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**Interpretations on Quantum Fuzzy Computing:**

**Intuitionistic Fuzzy Operations**

*×*

**Quantum Operators**

Renata Reiser[1](#_bookmark0), Alexandre Lemke[1](#_bookmark0), Anderson Avila[1](#_bookmark0), Ju´lia Vieira[1](#_bookmark0), Maur´ıcio Pilla[1](#_bookmark0), Andr´e Du Bois[1](#_bookmark0)

*Centro de Desenvolvimento Tecnol´ogico, CDTEC Universidade Federal de Pelotas*

*Pelotas, RS, Brazil*

Abstract

Quantum processes provide a parallel model for fuzzy connectives. Calculations of quantum states may be simultaneously performed by the superposition of membership and non-membership degrees of each element regarding the intuitionistic fuzzy sets. This work aims to interpret Atanassov’s intuitionistic fuzzy logic through quantum computing, where not only intuitionistic fuzzy sets, but also their basic operations and corresponding connectives (negation, conjuntion, disjuntion, difference, codifference, implication, and

coimplication), are interpreted based on the traditional quantum circuit model.

*Keywords:* Quntum fuzzy computing, quantum computing, intuitionistic fuzzy logic, fuzzy implications, fuzzy difference.

# Introduction

Intuitionistic fuzzy logic (IFL)[[1](#_bookmark23)] and quantum computing (CQ) are relevant re- search areas consolidating the analysis and the search for new solutions for difficult problems faster than the classical logical approach or conventional computing.

Similarities between these areas in the representation and modelling of uncer- tainty have been explored [[2](#_bookmark24),[3](#_bookmark25),[4](#_bookmark26)]. The uncertainty of human being’s reasoning is modelled in fuzzy logic (FL) and its extensions as the fuzzy intuitionistic logic (IFL).

1 Email: *{*reiser,alemke,abdavila,jkvieira,pilla, dubois*}*@inf.ufpel.edu.br

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By making use of the Fuzzy Set Theories (FSTs) as mathematical models inher- iting the imprecision of natural language and uncertainty from membership and non-membership degrees, which are not necessarily complementary, intuitionistic fuzzy techniques help physicists and mathematicians to transform their imprecise ideas into new computational programs [[5](#_bookmark27)].

The uncertainty of the real world is concerned with fundamental concepts of quantum computing by making use of properties of quantum mechanics as the su- perposition, suggesting an improvement in the efficiency regarding complex tasks. In addition, simulations using classical computers allow the development and valida- tion of basic quantum algorithms (QAs), anticipating the knowledge related to their behaviours when executed in a quantum computer. Quantum interpretations come from measurement operations performed on the corresponding quantum states.

In spite of quantum computers being restricted to a few research centers and laboratories, the studies from quantum information and quantum computation (QC) are a reality nowadays. In this context, new methods dealing with quantum fuzzy applications have been proposed [[3](#_bookmark25)]. In this work, the modelling and interpretation of intuitionistic fuzzy logic via quantum computing is considered, providing the description of representable fuzzy connectives by using pairs of quantum states and quantum registers from the traditional model of quantum circuits(qCs).

Some results of this research approach mainly related to interpretation of fuzzy connectives via quantum computing, as negation, conjuntion, disjuntion and impli- cations can be found in [[6](#_bookmark28),[7](#_bookmark29),[8](#_bookmark30),[9](#_bookmark31)] and more recently, by considering fuzzy exclusive or operators, in [[10](#_bookmark32)]. This work is the first step in order to extend this approach, towards an interpretation of intuitionistic fuzzy connectives from quantum comput- ing.

In this regard, this paper considers the interpretation of uncertainty described by the membership and non-membership degrees of IFSs defined by intuitionistic fuzzy connectives by quantum states and quantum operators.

Moreover, it contributes to increase the interest in quantum algorithm applica- tions representing IFSs operations, by exploring potentialities as quantum paral- lelism, entanglement and superposition of quantum states which can provide theo- retical foundation for modelling humanoid behaviour based on intuitionistic fuzzy logic [[11](#_bookmark33),[12](#_bookmark34),[13](#_bookmark35)].

This paper is organized as follows:

Section [2](#_bookmark1) presents the fundamental concepts and properties of connectives of FL and IFL. Moreover, *IFSs* can be obtained by intuitionistic fuzzy operators as presented in subsection [2.2](#_bookmark3). Section [3](#_bookmark12) brings the main concepts of *QC* which will be considered in the development of this work. In Section [4](#_bookmark15), a discussion about IFSs which can be obtained from quantum computing following the same methodology from previous work in order to model fuzzy operations from quantum registers. In Section [5](#_bookmark17), the approach for describing *IFSs* using the *QC* is depicted. An interpre- tation of classical intuitionistic fuzzy sets from quantum states is also considered, presenting the operations on *IFSs* modelled from quantum transformations, relat- ing the intuitionistic fuzzy approach to the difference and implication operators,

also including their dual constructions. Finally, conclusions and further studies are discussed in Section [6](#_bookmark22).

# Preliminaries

This section reports main concepts of fuzzy logic (FL) and its corresponding exten- sion, the Atanassov intuitionistic fuzzy logic (IFL).

* 1. *Fuzzy connectives*

FL connectives are studied from QC operators, overcoming the limitations related to quantum transitions as modelling smoothly their logical properties.

A **membership function** (MF) *fA*(*x*): *X →* [0*,* 1] determines the membership degree (*MD*) of the element *x∈ X* to the fuzzy set *A*, such that 0 *≤ fA*(*x*) *≤* 1. Thus, a **fuzzy set A** related to a set *X /*= *∅* is given as *A* = *{*(*x, fA*(*x*)) : *x ∈ X}*.

A function *N* : [0*,* 1] *→* [0*,* 1] is a **fuzzy negation** (FN) when the following holds:

[N1] *N* (0) = 1 and *N* (1) = 0; [N2] If *x ≤ y* then *N* (*x*) *≥ N* (*y*), for all *x, y ∈* [0*,* 1]. Fuzzy negations verifying the involutive property: [N3] *N* (*N* (*x*)) = *x*, for all *x ∈*

[0*,* 1], are called strong fuzzy negations. See, e.g., the standard negation *NS*(*x*) =

1 *− x*.

Fuzzy connectives can be represented by aggregation functions. Herein, we con- sider triangular norms (t-norms) and triangular conorms (t-conorms).

