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Jordan Areas and Grids

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**Abstract**

Jordan curves can be used to represent special subsets of the Euclidean plane, either the (open) interior of the curve or the (compact) union of the interior and the curve itself . We compare the latter with other representations of compact sets using grids of points and we are able to show that knowing the length of a rectifiable curve is sufficient to translate from the grid representation to the Jordan curve.

*Keywords:* representations of compact sets, Jordan curves, Jordan-Schoenflies theorem, TTE

# 1 Introduction

In computable analysis, several definitions of computability of subsets of R*n* have been discussed in the near past, especially for bounded sets, e.g. [[6](#_bookmark15),[7](#_bookmark16),[8](#_bookmark17)], [[13](#_bookmark20)], [[3](#_bookmark12)], [[16](#_bookmark21),[17](#_bookmark26)].

Many of theses definitions apply at the same time to a set *S* and its closure *S*, so we will restrict ourselves to compact (i.e. bounded and closed) sets. Here a representation using grids of points with a decreasing Hausdorff distance to the represented set is of interest.

For the special case of the Euclidean plane, another non-equivalent representa- tion of special compact sets can be based on Jordan curves, although the papers

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by Chou, Ko and Yu [[6](#_bookmark15)],[[7](#_bookmark16)],[[8](#_bookmark17)],[[13](#_bookmark20)] rather use the curves to represent their open interior.

During the CCA conference 2007, two independent papers [[14](#_bookmark22)],[[17](#_bookmark26)] have been presented showing results that were quite similar although based on the non-equivalent approaches via grids or Jordan curves. In this paper we show that using the (finite) length of a Jordan curve as additional information allows the translation between the corresponding representations. This helps to explain the similarities. In con- trast to [[6](#_bookmark15)],[[7](#_bookmark16)],[[8](#_bookmark17)],[[13](#_bookmark20)] where the emphasis was on computational complexity, we concentrate on uniform solutions, i.e. on possible reductions between the different representations.

We will use the following notations and definitions to specify the computational model. Details have been omitted, as we don’t consider computational complexity.

* The set B := *{*0*,* 1*}* will be used as basic alphabet for all strings.
* Based on an encoding of the set *{*0*,* 1*,* #*}* by pairs of bits, with # acting as a delimiter, we will use naive bijections (B*∗*)*n ↔* B*∗* and (B*∗*)*∗ ↔* B*∗*.
* Natural numbers will be written in binary, i.e. as strings from B.
* Dyadic numbers are defined as D = *{z ·* 2*−n | z ∈* Z*,n ∈* N*}*, they are a dense subset of R. We let D*n* = *{z·* 2*−n | z ∈* Z*}*. Here we will use an encoding by words from *{*+*, −} ◦* B*∗ ◦ {•} ◦* B*∗* (for sign, integer part, position of the fractional point, and the fractional part itself; so obviously denoting a dyadic number). These words again can easily be encoded using strings from B*∗*.
* *F* (*M* ) denotes the finite subsets of a set *M* : We will especially use the notation

*F* (D2) for the finite sets of dyadic points in the plane and *F* (D2 ) for the finite

*n*

sets of dyadic points on a grid with width 2*−n*.

* The Euclidean distance of points in R*n* will be written as dist(*x, y*).
* Open balls in R*n* with center *x ∈* R*n* and radius *ε ∈* R+ will be written as *B*(*x*; *ε*) := *{y ∈* R*n |* dist(*x, y*) *< ε}*. This includes the special cases of open discs in R2 and open intervals in R.
* Let *B*(D*n*) := *{B*(*d*; *ε*) *| d ∈* D*n,ε ∈* D*,ε >* 0*}* be the set of basic open balls in

R*n*. Again a naive encoding with a delimiter symbol is assumed.

* As computational model we will use oracle Turing machines, with input alphabet and oracle alphabet B. Formally, oracles for and the functions computed by these machines are of type *φ* :*⊆* B*∗ →* B*∗*. In general, we will implicitly use naive notations like above and directly work with functions like *φ* :*⊆* N *→* D or even like *φ* :*⊆ F* (D2) *→* N *× B*(D*, n*).
* For any set *S ⊆* R*n* , let *∂S* denote the boundary of *S*, i.e., the set of all points *z ∈* R*n* such that any open ball *B*(*z*; *ε*) around *z* contains both points in *S* and points not in *S*. *int*(*S*) denotes the interior of S; *ext*(*S*) denotes its exterior.
* Let S := *∂B*((0*,* 0); 1) = *{x ∈* R2 *|* dist(*x,* (0*,* 0)) = 1*}* be the unit circle in R2. Let *σ*S : [0; 1] *→* S be defined as *σ*S(*t*) = (sin(2*πt*)*,* cos(2*πt*)).
  + The Hausdorff distance of sets is denoted by

dist*H* ((*X, Y* ) = max*{* sup inf dist(*x, y*)*,* sup inf dist(*x, y*) *}*

*x∈X y∈Y y∈Y x∈X*

We assume that the reader has some (basic) knowledge about TTE [[16](#_bookmark21)], nevertheless we repeat some definitions here:

