3], there exists *d ∈ D* such that *↑Fd ⊆ U* . Whence

*↑Fd ∈ UF* . Hence *UF ∈ σ*(**Fin** *P* ). *2*

**Lemma 3.10** *Let P be a poset and U ∈ σ*(**Fin** *P* )*. Then U* = S *U* = S*{↑F ∈* **Fin**

*P* : *↑F ∈ U}∈ σ*(*P* )*.*

**Proof.** Let *y ∈ ↑U* . Let *x ∈ U* such that *x ≤ y*. Then there exists *↑F ∈ U* such that *x ∈ ↑F* . Thus *↑y ⊆ ↑x ⊆ ↑F* , i.e., *↑F ≤ ↑x ≤ ↑y*. Since *U ∈ σ*(**Fin** *P* ),

*↑y ∈ U* . Whence *y ∈ U* . Hence *↑U* = *U* .

For all directed sets *D ⊆ P* with W *D ∈ U* , we have *↑d* = *↑* W *D ∈ U* . Thus

*d∈D*

there exists *d ∈ D* such that *↑d ∈ U* . So *d ∈ U* . *2*

**Theorem 3.11** *Let P be a dcpo. If σ*(*P* ) *is hypercontinuous or σ*(**Fin** *P* ) *is com- pletely distributive, then σ*(*P* )*F =σ*(**Fin** *P).*

**Proof.** Let *↑F ∈ U ∈ σ*(**Fin** *P* ). If *σ*(*P* ) is hypercontinuous, then *↑F* = *{↑G ∈* **Fin** *P* : *↑F ⊆ intσ*(*P* )*↑G}* and *{↑G ∈* **Fin** *P* : *↑F ⊆ intσ*(*P* )*↑G}* is directed. Thus there exists *↑G ∈* **Fin** *P* such that *↑F ⊆ intσ*(*P* )*↑G ⊆ ↑G ∈ U* . Let *V* = *intσ*(*P* )*↑G*. Then *↑F ∈ VF ⊆ ↑***Fin***P* (*↑G*) *⊆ U* . Hence *U ∈ σ*(*P* )*F* .

If *σ*(**Fin** *P* ) is completely distributive, then there is *↑H ∈* **Fin** *P* with *↑F ∈ intσ*(**Fin***P* )*↑***FinP**(*↑H*) *⊆ ↑***FinP**(*↑H*) *⊆ U* . Let *W* = S *intσ*(**Fin***P* )*↑***FinP**(*↑H*). Then by Lemma [3.10](#_bookmark14), *W ∈ σ*(*P* ) and *↑F ∈ WF ⊆ ↑***Fin***P* (*↑H*) *⊆ U* . Therefore, *U ∈*

*σ*(*P* )*F* . By Lemma [3.9](#_bookmark13), we have *σ*(*P* )*F* =*σ*(**Fin** *P* ). *2*

### By Theorem [3.5](#_bookmark10) and Theorem [3.11](#_bookmark15), we get the following

**Corollary 3.12** *Let P be a dcpo. Then the following two conditions are equivalent:*

1. *σ*(*P* ) *is a hypercontinuous lattice;*
2. *σ*(**Fin** *P* ) *is a completely distributive lattice.*

### By Theorem [1.3](#_bookmark1), Theorem [2.3](#_bookmark5) and Corollary [3.12](#_bookmark16), we have the following

**Corollary 3.13 ([**[**7**](#_bookmark24)**])** *Let P be a dcpo. Then the following two conditions are equiv- alent:*

1. *P is a quasicontinuous domain;*
2. **Fin** *P is a continuous poset.*

### By Theorem 5.6 of [[10](#_bookmark27)] and Proposition [3.8](#_bookmark12), we have the following

**Corollary 3.14** *Let P be a poset. Then the following two conditions are equivalent:*

1. **Fin** *P is a continuous poset;*
2. **Fin** *P is a quasicontinuous poset.*

### By Theorem [2.6](#_bookmark7) and Proposition [3.8](#_bookmark12), we have the following

**Corollary 3.15** *Let P be a poset. Then the following two conditions are equivalent:*

1. **Fin** *P is an algebraic poset;*
2. **Fin** *P is a quasialgebraic poset.*

### A poset *P* is said to have property *DINT* (see [[9](#_bookmark26)]) if every set closed in the lower topology is a directed intersection of finitely generated upper sets.

**Theorem 3.16** *Let P be a poset satisfying property DINTop. Then υ*(**Fin** *P)=υ*(*P* )*F .*

**Proof.** For all *↑F ∈* **Fin** *P* , we have **Fin** *P \↓***Fin***P* (*↑F* ) = S (*P\↓u*)*F* . Thus

*u∈F*

*υ*(**Fin** *P* ) *⊆ υ*(*P* )*F* .

Conversely, it is clear that *PF* = **Fin** *P ∈ υ*(**Fin** *P* ). For any nonempty set *U ∈ υ*(*P* ) with *U /*= *P* , we show *UF ∈ υ*(**Fin** *P* ). Since *P* satisfies property DINT*op*, there exists a directed family *{↓Fd* : *Fd ∈ P* (*<ω*) and *d ∈ D}* such that

*U* = *P\* *↓Fd* = S (*P\↓Fd*). Thus *UF* =( S (*P\↓Fd*))*F* = S (*P\↓Fd*)*F* . We

*d∈D*

*d∈D*

*d∈D*

*d∈D*

claim that (*P\↓Fd*)*F ∈ υ*(**Fin** *P* ) for all *d ∈ D*. For all *↑G ∈* (*P\↓Fd*)*F* , we have

*↑G ⊆ P\↓Fd*, i.e., *↑h* ¢ *↑G* for all *h ∈ Fd*. Thus *↑G ∈* (**Fin** *P\↓***Fin***P* (*↑h*)) =

**Fin** *P\*

S

*h∈Fd*

*h∈Fd*

*↓***Fin***P* (*↑h*) = **Fin** *P\↓***Fin***P {↑h* : *h ∈ Fd} ∈ υ*(**Fin** *P* ). Therefore,

*UF ∈ υ*(**Fin** *P* ). *2*

**Problem 3.17** *Is property DINTop necessary to derive Theorem* [*3.16*](#_bookmark17)*?*

### By Theorem [1.4](#_bookmark2), Theorem [2.5](#_bookmark6), Theorem [3.5](#_bookmark10) and Theorem [3.16](#_bookmark17), we have the following

**Corollary 3.18** *Let P be a poset satisfying property DINTop. Then the following* two conditions are equivalent:

1. *P is a quasi-hypercontinuous poset;*
2. **Fin** *P is a hypercontinuous poset.*

**Corollary 3.19** *Let P be a semilattice. Then the following two conditions are* equivalent:

* 1. *P is a quasi-hypercontinuous poset;*
  2. **Fin** *P is a hypercontinuous poset.*

### Similarly, we have the following two corollaries.

**Corollary 3.20** *Let P be a poset. Then the following two conditions are equivalent:*

1. **Fin** *P is a hypercontinuous poset;*
2. **Fin** *P is a quasi-hypercontinuous poset.*

**Corollary 3.21** *Let P be a poset. Then the following two conditions are equivalent:*

1. **Fin** *P is a hyperalgebraic poset;*
2. **Fin** *P is a quasi-hyperalgebraic poset.*

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