A **triangular (co)norm** is an operation (*S*)*T* : [0*,* 1]2 *→* [0*,* 1] such that, for all *x, y, z ∈* [0*,* 1], the commutative and associative properties are verified along with the fact that it is an increasing function with neutral element (0) 1 .

Among different definitions of t-norms and t-conorms [[14](#_bookmark36)], in this work we con- sider the *Algebraic Product* and *Algebraic Sum*, respectively given as:

*TP* (*x, y*)= *x · y*; and *SP* (*x, y*)= *x* + *y − x · y, ∀x, y ∈* [0*,* 1]*.* (1)

A binary function *I* : *U* 2 *→ U* is an implication operator (implicator) if the following conditions are satisfied:

I0: *I*(1*,* 1) = *I*(0*,* 1) = *I*(0*,* 0) = 1 and *I*(1*,* 0) = 0.

Additional properties are considered to define a fuzzy implication from an implica- tor:

A **fuzzy implication** *I* : *U* 2 *→ U* is an implicator verifying the antitonicity in the first argument, isotonicity in the second argument, falsity dominance in the antecedent and truth dominance in the consequent are verified.

In analogous manner, one can define a fuzzy implication *J* : *U* 2*→U* , which also satisfies the boundary conditions:

J0: *I*(1*,* 1) = *I*(1*,* 0) = *I*(0*,* 0) = 0 and *I*(0*,* 1) = 1.

A (*S, N* )**-(co)implication** is a fuzzy (co)implication *I*(*S,N* ) : *U* 2 *→ U* defined

by:

*I*(*S,N* )(*x, y*)= *S*(*N* (*x*)*, y*)*,* (2)

*J*(*T,N* )(*x, y*)= *T* (*N* (*x*)*, y*)*, ∀x, y ∈ U.* (3)

When *N* is involutive then *I*(*S,N* ) is an *S*-implication. The Reichenbach (co)implication is given as:

*IRB*(*x, y*)=1 *− x* + *xy,* (4)

*IRB*(*x, y*)= *x* + *y − xy, ∀x, y ∈ U,* (5)

is an *S*-implication (*T* -coimplication) obtained by a fuzzy negation *NS*(*x*)=1 *− x* and a t-conorm *SP* (*x, y*)= *x* + *y −x· y* (t-norm *TP* (*x, y*)= *xy*) previously presented in Eqs. ([1](#_bookmark2)a) and ([1](#_bookmark2)b), respectively.

* 1. *Intuitionistic fuzzy connectives*

An intuitionistic fuzzy set (IFS) *AI* in a non-empty and finite universe *X* , expressed as

*AI* = *{*(x*,* (*μAI* (x)*, νAI* (x))) : x *∈X , μAI* (x) + *νAI* (x)) *≤* 1*}*,

extending a fuzzy set *AI* = *{*(*x, μAI* (*x*)*,* 1 *− μAI* (*x*)) : x *∈ X}*, since the non- membership degree (NMD) *νAI* (*x*) of an element x *∈X* is less, at most equal to its complement, the membership degree (MD) *μAI* (x). Hence, it does not necessarily equal to one.

Let *U*˜ *⊂* [0*,* 1] *×* [0*,* 1], *U*˜ = *{x*˜ = (*x*1*, x*2) *∈ U*˜ : *x*1 + *x*2 *≤* 1*}* be the set of all pairs of MDs and NMDs and *lU*˜ *, rU*˜ : *U*˜ *→U* be two projection functions on *U*˜, which are given by *lU*˜ (*x*˜)= *lU*˜ (*x*1*, x*2)= *x*1 and *rU*˜ (*x*˜)= *rU*˜ (*x*1*, x*2)= *x*2, respectively.

Thus, for all **x**˜ = (*x*˜1*,..., x*˜*n*) *∈ U*˜*n*, such that *x*˜*i* = (*xi*1*, xi*2) and *xi*1 = *NS*(*xi*2)

when 1 *≤ i ≤ n*, let *lU*˜*n , rU*˜*n* : *U*˜*n × U*˜*n → U*˜*n* be the projections given by:

*lU*˜*n* (**x**˜)= (*x*11*, x*21*,... xn*1) and *rU*˜*n* (**x**˜)= (*x*12*, x*22*,... xn*2)*.* (6)

Consider the order relation *x*˜ *≤U*˜ *y*˜ given by *x*1 *≤ y*1 and *x*2 *≥ y*2 such that

˜0= (0*,* 1) *≤U*˜ *x*˜ and ˜1= (1*,* 0) *≥U*˜ *x*˜, for all *x*˜*, y*˜ *∈ U*˜ [[1](#_bookmark23)].

The complement, intersection and union, implication and coimplication, differ- ence and codifference operations are reported in the following.

* + 1. *Complement operation*

An *intuitionistic fuzzy negation* (IFN) is a function *NI* : *U*˜ *→ U*˜ verifying:

[N1I]: If *NI* (˜0) = *NI* (0*,* 1) = ˜1 and *NI* (˜1) = *NI* (1*,* 0) = ˜0;

[N2I]: If *x*˜ *≥U*˜ *y*˜ then *NI* (*x*˜) *≤U*˜ *NI* (*y*˜), for all *x*˜*, y*˜*∈ U*˜.

And, *NI* is a *strong intuitionistic fuzzy negation* (SIFN) if it is also an involutive function:

[N3I]: *NI* (*NI* (*x*˜)) = *x*˜, *∀x*˜ *∈ U*˜.

In this work, we consider the standard intuitionistic fuzzy negation expressed as

:

*NIS* (*x*˜)= *NIS* ((*x*1*, x*2)) = (*x*2*, x*1)*,* for all *x*˜ = (*x*1*, x*2) *∈ U*˜*.* (7)

**Definition 2.1** Let *NI* be an IFN and *AI* be IFS. The **complement of** *AI* with respect to *U*˜ is a *IFS* indicated as *AjI* and given by

*AjI* = *{*(x*, NI* (*μAI* (x)*, νAI* (x))) : x *∈ X}.* (8)

By the representation theorem [[15](#_bookmark37)], a SIFN *NI* on *U*˜ can be given as

*NI* (*μAI* (x)*, νAI* (x)) = (*N* (*NS*(*νAI* (x))*, NS*(*N* (*μAI* (x)))*.* (9)

Additionally, when *NS* = *N* in Eq.([9](#_bookmark4)), *NI* = *NIS*.