* + A representation of a set *M* is a partial surjective function *δ* : *⊆*(*⊆*B*∗ →* B*∗*) *→ M* .
  + The pairing of words *φ, ψ* is denoted by *⟨φ, ψ⟩* Here *φ* and *ψ* may have finite or infinite lengths. We will extend this notion to the represented sets.
  + *Q* will be the representation of real numbers, where a name *φ* of *x ∈* R consists of a list *{φ*(*i*) *| i ∈* N*}* of open intervals *Oi* with dyadic endpoints, where *x ∈ Oi* and the diameter of *Oi* is *≤* 2*−i*. We will use a similar representation *Qn* for R*n*, now consisting of open balls with dyadic center and dyadic diameter.
  + *Q<* is a representation of real numbers (including infinities) using approximations from the left: For *x ∈* R = R *∪ {−∞,* +*∞}* we let *Q<*(*φ*) = *x ⇔ {φ*(*n*) *| n ∈* N*}* =

*{d ∈* D *| d < x}*

* + *Q>* represents R is a similar way, but now using approximations from the right.

Computable translations between different representations *ρ*1, *ρ*2 will be expressed as reducibilities ‘*ρ*1 *≤ ρ*2’; the equivalence ‘*ρ*1 *≡ ρ*2’ expresses reducibility in both directions. ‘*ρ*1 *≤t ρ*2’ denotes a topological reduction, i.e. the translation is only shown to be continuous, but not necessarily computable.

# 2 Jordan curves

Jordan curves are continuous functions *γ*, usually from [0; 1] to R2, that are closed (i.e. *γ*(0) = *γ*(1)) and furthermore one-to-one (with exception of the identity at the endpoints). They have been studied for about 200 years now (already Bolzano tried a proof of the Jordan curve theorem).

The topological properties of these curves (e.g. the Jordan-Scho¨nflies theorem, see below) suggest an additional approach: Instead of functions *γ* : [0; 1] *→* R2 we could use functions *γ* : S *→* R2 that additionally are one-to-one, i.e. the curves could also be viewed as images of the unit circle under a continuous injective function.

S is a compact and, moreover, recursive subset of R2. The same obviously holds for the closed interval [0; 1] *⊆* R. So, using the notation of [[16](#_bookmark21)](153ff), it is near at hand to use the following representations in connection with Jordan curves:

## Definition 2.1

* + Let *C*(S*,* R2) := *{f* : *⊆*R2 *→* R2 *| f* continuous and dom(*f* ) = S*}* be the set of all continuous partial functions from R2 to R2 with domain S. In the same way define *C*([0; 1]*,* R2).
  + As representations for *C*(S*,* R2) or *C*([0; 1]*,* R2) we will use *δ*S and *δ*[0;1], which

*→*

*→*

are standard representations of these functions spaces (from [[16](#_bookmark21)]).

Using these representations, a function is essentially encoded as a list of pairs of open balls (*Oi, Ui*) with certain convergence properties, for details see [[16](#_bookmark21)].

From time to time, it will be convenient to switch between *C*([0; 1]*,* R2) (used e.g. in [[12](#_bookmark23),[14](#_bookmark22)]) and *C*(S*,* R2). As we are only considering closed curves, this can be done using the simple transformation *σ*S : [0; 1] *→* S defined by *σ*S(*t*) = (sin(2*πt*)*,* cos(2*πt*)). In the following, *γ* will always be from *C*(S*,* R2), while we use *γ* to denote a function from *C*([0; 1]*,* R2).

Many proofs concerning Jordan domains rely on the Jordan curve theorem. For our purpose, we need to use a rather strong version of this theorem, namely the Jordan-Scho¨nflies theorem. A suitable formulation can e.g. be found in [[15](#_bookmark24)](Thm 40.15); a translation of this version reads as follows:

**Theorem 2.2 (Jordan-Sch¨onflies)** *Each Jordan curve C partitions the plane into exactly two parts. Each homeomorphism from C onto the unit circle* S *can be extended to a homeomorphism of the plane onto itself, where the interior of C is mapped onto the interior of* S *and the exterior of C is mapped onto the exterior of* S*.*

In our case, we do not use functions from *C* to S, but from S to *C*. As we are dealing with homeomorphisms, this is not really a problem, as they are bijective and continuous in both directions. So we may use the following property:

**Corollary 2.3** *For any Jordan curve γ* : S *→* R2 *an extension* Γ : R2 *→* R2 *of γ*

*exists which is a homeomorphism.*

Of course, this homeomorphic extension is not uniquely determined. In this context, we want to mention the following papers treating Jordan curves from a constructive view:

* Berg et.al. [[2](#_bookmark11)] consider a constructive version of the (plain) Jordan curve theorem, which unfortunately is not enough for our considerations: A point *x* outside of the curve and another point *y* inside of the curve can be constructed from the curve, and that additionally, for every point *z* not on the curve, there is a polygonal path not touching the curve that either connects *x* and *z* or *y* and *z*.
* Hertling [[10](#_bookmark19)] shows an effective version of the Riemann open mapping theorem, a theorem that can be used to prove the Jordan curve theorem. This gives us a homeomorphism between the interior of the curve and the interior of S considered as sets, and doesn’t reflect the curve itself: If *γ* is not differentiable, for example, we will of course not be able to get a holomorphic extension to R2, which is a core property of that theorem.

However, below we need a homeomorphism valid on the whole plane; we do not know whether a constructive version of the above theorem has already been shown. Fortunately, we do not need such a constructive version here.

Later we want to compare different representations using Jordan curves, where we will use the length of a curve as important additional information. A suitable definition of this length is (see [Wikipedia], e.g.; we will only use the case *X* = R2):

**Definition 2.4** If *X* is a metric space with metric ‘dist’, then the length of a curve

*γ* : [0*,* 1] *→ X* is defined by

Length(*γ*) = sup

*k*

*i*=1

Σ

dist(*γ*(*ti*)*, γ*(*ti−*1)) : *k ∈* N and 0 = *t*0 *< t*1 *< ··· < tk* = 1

A rectifiable curve is a curve with finite length.

In general, the length of a (computable) Jordan curve can be infinite. Even for rectifiable Jordan curves, the length might be non-recursive; this has already been shown in [[5](#_bookmark14)](9): The authors construct a computable curve with a non-recursive length. It is quite easy to see that the length of the constructed curve is left- computable but not right-computable. This leaves the question whether the length must always be left-computable. We are able to give a positive answer. Please note that this result holds for all curves; it is not necessary that they are Jordan curves. So for each computable Jordan curve *γ*, Length(*γ*) is left-computable.

**Theorem 2.5** *The length of curves is* (*δ*[0;1]*, Q* )*-computable, but even restricted to*

*→*

*<*

*Jordan curves it is not* (*δ*[0;1]*, Q* )*-continuous.*

*→ >*

Please note that the construction in the proof below even works if we restrict the domain of the length operator to Jordan curves with finite length.

**Proof.** To show the computability, consider *γ ∈ C*([0; 1]*,* R2). For any *l <* Length(*γ*), definition [2.4](#_bookmark1) implies that there are 0 = *t*0 *< t*1 *< . . . < tk* = 1 and *m ∈* N such that

*k*

Σ

Length(*γ*) *≥* dist(*γ*(*ti*)*, γ*(*ti−*1)) *> l* + 22*−m*

*i*=1

As *γ* is continuous, there are dyadic values *di* such that dist(*γ*(*di*) *− γ*(*ti*)) *≤*

2*−m/k* for each *i*. So also Σ*k* dist(*γ*(*di*)*, γ*(*di−*1)) *> l* + 21*−m*.

*i*=1

Now consider the following oracle machine *M* . Its oracle *φ* is interpreted as a curve *γ* using *δ*[0;1]. Internally, *M* will use a (computable) bijection *π* : N *→* D*∗ ×* N.

*→*

* + Given any input *n ∈* N, *M* first computes *π*(*n*) = ((*d*0*,..., dk*)*, m*) and checks whether 0 = *d*0 *< di < . . . < dk* = 1. If not, *M* returns 0 *∈* D (as a valid lower bound for the curve length) and stops.
  + Otherwise, *M* uses the oracle *φ* to compute approximations *gi* to *γ*(*di*) with an error of *≤* 2*−m−*1*/k* for each *i*.
  + Finally *M* computes *sn* := *−*2*−m* + Σ*k* dist(*gi, gi−*1)(*∈* D) and returns *sn*.

*i*=1

By construction, we have *sn ≤* Length(*γ*). On the other hand, the discussion above shows that for any *l <* Length(*γ*) there is an input *n* such that *l < sn*.

So for any *φ* in the domain of *δ*[0;1] this algorithm computes a sequence (*s*

*→*

*n*)*n∈*N

[0;1]

with sup*n sn* = Length(*δ* (*φ*)), i.e. Length is left-computable.

*→*

Standard arguments can be used to show non-(*δ*[0;1]*, Q* )-continuity of Length.

*→ >*

# 3 Jordan domains/areas

As Jordan curves have a non-empty, open interior, they can be used to define regions in the Euclidean plane. In [[6](#_bookmark15),[12](#_bookmark23),[14](#_bookmark22)] a bounded open set *S ⊆* R2 is called a Jordan domain if its boundary *∂S* is a Jordan curve. The main focus of these papers was on Jordan domains computable in polynomial time, i.e. there had to be at least one polynomial time computable Jordan curve *ц* such that *ц*(S) = *∂S*. Of course, due to the Jordan-Scho¨nflies theorem, the Jordan domains are topologically just homeomorphic images of the open unit disc.

On the other hand, in computable analysis several definitions of computability of subsets of R*n* have been discussed in the near past, especially for bounded sets. Many of these definitions always apply at the same time to sets *S* and their closure *S* as well: [[6](#_bookmark15),[7](#_bookmark16),[8](#_bookmark17),[13](#_bookmark20)], [[3](#_bookmark12)], [[16](#_bookmark21),[17](#_bookmark26)]. So we should also consider the union of the domain and its boundary, i.e. the homeomorphic images of the closed unit disc. We will call them Jordan areas:

**Definition 3.1** A bounded subset *S ⊂* R2 is called a Jordan domain, iff it is a homeomorphic image of the open unit disc. *S* is called a Jordan area, iff it is a homeomorphic image of the closed unit disc. Let *Jo* be the set of all Jordan domains and let *Jc* be the set of Jordan areas.