* 1. *Union and intersection operations*

A function (*SI* )*TI* : *U*˜2 *U*˜ is a fuzzy triangular (co)norm (t-(co)norm shortly), if it is a commutative, associative and increasing function with neutral element (˜0) ˜1, meaning that for all *x*˜*, y*˜*, z*˜ *∈ U*˜, the following properties hold:

*→*

[TI1]: *TI* (*x*˜*,* ˜1) = *x*˜; [SI2]: *SI* (*x*˜*,* ˜0) = *x*˜;

[TI2]: *TI* (*x*˜*, y*˜)= *TI* (*y*˜*, x*˜); [SI2]: *SI* (*x*˜*, y*˜)= *SI* (*y*˜*, x*˜));

[TI3]: *TI* (*x*˜*, TI* (*y*˜*, y*˜)) = *TI* (*TI* (*x*˜*, y*˜)*, y*˜); [SI3]: *SI* (*x*˜*, SI* (*y*˜*, z*˜)) = *SI* (*SI* (*x*˜*, y*˜)*, z*˜));

[TI4]: if *x*˜ *≤ x*˜*j*, *y*˜ *≤ y*˜*j*, *TI* (*x*˜*, y*˜) *≤ TI* (*x*˜*j, y*˜*j*). [SI4]: if *x*˜ *≤ x*˜*j*,*y*˜ *≤ y*˜*j*, *SI* (*x*˜*, y*˜) *≤ SS*(*x*˜*j, y*˜*j*).

Based on results of [[16](#_bookmark38), Definition 3], an intuitionistic t-norm (*SI* )*TI* : *U*˜2 *→ U*˜ is t-representable if there exist a t-norm *T* : *U* 2 *→ U* and a t-conorm *S* : *U* 2 *→ U* such that *T* (*x, y*) *≤ NS*(*S*(*NS*(*x*)*, NS*(*y*))), it is given as:

*TI* (*x*˜*, y*˜)= *TI* ((*x*1*, x*2)*,* (*y*1*, y*2)) = (*T* (*x*1*, y*1)*, S*(*x*2*, y*2)); (10)

*SI* (*x*˜*, y*˜)= *SI* ((*x*1*, x*2)*,* (*y*1*, y*2)) = (*S*(*x*1*, y*1)*,T* (*x*2*, y*2))*.* (11)

We consider both intuitionistic aggregations: the Product t-norm *TIP* and Alge- braic Sum *SIP* , respectively described by Eqs. ([12](#_bookmark5)) and ([13](#_bookmark5)), for all *x*˜ = (*x*1*, x*2)*, y*˜ = (*y*1*, y*2) *∈ U*˜ in the following:

*TIP* (*x*˜*, y*˜)= (*TP* (*x*1*, y*1)*, SP* (*x*2*, y*2)); (12)

*SIP* (*x*˜*, y*˜)= (*SP* (*x*1*, y*1)*, TP* (*x*2*, y*2))*.* (13)

**Definition 2.2** Let *SI, TI* : *U*˜2 *→ U*˜ be an intuitionistic fuzzy t-norm (IFT)

and t-conorm (IFS). The **intersection** and **union** between the *IFSs AI* and *BI* , both defined with respect to *x ∈ X* and such that *x*˜ = (*μAI* (x)*, νAI* (x))*, y*˜ = (*μBI* (x)*, νBI* (x)) *∈ U*˜, resulting in the corresponding intuitionistic fuzzy sets:

*AI ∩ BI* = *{*(x*,* (*μAI∩BI* (x)*, νAI∩BI* )(x))) : x *∈ X}*; (14)

*AI ∪ BI* = *{*(x*,* (*μAI∪BI* (x)*, μAI∩BI* (x))) : x *∈ X}.* (15)

By the t-representability [[15](#_bookmark37), Definiton 5] of an IFT and IFS in terms ofa t-norm

*T* and a t-conorm *S*, Eqs. ([14](#_bookmark6)a) and ([14](#_bookmark6)b) can be expressed as:

*AI ∩ BI* = *{*(x*,T* (*μAI* (x)*, μBI* (*x*))*, S*(*νAI* (*x*)*, νBI* (x))) : x *∈ X}*; (16)

*AI ∪ BI* = *{*(x*, S*(*μAI* (x)*, μBI* (x))*,T* (*νAI* (x)*, νBI* (x))) : x *∈ X}.* (17)

In the following, by taking *x*˜

= (*x*1*, x*2) and *y*˜

= (*y*1*, y*2) such that *μAI* (*x*) =

*x*1*, νAI* (*x*) = *x*2*, μBI* (*x*) = *y*1*, νAI* (*x*) = *y*2 *∈ U* the MFs and NMFs related to the

intersection and union of IFSs *AI* and *BI* in Eqs.([16](#_bookmark6)) and ([17](#_bookmark6)) are, respectively, given as:

*μAI∩BI* (x) = *x*1 *· y*1; *νAI∩BI* (x) = *x*2 + *y*2 *− x*2 *· y*2; (18)

*μAI∪BI* (x) = *x*1 + *y*1 *− x*1 *· y*1; *νAI∪BI* (x) = *x*2 *· y*2*.* (19)

* + 1. *Difference and codifference operations*

According with [[17](#_bookmark39)], the difference between IFSs *A* and *B* is given by

*A* *− B* = *{*(*μA−B*(*x*)*, νA−B*(*x*))*|x ∈ χ,* 0 *≤ μA−B*(*x*)+ *νA−B*(*x*) *≤* 1*}*

where *μA−B, νA−B* : *χ → U* are the membership and non-membership function of *x ∈ χ* in the A-IFS *A − B*. By [[17](#_bookmark39), Definition 3] an A-IFS *A − B* can be expressed as:

*DI* ((*μA*(*x*)*, νA*(*x*))*,* (*μB*(*x*)*, νB*(*x*))) = (*μA−B*(*x*)*, νA−B*(*x*))*.*

The next definition extends the results in [[18](#_bookmark40)]:

**Definition 2.3** [[17](#_bookmark39), Definition 4] *DI* (*EI* ) : *U*˜2 *→ U*˜ is an Atanassov intuitionistic

fuzzy (co)difference (A-IFD (A-IFE)) if it satisfies the following axioms:

[DI1]: *DI* (*x*˜*, y*˜) *≤ x*˜; [EI1]: *EI* (*x*˜*, y*˜) *≥ x*˜;

[DI2]: *DI* (*x*˜*,* ˜0) = *x*˜; [EI2]: *EI* (*x*˜*,* ˜1) = *x*˜;

[DI3]: If *y*˜ *≤ z*˜, *DI* (*x*˜*, y*˜) *≥ DI* (*x*˜*, z*˜); [EI3]: If *y*˜ *≤ z*˜, *EI* (*x*˜*, y*˜) *≥ EI* (*x*˜*, z*˜);

[DI4]: If *x*˜ *≤ y*˜, *DI* (*x*˜*, z*˜) *≤ DI* (*y*˜*, z*˜)); [EI4]: If *x*˜ *≤ y*˜, *EI* (*x*˜*, z*˜) *≤ EI* (*y*˜*, z*˜));

[DI5]: *DI* (˜1*, x*˜)= *NI* (*x*˜), *NI* is a IFN; [EI5]: *EI* (˜0*, x*˜)= *NI* (*x*˜), *NI* is a IFN.