The first two of the following representations are immediate from the definition of Jordan domains in [[6](#_bookmark15),[12](#_bookmark23),[14](#_bookmark22)]. The third is essentially a restriction of a representation of compact sets using a grid of points from [[16](#_bookmark21)](143ff). Some properties of this approach (concerning computational complexity) can be found in own paper [[17](#_bookmark26)]. In [[4](#_bookmark13)] a comparison of several representation of compact sets can be found.

**Definition 3.2** Representations for the set *Jc* of Jordan areas:

1. The ‘Jordan curve’ representation *Q→* is defined by: *Q→*(*φ*) = *S* for a *S ∈ Jc*

iff

*ц* := *δ*[0;1](*φ*) is one-to-one on [0; 1)*, ц*(0) = *ц*(1) and *ц*[0; 1] = *∂S*

*→*

1. The ‘Jordan sphere’ representation *Q* is defined by: *Q*(*φ*) = *S* for a *S ∈ Jc*

iff

*ц* := *δ*S (*φ*) is one-to-one on S and *ц*(S) = *∂S*

*→*

1. The ‘grid name’ representation *Q*# is defined by: *Q*#(*φ*) = *S* for a *S ∈ Jc* iff

*φ* : N *→ F* (D2) satisfies (*∀n ∈* N) (*φ*(*n*) *∈ F* (D2 ) *∧ dH* (*S, φ*(*n*)) *≤* 2*−n*)

*n*

Essentially, *Q→* and *Q* are defined by combination of a restriction of the standard representations of function spaces (to [0; 1] or S) and an additional equivalence relation on the functions (where *ƒ ≡ g* iff *ƒ* [0; 1] = *g*[0; 1] or iff *ƒ* (S) = *g*(S) ).

So the same set is represented by many different functions: (1) we can choose any point on the boundary to be the ‘starting point’ of the Jordan curve, (2) the orientation of the curve might be clockwise or anticlockwise, and (3) the ‘speed’ of the curves can differ.

Obviously it is quite easy to translate between *Q→* and *Q*, but the relations between *Q* and *Q*# will be the main topic of this paper.

**Lemma 3.3** *Q→ ≡ Q*

**Proof.** Omitted.

In the following we first want to show that we are able to translate *Q* to *Q*#. Using the Jordan curve theorem, we essentially have to determine whether a point is in the exterior of the curve *ц*, in its interior, or whether it lies on the curve itself (or more precisely: is near the curve). Neither the exterior nor the neighborhood of the curve place a problem; we only have to be sure when we look at the interior. Here the notion of the winding number (or index) of a curve is important; the basic idea has already be used in [[2](#_bookmark11)], in [[5](#_bookmark14),[6](#_bookmark15)] Ko and Chou analyzed aspects of its complexity. The winding number for a closed curve *ц* and a point *z*0 may be defined via a complex integral (i.e. we identify R2 with C):

*Ind*

(*z* ) := 1

*dz*

*γ* 0 2*πi*

*γ*

*z − z*0

Constructive aspects of the winding number have been addressed already in [[1](#_bookmark10)], a fully computerized proof of some of its properties can be found in [[9](#_bookmark18)].

Please note that here we essentially have a path integral in the set of complex numbers. If *z*0 does not lie on the curve, then the value of the integral is well- defined, and must be an integer. In case of Jordan curves, its value is either 0

(for points in the exterior of *ц*), +1 (for points in the interior of a curve *ц* that is positively oriented) or *−*1 (for points in the interior of a negatively oriented *ц*). As the value must be an integer, it is sufficient to approximate the integral with a

precision fixed to 2*−*1. So for any point not on the curve, we are able to determine

the value in finite time. The dependency of the computation time on the distance to curves, that are polynomial time computable, has already been studied in [[6](#_bookmark15)]. Here we present a uniform formulation of the result:

**Lemma 3.4** *The winding number Ind is* ((*δ*S *, Q*2)*, ν*N)*-computable on the set*

*→*

*{*(*ц, z*0) *| ц ∈ C*(S*,* R2)*, z*0 */∈ ц*(S)*}*

**Proof.** Omitted.

In the following we will use the computability of the winding number to translate between *Q* and *Q*#. In [[12](#_bookmark23)], Ko and Yu analyzed the complexity of a further algorithm for the membership problem for curves computable in polynomial time, which could also be used as (still non-uniform) step towards the reduction between *Q* and *Q*#.

**Definition 3.5** Define the ‘boundary grid’ representation *Q∂* of Jordan areas by:

*Q∂⟨d*0*, φ⟩* = *S* for a *S ∈ Jc* iff *d*0 denotes a dyadic point in the interior of *S* and

*φ* determines the boundary of *S* as follows:

*φ* : N *→ F* (D2) satisfies (*6n ∈* N) (*φ*(*n*) *∈ F* (D2 ) *Λ dH* (*∂S, φ*(*n*)) *≤* 2*−n*) Please note the difference: *Q∂* uses just the boundary of *S* and an interior point,

*n*

while *Q*# uses the whole set *S*!

**Lemma 3.6** *Q* *≤ Q∂ ≤ Q*#

**Proof.** Omitted.

It can easily be seen that at least one part of the inverse of the previous lemma is not true: There is no continuous translation from *Q*# to *Q∂*. Unfortunately, we do not know whether *Q∂ /≤t Q→* or not.

**Lemma 3.7** *Q*# */≤t Q∂, Q*# */≤t Q→*

**Proof.** To prove *Q*# */≤t Q∂*, we can use a standard argument for non-continuity: Just consider the unit circle as a special Jordan area and the special *Q*#-name *φ*

with *φ*(*n*) = *{d ∈* D2 *|* dist(*d,* (0*,* 0)) *≤* 1*}*. If there were a translation from *Q*# to

*n*

*Q∂*, there would be a *n*0 such that only *φ*(*n*) with *n < n*0 are used to determine a

dyadic *d* in the interior of the unit circle. But if we now remove an open strip from the unit circle that has a width of 2*−n*0 and contains *d*, then the resulting Jordan area would have a name *ψ* that coincides with *φ* for *n < n*0, but doesn’t contain *d*. So the translation would be wrong for *ψ*.

*Q*# */≤t Q→* then is a simple consequence of *Q→ ≤ Q∂* and *Q*# */≤t Q∂*.

In the following we want to find additional constraints that allow to translate from the grid name representation *Q*# to the Jordan curve representation *Q∂*. First, we have a closer look at the length of Jordan curves (i.e. at the length of the boundary of Jordan areas) again:

In the proof on the length of a Jordan curve in the previous chapter, it was very easy to find an ‘ordered’ set of values on the curve: We simply took *ц*(*ti*) for an increasing sequence (*ti*). Using the grid representation *Q*#, it is harder to ensure this ordering. The following two lemmata will give us the necessary tools. The first one is very technical and will ensure that we are able to find valid lower approximations for Length(*ц*); many of its preconditions are depicted in figure [1](#_bookmark5)(a). The second lemma will be used to show the convergence of the approximations.

## Lemma 3.8

1. *Let S be a Jordan area and let ц be a Jordan curve with ∂S* = *ц*(S)*.*
2. *Let θ* : [0; 1] *→* R2 *be a Jordan curve in the form of a square such that S lies in the interior of θ.*
3. *Let* 0 *< t*1 *< . . . < tk <* 1 *be given for a k >* 1*.*
4. *Let k pairwise disjoint curves θi* : [0; 1] *→* R2 *be given such that:*
   * *θi*(0) = *θ*(*ti*)*, i.e. θi starts in θ*(*ti*) *on the square.*
   * *All the other points of θi are in the interior of θ and in the exterior of S.*
5. *Let ε ∈* R+ *be given.*
6. *Deﬁne pi, si, Oi and Ci as follows for each i from* 1*,..., k:*
   * *si* := *θi*(1) *is the endpoint of θi, pi* := *θi*(0) *is its initial point.*
   * *Oi* := *B*(*si*; *ε*) *is an open disc around si with radius ε.*
   * *Ci* := *∂Oi is the circle with center si with radius ε.*
7. *Suppose that Oi ∩ S /*= *∅.*
8. *Suppose that the Ci are pairwise disjoint, lie in the interior of θ and that Ci*

*and θj are disjoint for i /*= *j.*

*Then the length of ц can be approximated as follows:*

*Length*(*ц*) *≥ Length*(*s*1*s*2 *... sks*1) *—* 2*k · ε*

**Proof.** First we want to find a set *X* of points *{xi |* 1 *≤ i ≤ k}* on the curve *ц* that are ‘connected’ to the curves *θi*: Let *i* be arbitrary between 1 and *k*. By conditions

1. and (vii) we know that *Oi ∩ ц*(S) */*= *∅*, so a part of *ц*(S) will be inside of *Ci*. As *k >* 1, there is a second index *j* with *j /*= *i* and *Oj ∩ ц*(S) */*= *∅*. So, by (viii), another part of *ц*(S) must be outside of *Ci*. As *ц* is continuous, *ц*(S) *∩ Ci /*= *∅* follows.

As both *Ci* and *θi* are closed sets, their intersection must also be closed. So on

*θi* there is a *ﬁrst* point *zi* of intersection with *Ci*:

*zi* := min*{θi*(*t*) *|* 0 *≤ t ≤* 1 *Λ θi*(*t*) *∈ Ci}*

*t*

With a similar argument, we can show that there is a first point *xi* on *Ci*, clockwise after *zi*, that lies on *ц*(S). The situation is depicted in figure [1](#_bookmark5)(a).

*p*2

*pi*

*θi*

*zi*

*Ci*

*si*

*xi*

*γ*

*θ*

*z*2

*x*2

*z*1 *zη*

*p*1 *x*1

*xη*

*γ*

*xν*

*θ zν*

*pν*

(a) Construction of *xi* from preconditions in Lemma [3.8](#_bookmark4) (b) Main argument for point order Fig. 1. Relations between Jordan curve *γ* and boundary curve *θ*

In the following we want to show that the points *{xi |* 1 *≤ i ≤ k}* are already ordered on the curve *ц* according to the index *i*: Any pair of points *xi* and *xj* with *i /*= *j* partitions the Jordan curve *ц* into two disjoint sub-curves *Pi,j* (leading

clockwise from *xi* to *xj*) and *Pj,i* (leading clockwise from *xj* to *xi*, so consisting of the rest of the curve *ц*). These sub-curves *Pi,j* also define sets of indices *Ii,j* := *{ν | xν* lies on *Pi,j,ν /*= *i, ν /*= *j}* of the points from *X* on that sub-curve.