**Proposition 2.4** *For all x*˜*, y*˜ *∈ U, the operator DI* 1(*EI* 1): *U* 2 *→ U given by*

*DI* 1(*x*˜*, y*˜)= (*TI* (*x*˜*, NI* (*y*˜)); (20)

*EI* 1(*x*˜*, y*˜)= (*SI* (*x*˜*, NI* (*y*˜))*.* (21)

*is an intuitionistic fuzzy (co)difference operator in the sense of Proposition*[*2.3*](#_bookmark7)*.*

**Proof** Straightforward. *2*

**Proposition 2.5** *Let TI(SI) be a representable A-IFT (A-IFS). A function DI* (*EI* ) : *U* 2 *→ U in Eq.(*[*20*](#_bookmark8)*) and (*[*21*](#_bookmark8)*) can be expressed, for all x*˜ = (*x*1*, x*2)*, y*˜ = (*y*1*, y*2) *∈ U*˜*, as follows:*

*DI* (*x*˜*, y*˜)= (*T* (*x*1*, y*2)*, S*(*x*2*, y*1)); (22)

*EI* (*x*˜*, y*˜)= (*S*(*x*1*, y*2)*,T* (*x*2*, y*1))*.* (23)

**Proof** It follows from Eq.([10](#_bookmark5)a) (Eq.([10](#_bookmark5)b)). *2*

**Example 2.6** By Eqs.([10](#_bookmark5)) and ([11](#_bookmark5)) and results in Proposition [2.5](#_bookmark9), we have that:

*DTP ,NS I* (*x*˜*, y*˜)= (*x*1*y*2*, x*2 + *y*1 *− x*2*y*1); (24)

*ESP ,NSI* (*x*˜*, y*˜)= (*x*1 + *y*2 *− x*1*y*2*, x*2*y*1)*,* (25)

is a representable A-IFT (A-IFS) obtained by the t-(co)norm *TP* (*SP* ) and standard negation *NS*.

* + 1. *Fuzzy intuitionistic (co)implications*

The Atanassov’s intuitionistic approach of fuzzy (co)-implications is considered in the following, discussing properties and projection functions in order to define rep- resentable fuzzy (co)-implications.

**Definition 2.7** An **intuitionistic fuzzy (co)implicator** ((A-IFC) A-IFI) (*JI* )*II* :

*U*˜2 *→ U*˜ is a binary function verifying the boundary conditions:

[II1]: *II* (˜0*,* ˜0) = *II* (˜0*,* ˜1) = *II* (1˜*,* ˜1) = ˜1 and *II* (˜1*,* ˜0) = ˜0; [JI1]: *JI* (˜0*,* ˜0) = *JI* (˜1*,* ˜0) = *JI* (˜1*,* ˜1) = ˜0 and *JI* (˜0*,* ˜1) = ˜1;

Definition ([2.7](#_bookmark11)) can be reduced to a fuzzy (co)implication if *x*˜ = (*x*1*, x*2) and *y*˜ = (*y*1*, y*2) *∈ U*˜, such that *x*1 = *NS*(*x*2) e *y*1 = *NS*(*y*2) [[1](#_bookmark23)]. Intuitionistic fuzzy (co)-implications are defined in the sense of J. Fodor and M. Roubens [[19](#_bookmark41),[20](#_bookmark42)].

**Definition 2.8** An ((A-IFC) A-IFI) (*JI* )*II* : *U* 2 *→ U*˜

˜

the conditions described as follows:

satisfies, for all *x*˜*, y*˜*, z*˜ *∈ U*˜,

[II2]: *x*˜ *≤ z*˜*⇒ II* (*x*˜*, y*˜) *≥ II* (*z*˜*, y*˜); [JI2:] *x*˜ *≤ z*˜*⇒ JI* (*x*˜*, y*˜) *≥ JI* (*z*˜*, y*˜);

[II3]: *y*˜*≤ z*˜*⇒ II* (*x*˜*, y*˜) *≤ II* (*x*˜*, z*˜); [JI3]: *y*˜*≤ z*˜*⇒ JI* (*x*˜*, y*˜) *≤ JI* (*x*˜*, z*˜); [II4]: *II* (˜0*, y*˜)= ˜1; [JI4:] *JI* (˜1*, y*˜)= ˜0;

[II5]: *II* (*x*˜*,* ˜1) = ˜1; [JI5]: *JI* (*x*˜*,* ˜0) = ˜0;

In this work, for all *x*˜ = (*x*1*, x*2)*, y*˜ = (*y*1*, y*2) *∈ U*˜, we consider the Reichenbach intuitionistic fuzzy S-(co)implication:

*IRCI* (*x*˜*, y*˜)= *SP I* (*NSI* (*x*˜)*, y*˜)= (*x*2 + *y*1 *− x*2*y*1*, x*1*y*2) (26)

*JRCI* (*x*˜*, y*˜)= *TP I* (*NSI* (*x*˜)*, y*˜)= (*x*2*y*1*, x*1 + *y*2 *− x*1*y*2) (27)

which can be defined based on Eqs.([4](#_bookmark2)) and ([5](#_bookmark2)), respectively.

# Obtaining intuitionistic fuzzy sets from quantum computing

Firstly, quantum computing concepts are reported.

* 1. *Basic concepts of quantum computing*

In *QC*, the *qubit* is the basic information unit, being the simplest quantum sys- tem, defined by a unitary and bi-dimensional state vector. Qubits are generally described, in Dirac’s notation [[21](#_bookmark43)], by *|ψ⟩* = *α|*0*⟩* + *β|*1*⟩*, and the coefficients *α*

and *β* are complex numbers for the amplitudes of the corresponding states in the computational basis (state space), respecting the condition *|α|*2 + *|β|*2 = 1, which guarantees the unitary of the state vectors of the quantum system, represented by (*α, β*)*t* [[22](#_bookmark44)].

The state space of a quantum system with multiple *qubits* is obtained by the tensor product of the space states of its subsystems. Considering a quantum system with two *qubits*, *|ψ⟩* = *α|*0*⟩*+*β|*1*⟩* and *|ϕ⟩* = *γ|*0*⟩*+*δ|*1*⟩*, the state space comprehends the tensor product given by

*|*Π*⟩* = *|ψ⟩⊗ |ϕ⟩* = *α · γ|*00*⟩* + *α · δ|*01*⟩* + *β · γ|*10*⟩* + *β · δ|*11*⟩.*

The state transition of a quantum systems is performed by controlled and unitary transformations associated with orthogonal matrices of order 2*N* , with *N* being the number of *qubits* within the system, preserving norms, and thus, probability amplitudes [[23](#_bookmark45)].

For instance, the definition of the *Pauly X* transformation and its application over a one-dimensional and two-dimensional quantum systems are presented in Eq. ([28](#_bookmark13)).

⎛ 0 0 0 1 ⎞ ⎛ *α · γ* ⎞ ⎛ *α · γ* ⎞

*X|ψ⟩* = ⎛ 0 1 ⎞ ⎛ *α* ⎞ = ⎛ *β* ⎞ ; *X⊗*2*|*Π*⟩*=⎜ 0 0 1 0 ⎟ ⎜ *α · δ* ⎟ = ⎜ *α · δ* ⎟

⎝ 1 0 ⎠ ⎝ *β* ⎠

⎝ *α* ⎠

0 1 0 0 *β γ*

⎜⎝ ⎟ ⎜ *·* ⎟

1 0 0 0 ⎠ ⎝ *β · δ* ⎠

*β δ*

⎝ *β · γ* ⎠

⎜ *·* ⎟

(28)

Furthermore, a Toffoli transformation is also shown in Eq. ([3.1](#_bookmark13)), describing a controlled operation for athree dimensional quantum system. In this case, the *NOT* operator (*Pauly X* ) is applied to the third *qubit |σ⟩* when the current states of the first two *qubits |ψ⟩* and *|ϕ⟩* are both *|*1*⟩*.

Similarly to qTs of multiple *qubits* which were obtained by the tensor product performed over unitary transformations, Eq. ([3.1](#_bookmark13)) presents the matrix structure defining such transformation, when *|X⟩* is the initial state:

In order to obtain information from a quantum system, it is necessary to apply measurement operators, defined by a set of linear operators *Mm*, called projections. The index *m* refers to the possible measurement results. If the state of a quantum system is *|ψ⟩* immediately before the measurement, the probability of an outcome

occurrence is given by *p*(*|ψ⟩*) = *√*

*Mm|ψ⟩*

*⟨ψ|MmMm|ψ⟩*

*†*

. When measuring a *qubit |ψ⟩* with

*α, β /*= 0, the probability of observing *|*0*⟩* and *|*1*⟩* are, respectively, given by the

following expressions:

*p*(0)=*⟨φ|M*0*†M*0*|φ⟩*=*⟨φ|M*0*|φ⟩*=*|α|*2 and *p*(1)=*⟨φ|M*1*†M*1*|φ⟩*=*⟨φ|M*1*|φ⟩*=*|β|*2*.*

⎛ 1 0 0 0 0 0 0 0 ⎞ ⎛ *α* ⎞ ⎛ *α* ⎞

0 1 0 0 0 0 0 0 *β β*

⎜ ⎟ ⎜ ⎟ ⎜ ⎟

0 0 1 0 0 0 0 0 *γ γ T* = 0 0 0 1 0 0 0 0 *δ* = *δ* 0 0 0 0 1 0 0 0 *ϵ ϵ*

⎜ ⎟ ⎜ ⎟ ⎜ ⎟

*|X⟩* ⎜ ⎟ ⎜ ⎟ ⎜ ⎟

⎜ ⎟ ⎜ ⎟ ⎜ ⎟

0 0 0 0 0 1 0 0 *θ θ*

⎜ ⎟ ⎜ ⎟ ⎜ ⎟

0 0 0 0 0 0 0 1 *υ σ*

⎜⎜⎝ ⎟⎟ ⎜⎜ ⎟⎟ ⎜⎜ ⎟⎟

0 0 0 0 0 0 1 0 ⎠ ⎝ *σ* ⎠ ⎝ *υ* ⎠

After the measuring process, the quantum state *|ψ⟩* has *|α|*2 as the probability to be in the state *|*0*⟩* and *|β|*2 as the probability to be in the state *|*1*⟩*.

* 1. *Interpreting FS Operations from Quantum Transformations*

According to [[4](#_bookmark26)], fuzzy sets can be interpreted by quantum superposition of classical fuzzy sets (CFSs) associated with quantum states. Additionally, interpretations for fuzzy operations such as complement, intersection and union are obtained from the *NOT* , *AND* and *OR* quantum transformations.

In order to simplify the paper notation, the *MD* defined by *μA*(x), which is related to an element x *∈X* in the *FS A*, will be denoted by *μA*.

The modelling of fuzzy complement, intersection and union (*NOT* , *AND* and *OR* operators) are defined in [[10](#_bookmark32)] and summarized below. For that, let *μA, μB* : *X →* [0*,* 1] be membership functions (MFs) related to *FSs A* and *B*.

The corresponding pair (*|SμA ⟩, |SμB ⟩*) of CFSs is given as:

*|SμA ⟩* = √*μA*(x)*|*1*⟩* + √1 *− μA*(x)*|*0*⟩,* (29)

√ √

*|SμB ⟩* = *μB*(x)*|*1*⟩* + 1 *− μB*(x)*|*0*⟩.* (30) respectively.

The **complement of a** *FS* is performed by the standard negation, which is obtained by the ***NOT* operator**, based on Eq.([9](#_bookmark4)) and defined as

*NOT* (*|SμA ⟩*)= (√*μA*(x)*|*0*⟩* + √1 *− μA*(x)*|*1*⟩*) (31)

√ √

Moreover, when *|SμA ⟩* = *⊗*1*≤i≤N* 1 *− μA*(x*i*)*|*0*⟩* + *μA*(x*i*)*|*1*⟩* the complement

operator *NOTN* preserves an *N* -dimensional quantum superposition of 1-*qubit* states described as *C*2 in the computational basis, and expressed by Eq. ([32](#_bookmark14)) below:

*N*

*NOTN* (*|Sμ ⟩*)= *⊗*1*≤i≤N* √*μA*(x*i*)*|*0*⟩* + √1 *− μA*(x*i*)*|*1*⟩* (32)

*A*

Let *|SμA ⟩* and *|SμB ⟩* be quantum states given by Eqs. ([29](#_bookmark14)) and ([30](#_bookmark14)).