For simplicity, we will only consider the case *i* = 1 and the points *x*1, *x*2 in the following: The curves *P*1*,*2 and *P*2*,*1 divide the set *{i |* 3 *≤ i ≤ k}* into disjoint sets *I*1*,*2 and *I*2*,*1.

With an indirect argument we will show that one of the two index sets is empty: Suppose both were non-empty, so there exist a *ν ∈ I*1*,*2 as well as an *η ∈ I*2*,*1. So *P*1*,*2 is the union of *P*1*,ν* and *Pν,*2, while *P*2*,*1 consists of *P*2*,η* and *Pη,*1. These four

curves are all disjoint with the obvious exception of the endpoints, while together they must again yield *ц*. The situation is homeomorphic to the figure [1](#_bookmark5)(b).

*ц* may neither cross *θ* nor any *θi* (by conditions (ii) and (iv)) nor any of the arcs between *zi* and *xi* (by construction of the *xi*), with exception of the *xi* themselves. So obviously, two of the curves *P*1*,ν*, *Pν,*2 *P*2*,η*, *Pη,*1 must cross, which is a contradiction to *ц* being a Jordan curve. (A formal argument could be based e.g. on the Jordan curve composed by *Pν,*2, the line segments *p*2*z*2*x*2 and *pνzνxν* together with that part of *θ* connecting *p*2 and *pν* via *p*1 so that *x*1 lies in the interior and *xη* in its exterior.)

So either *I*1*,*2 or *I*2*,*1 is empty. Consider the case that *I*1*,*2 = *∅*. As 1 *∈ I*2*,*3 would imply 3 *∈ I*1*,*2, we know 1 *∈ I*3*,*2, so *I*3*,*2 */*= *∅*. Using the same argument as above, *I*2*,*3 must be empty. Inductively, the paths *P*1*,*2*, P*2*,*3*,..., Pk−*1*,k, Pk,*1 must be a disjoint (with exception of the endpoints) decomposition of *ц*. Here *ц* has the same orientation (positive or negative) as *θ*.

The same argument shows that in the case *I*2*,*1 = *∅* the curve *ц* is decomposed into *Pk,k−*1*, Pk−*1*,k−*2*,..., P*2*,*1*, P*1*,k*. Here *ц* and *θ* have different orientation.

In both cases the length of *ц* can be approximated from below by the length of the polygon given by the points *x*1*x*2 *... xkx*1. As dist(*xi, si*) *≤ ε*, we conclude

Length(*ц*) *≥* Length(*x*1*x*2 *... xkx*1)

*≥* Length(*s*1*s*2 *... sks*1) *—* 2*k · ε*

So from the ‘approximating’ polygon *s*1*s*2 *... sks*1 we are able to derive a valid lower bound for Length(*ц*).

**Lemma 3.9** *The boundary length is* (*Q*#*, Q<*)*-computable. So if a Jordan area S*

*has a computable Q*#*-name, then Length*(*∂S*) *is left-computable.*

**Proof.** Suppose *φ* is a *Q*#-name for a Jordan area *S*, let *ц* be a Jordan curve with

*ц*([0; 1]) = *∂S*.

To find a lower bound for Length(*∂S*), consider tuples (*ε, k, θ,* (*θi*)) of the fol- lowing objects: any *ε ∈* D, *k ∈* N, any square *θ* with center 0 and side length *r ∈* D, any piecewise linear curves *θi* with dyadic vertices. Let *pi* and *si* be the dyadic end points of *θi*, as well as *Oi* := *B*(*si*; *ε*). An additional *t ∈* N will act as a time bound. Then for at most *t* steps and using *φ* as an oracle for *S*, try to check whether this combination of values fulfills all properties from Lemma [3.8](#_bookmark4). If the time is sufficient and if the check gives a positive result, then return *l* := *Length*(*s*1*s*2 *... sks*1) *—* 2*kε*.

Otherwise return *l* := 0 (as a lower bound for *∂S*).

As the set of tuples (*ε, k, θ,* (*θi*)*, t*) is countable, we can use a pairing function to encode each tuple into a natural number *n*; so this ‘algorithm’ defines a functional *ƒ* : (*φ, n*) *'→ l*. By Lemma [3.8](#_bookmark4), *ƒ* (*φ, n*) *≤* Length(*∂S*) must hold.

We still have to show that the sup *ƒ* (*φ, n*) = *∂S*: So consider an arbitrary

*L <* Length(*∂S*). By definition [2.4](#_bookmark1), there must be a *ε >* 0 and values 0 *< t*1 *<*

*. . . < tk <* 1 such that *L <* Length(*y*1*y*2 *. .. yky*1) *— ε*, where *yi* := *ц*(*ti*). We want to show now that there is an *n* such that *L ≤ ƒ* (*φ, n*).