1. The **intersection** and **union** of FSs *A* and *B* are modelled by the ***AND*** and

***OR*** operators which are respectively expressed through the *Toffoli* transforma- tion as below:

*AND*(*|SμA ⟩, |SμB ⟩*)=*T* (*|SμA ⟩, |SμB ⟩, |*0*⟩*) (33)

*OR*(*|SμA ⟩, |SμB ⟩*)=*NOT* (*T* (*NOT|SμA ⟩,N OT|SμB ⟩, |*0*⟩*))*.* (34)

3

1. The fuzzy implication and coimplication of FSs *A* and *B*, are modelled by the ***IMP*** and ***COIMP*** operators which are respectively expressed through compo- sition of NOT and *Toffoli* transformations as in the following:

*IMP* (*|SμA ⟩, |SμB ⟩*)=*T* (*|SμA ⟩,N OT|SμB ⟩, |*1*⟩*)*.* (35)

*COIMP* (*|SμA ⟩, |SμB ⟩*)=*T* (*NOT|SμA ⟩,|SμB ⟩, |*0*⟩*))*.* (36)

1. The fuzzy difference and codifference of FSs *A* and *B*, are also modelled by the ***DIF*** and ***CODIF*** operators respectively expressed through composition of NOT and *Toffoli* transformations as follows:

*DIF* (*|SμA ⟩, |SμB ⟩*)=*T* (*|SμA ⟩,N OT|SμB ⟩, |*0*⟩*))*.* (37)

*CODIF* (*|SμA ⟩, |SμB ⟩*)=*T* (*NOT|SμA ⟩,|SμB ⟩, |*1*⟩*))*.* (38)

# Interpreting CIFs from quantum states

The description of *IFSs* from the *QC* viewpoint extends the work in [[4](#_bookmark26)] by modelling an element *x*˜ *∈ AI* which is given by *x*˜ = (*μAI* (x)*, νAI* (x)), such that x *∈ χ /*= *∅*, by a pair of quantum register (*|SμAI ⟩, |SνAI ⟩*) and fuzzy operators by quantum transformations.

Let *X* be a non-empty subset with cardinality *N* , meaning that *X /*= *∅*, *|X |* = *N* and *i ∈ NN* = *{*1*,* 2*, ..., N}*. Let *μAI , νAI* : *X →* [0*,* 1] *×* [0*,* 1] be the MF and NMF related to an element x*i ∈ X* in the *IFSs AI* , respectively, and *μAI* (x*i*)*, νAI* (x*i*) be their corresponding MD and NMD.

**Definition 4.1** A **classical intuitionistic fuzzy set** (CIFS) of N-*qubits* is a pair of *N* -dimensional quantum states, given by

(*|SμAI ⟩, |SνAI ⟩*)=

=⎛ 1*−μAI* (x*i*)*|*0*⟩*+ *μAI* (x*i*)*|*1*⟩* *,* 1*−νAI* (x*i*)*|*0*⟩*+ *νAI* (x*i*)*|*1*⟩* ⎞(39)

⎝1*≤i≤N*

such that *μAI* (x*i*)+ *νAI* (x*i*) *≤* 1.

1*≤i≤N* ⎠

A pair of CIFSs of N-*qubits* is an *N* -dimensional quantum state given by Eq.([39](#_bookmark16)). Taking *N*1, *μAI* (x1) = *x*1, *νAI* (x1) = *x*2 and *x*1*, x*2 *∈*]0*,* 1[, the quantum states corresponding to *IFSs* are obtained and expressed as

*|SμAI ⟩, |SνAI ⟩* = *√x*1*|*1*⟩* + *√*1 *− x*1*|*0*⟩, √x*2*|*1*⟩* + *√*1 *− x*2*|*0*⟩* (40) Moreover, by taking *N*2 and *μAI* (x*i*) = *x*1*i, νAI* (x*i*) = *x*2*i ∈*]0*,* 1[, the related

superposition of bi-dimensional quantum states is given by

*|SμAI ⟩, |SνAI ⟩*

and

*|SμAI ⟩* = (*√x*11*|*1*⟩* + *√*1 *− x*11*|*0*⟩*) *⊗* (*√x*12*|*1*⟩* + *√*1 *− x*12*|*0*⟩*)=

=√(1 *− x*11)(1 *− x*12)*|*00*⟩*+√(1 *− x*11)*x*12*|*01*⟩*+√*x*11(1 *− x*12)*|*10*⟩*+*√x*11*x*12*|*11*⟩*;

*S* = (*√x* 1 + *√*1 *x* 0 ) (*√x* 1 + *√*1 *x* 0 )

*| νAI ⟩* 21*| ⟩ −* 21*| ⟩ ⊗* 22*| ⟩ −* 22*| ⟩*

=√(1 *− x*21)(1 *− x*22)*|*00*⟩* + √(1 *− x*21)*x*22*|*01*⟩* + √*x*21(1 *− x*22)*|*10*⟩* + *√x*21*x*22*|*11*⟩.*

An application of an IMF *fI* to each element in the image-set *fI* [*X* ] defines a quantum state. Thus, a canonical orthonormal basis in *⊗N C× ⊗N C* denotes a pair of classical quantum registers of *N* -*qubits*, meaning that *μAI* (x*i*)*, νAI* (x*i*) *∈ {*0*,* 1*}* in Eq.([39](#_bookmark16)).

# Interpreting IFS operations from quantum transfor- mations

In the following, the composition of QTs applied to quantum registers also in- cludes an interpretation based on results obtained by the quantum measurement operations. Moreover, fuzzy operators are considered to obtain the corresponding modelling of IFS operation as a composition of QTs applied to quantum registers

* 1. *Modelling IFS complement operator by QTs*

The **complement operator** *NOTIN* applied to the state in an *N* -dimensional quantum superposition of 1-*qubit* states as described in Eq.([39](#_bookmark16)), is given by

*NOTIN* (*|Sμ*

=⎛ 1*−νAI* (x*i*)*|*0*⟩*+ *νAI* (x*i*)*|*1*⟩* *,* 1*−μAI* (x*i*)*|*0*⟩*+ *μAI* (x*i*)*|*1*⟩* ⎞(41)

*AI*

*⟩, |SνAI*

*⟩*)=

⎝1*≤i≤N* 1*≤i≤N* ⎠

By restricting a CIFs as the pair of one-dimentional qubits as given in Eq.([40](#_bookmark16)), the

**complement operator** of an *IFS AI* is modelled by the *NOTI* operator as:

*NOTI* *|SμAI ⟩, |SνAI ⟩* = *|SνAI ⟩, |SμAI ⟩* (42)

* 1. *Modelling IFS intersection operation by QTs*

Let *AI* and *BI* be IFSs both defined by *μAI* (x) = *x*1, *νAI* (x) = *x*2 and *μBI* (x) = *y*1, *μBI* (x) = *y*1 for x *∈ X* . Consider the pairs of CIFSs of one-dimensional quantum registers given by Eqs. ([43](#_bookmark18)) and ([44](#_bookmark18)) in the following:

*|SμAI ⟩, |SνAI ⟩* = *√x*1*|*1*⟩* + *√*1 *− x*1*|*0*⟩, √x*2*|*1*⟩* + *√*1 *− x*2*|*0*⟩* (43)

*|SμBI ⟩, |SνBI ⟩* = *√y*1*|*1*⟩* + √1 *− y*1*|*0*⟩, √y*2*|*1*⟩* + √1 *− y*2*|*0*⟩* (44)

By making use of the Toffoli transformation, the **intersection** *AI ∩ BI* between the *IFSs AI* and *BI* , is defined by the *ANDI* operator and described in the fol- lowing:

*ANDI* *|SμAI ⟩, |SνAI ⟩* *,* *|SμBI ⟩, |SνBI ⟩*

= *AND* *|SμAI ⟩, |SμBI ⟩* *, OR* *|SνAI ⟩, |SνBI ⟩* (45)

Expanding the components of Eq.([45](#_bookmark18)) based on the fuzzy operators in Eqs.([33](#_bookmark14)) and ([34](#_bookmark14)), we firstly consider the *AND* operator also expressed through the *Toffoli* quantum transformation as given in the next expressions:

*AND* *|SμAI ⟩, |SμBI ⟩* = *T* *|SμAI ⟩, |SμBI ⟩, |*0*⟩* =

=*T* (*√x*1*|*1*⟩* + *√*1 *− x*1*|*0*⟩*) *⊗* (*√y*1*|*1*⟩* + √1 *− y*1)*|*0*⟩*) *⊗ |*0*⟩*)

=(*√*1 *−x*1√1*−y*1*|*000*⟩*+(*√*1*−x*1*√y*1*|*010*⟩*+(*√x*1√1*−y*1*|*100*⟩*+(*√x*1*√y*1*|*111*⟩*)(46)

Thus, a measurement performed over the third *qubit* (*|*1*⟩*) in the first component of the final quantum state, which is expressed by Eq. ([51](#_bookmark20)), provides the output with probability *pAND*(1) = *x*1 *· y*1.

Analogously, a measurement of such third *qubit* (*|*0*⟩*) in Eq. ([51](#_bookmark20)), returns an output state given with probability *pAND*(0) = 1 *− x*1 *· y*1.

Now, by applying the composition between *NOT* 3 and Toffoli operators we are able to express *OR*(*|SνAI ⟩, |SνAI ⟩*) with similar calculations as the following:

3

*OR*(*|SνAI ⟩, |SνAI ⟩*)= *NOT T* (*NOT* (*|Sν ⟩*)*,NOT* (*|SνAI ⟩*)*, |*0*⟩*) =

√

*AI*

=*NOT* 3(*T* ((*√x*2*|*0*⟩* + *√*1 *− x*2*|*1*⟩*) *⊗* (*√y*2*|*0*⟩* + 1 *− y*2*|*1*⟩*) *⊗ |*0*⟩*))

√ √ √

=*√x*2*y*2*|*001*⟩*+ *x*2(1*−y*2)*|*011*⟩*+ (1*−x*2)*y*2*|*101*⟩*+ (1*−x*2)(1*−y*2)*|*110*⟩* (47)

Observing a measure of the quantum state given in Eq. ([50](#_bookmark20)) performed on the third *qubit* :

1. related to *|*1*⟩*, returns *√*

1

*x*2+*y*2*−x*2*y*2

*√x*2*y*2*|*001*⟩*+√*x*2(1*−y*2)*|*011*⟩*+√(1*−x*2)*y*2*|*101*⟩*

and the corresponding probability *pOR*(1) = *x*2 + *y*2 *− x*2 *· y*2.

1. related to *|*0*⟩* results on *|*110*⟩* with probability *pOR*(0) = (1 *− x*2) *·* (1 *− y*2). Based on Eqs. ([51](#_bookmark20)) and ([50](#_bookmark20)), it results in the state with corresponding compo-

nents probabilities (*pAND*(1)*, pOR*(1)) which is related to Eq. ([18](#_bookmark6)),

(*μAI∩BI* (x)*, νAI∩BI* (x)) = (*x*1 *· y*1*, x*2 + *y*2 *− x*2 *· y*2) interpreting MD of *x ∈X* belongs to the intersection *AI ∩ BI* .

* 1. *Modelling IFS union operation by QTs*

The **union** *AI ∪ BI* between the *IFSs AI* and *BI* both defined with respect to *U*˜, is analogously obtained. It is modelled by the *ORI* operator described as:

*ORI* *|SμAI ⟩, |SνAI ⟩* *,* *|SμBI ⟩, |SνBI ⟩* =

*OR* *|SμAI ⟩, |SμBI ⟩* *, AN D* *|SνAI ⟩, |SνBI ⟩* (48)

Similar to *ANDI* operator, the *ORI* operator described in Eq.([48](#_bookmark19)) can also be obtained based on Eqs. ([51](#_bookmark20)) and ([50](#_bookmark20)), resulting the state with corresponding

components probabilities: (*pOR*(1)*, pAND*(1)) which is related to Eq. ([19](#_bookmark6)) and given as follows:

(*μAI∪BI* (x)*, νAI∪BI* (x)) = (*x*1 + *y*1 *− x*1 *· y*1*, x*2 *· y*2) interpreting the MD and NMD of an element *x ∈X* in the union *AI ∪ BI* .