Now instead of *ц* with *ц*([0; 1]) = *∂S* consider *ц* with *ц ◦σ*S = *ц*. This transforms the *ti* into points *{ri |* 1 *≤ i ≤ k}* ordered clockwise on S such that *ц*(*ri*) := *yi*. Due to the Jordan-Scho¨nflies theorem, there is a homeomorphism Γ on R2 that extends *ц*, i.e. Γ(*ri*) = *ц*(*ri*) = *yi*. Define infinite curves *ηi* : *{t | t ≥* 1*} →* R2 by *ηi* := *ri · t*. These curves start at the *ri* on the unit circle. Additionally, they are unbounded and pairwise disjoint. As Γ is homeomorphic, this must also hold for their images Γ(*ηi*). In consequence, any square *θ* containing *S* has a non-empty intersection with each of the Γ(*ηi*).

*η*2

*η*3

*η*4

*r r*3

2 *r*4

*η*1

*r*1



Γ(*η*3 )

Γ

*y*3

*y*2

*y*4 Γ(*η*4 )

Γ(*η*2 )

*y*1

Γ(*η*1 )

*θ*

*θ*3

*y*3

*y*2

*y*4 *θ*4

*θ*2

*y*1

*θ*1

Fig. 2. Homeomorphism Γ and resulting *θi*

Just fix one ‘large’ square having a dyadic side length. Then let *θi* be the restriction of Γ(*ηi*) from the starting point *yi* to the first point intersection point *pi* with *θ*. Using a re-parameterization, we may as well assume *θi* : [0; 1] *→* R2, see figure [2](#_bookmark7). These curves *θi* must be pairwise disjoint, so there is a dyadic *ε' >* 0 smaller than the distance between any of these curves. We may assume *ε' <* *ε* . The closed sets *θi*[0; 1] *\ B*(*yi, ε'*) and *ц*(S) are also pairwise disjoint, so there is a dyadic *δ* with 0 *< δ < ε'* that is smaller than the distance between them.

4*k*

Consider the open sets *Si* := *z∈θ*[0;1] *B*(*z, δ*): It is easy to see that each of these sets contains dyadic polygons *θ'* touching *θ* on the one end in a point *p'* , while the

*i* *i*

other end point *s'* is closer to *yi* than *ε'*, and additionally that the distance between

*i*

*ц*(S) and *θ'*[0; 1] is at least *δ*.

*i*

The tuples (*ε', k, θ,* (*θ'*)) have all the properties necessary for Lemma [3.8](#_bookmark4). Be- cause of the lower bound *δ* we are able to check these properties in a finite amount

*i*

*t* of time, just using *φ*; so there is an *n* with *ƒ* (*φ, n*) = Length(*s' s' ... s' s'* ) *—* 2*k · ε'*.

1 2 *k* 1

On the other hand, each *s'* is closer to *yi* than *ε*, so Length(*s' s' ... s' s'* ) *≥*

*i * 1 2 *k* 1

Length(*y*1*y*2 *... yky*1) *—* 2*kε'*. Using *ε' <* *ε* , this implies *L < ƒ* (*φ, n*).

4*k*

To be more precise, we reformulate this result using representations.

## Definition 3.10

* Let *Jc* be the set of all Jordan areas with a boundary of finite length.

*f*

*f*

*f*

* Let *Q*#

*f*

be a representation of *Jc*

such that *Q*#*⟨φ, ψ⟩* = *S* iff *Q*#(*φ*) = *S* and

*Q*(*ψ*) = Length(*∂S*).

*f*

*f*

* Let *Q→*

*f*

be a representation of *Jc*

such that *Q→⟨φ, ψ⟩* = *S* iff *Q→*(*φ*) = *S* and

*Q*(*ψ*) = Length(*∂S*).

* Let *Q→* be a representation of *Jc* defined by restriction of *Q→*, i.e. *Q→*(*φ*) := *Q→*(*φ*)

*r f r*

iff *Q→*(*φ*) *∈ Jc*.

*f*

Please note: As we already know that the boundary length must be computable from the left, it would be sufficient if we additionally are able to compute it from the right.

Now we are able to formulate the main result of this paper:

**Theorem 3.11** *The following reductions are valid:*

* *Q*# *≤ Q→*

*f r*

* *Q*# *≡ Q→*

*f f*

* *Q→ /≤t Q→*

*r f*

*So given the (ﬁnite!) length of the boundary of a Jordan area and a grid name of the area, we are able to ﬁnd a Jordan curve describing this boundary.*

**Proof.** We only have to show *Q*# *≤ Q→* here. Then *Q*# *≡ Q→* is a trivial consequence

*f r f f*

of Lemma [3.6](#_bookmark3), while *Q→ /≤t Q→* follows from [2.5](#_bookmark2).

*r f*

Let *⟨φ, ψ⟩* be a *Q*#-name for a Jordan domain and let *S* := *Q*#(*φ*) be the Jordan area given by *φ*, i.e. *l* := *Q>*(*ψ*) is the length of any of the (uncountably many) Jordan curves *ц* with *ц*[0; 1] = *∂S*.