* 1. *Modelling IFS implication operation by QTs*

By making use of the Toffoli transformation, the **implication** *AI → BI* between the

*IFSs AI* and *BI* , is defined by the *IMPI* operator and described in the following:

*IMPI*  *|SμAI ⟩, |SνAI ⟩* *,* *|SμBI ⟩, |SνBI ⟩*

= *OR* *|SνAI ⟩, |SμBI ⟩* *, AN D* *|SμAI ⟩, |SνBI ⟩* (49)

Expanding the components of Eq.([52](#_bookmark21)), we firstly consider the *OR* operator also expressed through the *Toffoli* QT as given in next expression:

*OR*(*|SνAI ⟩, |SμBI ⟩*)= *T* (*NOT* (*|SνAI ⟩*)*,NOT* (*|SμBI ⟩*)*, |*1*⟩*)=

=*T* ((*√x*2*|*0*⟩* + *√*1 *− x*2*|*1*⟩*) *⊗* (*√y*1*|*0*⟩* + √1 *− y*1*|*1*⟩*) *⊗ |*1*⟩*))

=*√x*2*y*1*|*001*⟩*+√*x*2(1*−y*1)*|*011*⟩*+√(1*−x*2)*y*1*|*101*⟩*+√(1*−x*2)(1*−y*1)*|*110*⟩* (50) Observing a measure of the quantum state in Eq. ([50](#_bookmark20))

*√x*2*y*1*|*001*⟩*+

performed

*√*

on the

third *qubit*

√ √

*x*2(1*−y*1)*|*011*⟩*+

(1*−x*2)*y*1*|*101*⟩*

related t o *|*1*⟩*, returns

and the corre-

sponding probability *pOR*(1) = *x*2 + *y*1 *− x*2 *· y*1. And, a measure operation related

to *|*0*⟩* results on *|*110*⟩* with probability *pOR*(0) = (1 *− x*2) *·* (1 *− y*1).

1

*x*2+*y*1*−x*2*y*1

Now, by applying Toffoli QT we express *AND*(*|SμAI ⟩, |SνBI ⟩*) in a similar way:

*AND* *|SμAI ⟩, |SνBI ⟩* = *T* *|SμAI ⟩, |SνBI ⟩, |*0*⟩* =

=*√*1*−x*1√1*−y*2*|*000*⟩*+*√*1*−x*1*√y*2*|*010*⟩*+*√x*1√1*−y*2*|*100*⟩*+(*√x*1*√y*2*|*111*⟩* (51)

Thus, a measurement performed over the third *qubit* (*|*1*⟩*) in the first component of the final quantum state, which is expressed by Eq. ([51](#_bookmark20)), provides the output with probability *pAND*(1) = *x*1 *· y*2. Analogously, a measurement of such third *qubit* (*|*0*⟩*) in Eq. ([51](#_bookmark20)), returns an output state given with probability *pAND*(0) = 1 *− x*1 *· y*2.

Based on Eqs. ([51](#_bookmark20)) and ([50](#_bookmark20)), it results in the state with corresponding components probabilities (*pOR*(1)*, pAND*(1)) such that (*μAI→BI* (x)*, νAI→BI* (x)) = (*x*2 + *y*1 *− x*2 *· y*1*, x*1 *· y*2) interpreting MD of *x ∈ X* belongs to the **intuitionistic implication** *AI → BI* .

* 1. *Modelling IFS coimplication operation by QTs*

By making use of the Toffoli transformation, the **co-implication** *AI → BI* between the *IFSs AI* and *BI* , is defined by the *COIMPI* operator and described in the following:

*COIMPI* *|SμAI ⟩, |SνAI ⟩* *,* *|SμBI ⟩, |SνBI ⟩* =

*AND* *|SνAI ⟩, |SμBI ⟩* *, OR* *|SμAI ⟩, |SνBI ⟩* (52) Finally, analogous calculations result in the state with corresponding compo-

nents probabilities (*pAND*(1)*, pOR*(1)) such that

(*μAI→BI* (x)*, νAI→BI* (x)) = (*x*2 *· y*1*, x*1 + *y*2 *− x*1 *· y*2)

= (*pAND*(0)*, pOR*(0))

interprets MD of *x ∈X* belongs to the **intuitionistic co-implication** *AI → BI* .

* 1. *Modelling IFS difference operation by QTs*

The **difference** *AI − BI* between *IFSs AI* and *BI* both defined with respect to *U*˜, is analogously obtained. It is modelled by the *DIFI* operator described as:

*DIFI* *|SμAI ⟩, |SνAI ⟩* *,* *|SμBI ⟩, |SνBI ⟩* =

*AND* *|SμAI ⟩, |SνBI ⟩* *, OR* *|SνAI ⟩, |SμBI ⟩* (53)

Therefore, a measure performed over the operator *DIFI* results on state with corresponding components probabilities (*pAND*(1)*, pOR*(1)) which is related to Eq. ([24](#_bookmark10)) and given as follows,

(*pAND*(0)*, pOR*(0)) = (1 *− x*1 *· y*1*,* 1 *− x*2 *− y*2 + *x*2 *· y*2)

= (*μ*(*A−B*)(*x*)*, ν*(*A−B*)(*x*)) (54)

interpreting the MD and NMD of an element *x ∈X* in the difference *AI − BI* .

* + 1. *Modelling IFS codifference operation by QTs*

The **codifference** operation *AI c BI* between *IFSs AI* and *BI* both defined with respect to *U*˜, is analogously obtained. It is modelled by the *CODIFI* operator described as:

*—*

*CODIFI* *|SμAI ⟩, |SνAI ⟩* *,* *|SμBI ⟩, |SνBI ⟩* =

*OR* *|SμAI ⟩, |SνBI ⟩* *, AN D* *|SνAI ⟩, |SμBI ⟩* (55) Analogously the above operators, the measure performed over the

*CODIFI* operator results on state with corresponding components probabilities

(*pOR*(1)*, pAND*(1)) which is related to Eq. ([25](#_bookmark10)) as the follows:

(*pAND*(0)*, pOR*(0)) = (1 *− x*1 *· y*1*,* 1 *− x*2 *− y*2 + *x*2 *· y*2)

= (*μ*(*A−cB*)(*x*)*, ν*(*A−cB*)(*x*)) (56)

It provides interpretation to the MD and NMD of an element *x ∈ X* in the codif- ference *AI −c BI* .

# Conclusion and Final Remarks

This work is mainly focussed on the interpretation of Atanassov’s intuitionistic fuzzy logic via *QC*, where not only the intuitionistic fuzzy sets but also complement, inter- section, union, difference and codifference operations are interpreted based on the quantum circuit model, including IFSs obtained by representable (co)implications. Further work aims to consolidate this specification including not only other fuzzy connectives but also constructors (e.i. automorphisms and reductions) and the corresponding extension of (de)fuzzyfication methodology from formal structures

provided by *QC*.

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