*f*

One part of the problem is to determine one ‘initial’ point *α* on the boundary

*∂S*; however, the location of this point will depend computably(!) on the name *φ*. Consider the special curve *цα* with:

* clockwise orientation,
* *цα*(0) = *цα*(1) = *α* and
* Length(*цα*( *i* )) = *l ·* *i* .

*n n*

Having fixed *α*, *цα* is uniquely determined by the properties above. In the following we will construct both *α* and the curve *цα* in parallel.

*⟨φ, ψ⟩* will be used as the oracle for an oracle Turing machine. On input *n ∈* N, this machine should return converging approximations to *цα*.

In the following we will just show instead that we are able to evaluate *цα*(*t*) to any precision for any argument *t*, which is equivalent to the approximation of *цα*.

Later in the proof we will need increasing sequences of natural numbers *m* to- gether with positive dyadic values *εm* and a second sequence of closed dyadic polyg-

onal paths *smsm...smsm* such that for any of these *m* there is a corresponding

1 2 *m* 1

path *xmxm...xmxm* of points from *∂S* (in clockwise orientation) with *xm ∈ Om* :=

1 2 *m* 1 *i* *i*

*B*(*sm*; *εi*). For any of the *xm* let *tm* be such that *цα*(*tm*) = *xm*. We may easily

*i i i i* *i*

ensure that always *εm ≤* 2*−m*.

The construction of the sequences will be done iteratively (and independent from *t*, so the resulting point *α* will only depend on *φ*): We start with *m* = 1 and *ε*1 := 1*/*2. Then we use *φ* to find an arbitrary dyadic point *s*1 from ext(*S*) such

1

that *B*(*s*1; 1*/*2) *∩ S /*= *∅*, so there is a *x*1 *∈ O*1.

1 1 1

*m'*

If *εm'* and *s*1 are already determined, we can use *φ* and the exhaustive con-

struction from the proof of [3.9](#_bookmark6) to find new values *m > m'*, *εm* and *smsm...smsm*. We

1 2 *m* 1

only restrict this search as follows: We demand *εm ≤* 2*−m*, a nondecreasing length

of the path *smsm...smsm*, and additionally *sm ∈ Om ⊂ Om'* . This will ensure that

1 2 *m* 1 1 1 1

the sequence *Om* converges to a single point *α* with *α ∈ B*(*sm*; *εm*) for any *m*. As

1

*m' m'*

*m' m'* 1 *' m' m'*

*x*1 *∈ O*1 , choosing *y*1*y*2 *... ymy*1 as a refinement of *x*1 *x*2 *xm' x*1 with *y*1 = *x*1

*m*

in the proof of the convergence in [3.9](#_bookmark6) shows that even with this restriction we still

get paths *smsm...smsm* having length arbitrarily close to *l*.

1 2 *m* 1

As the *xm* are close to the *sm*, for any *i < j* we have

*i* *i*

Length(*цα*([*tm*; *tm*]) *≥* Length(*xmxm ... xm*

*xm*)

(1)

*i j i i*+1

*j−*1 *j*

*≥* Length(*smsm ... sm sm*) *—* 2*mεm*

*i i*+1 *j−*1 *j*

for the sub-curve *цα*([*tm*; *tm*]) between *xm* and *xm*.

*i j i j*

Now let *ϵ*(*m*) := 2(*m*+1)*εm*. So using *lm* := Length(*smsm ... smsm*) *— ϵ*(*m*) we

*<*

get *lm <* Length(*xmxm ... xmxm*) *≤ l*.

1 2 *m* 1

*<* 1 1 *m* 1

To get approximations *lm* converging to *l* from the right is much easier: We

*>*

only have to use *ψ*.

Now consider an arbitrary *t ∈* [0; 1] for which we want to approximate *цα*(*t*). (For simplicity we will use *t* as a fixed, exact value, but the following could be formulated using approximating intervals as well, on the cost of the ease of reading.) The precision of the approximation shall be 2*−n*.

We start the approximation by searching a value *m* such that *ϵ*(*m*) *<* 2*−n−*4 and additionally *lm — lm <* 2*−n−*4. This search will stop after finite time, as both *lm*

*> < >*

and *lm* converge to *l*. We may assume that at the end also all lengths *smsm* are

*< i i*+1

smaller than 2*−n−*3, as we can reduce the size of the sets *Oi* in the construction in

[3.9](#_bookmark6) and then insert new points into the sequence if needed.

By definition of *цα* we have Length(*цα*[0; *t*]) = *l · t*. Let *j* be maximal such

that Length(*smsm ... smsm*) *≥* (1 *— t*) *· lm* + *ϵ*(*m*) and *i* be minimal such that

*j j*+1 *m* 1 *>*

Length(*smsm ... sm*) *≥ t · lm* + *ϵ*(*m*). As the whole curve has a length close to *lm*,

1 2 *i > >*

we will assume 1 *< j < i < m* for simplicity in the following.

Then using equation ([1](#_bookmark9)) and *α ∈ B*(*sm*; *εm*), we have (for *i*) Length(*цα*([0; *tm*])) *≥*

1 *i*

*t · lm > t · l*, and (for *j*) Length(*цα*([*tm*; 1]) *≥* (1 *— t*) *· lm >* (1 *— t*) *· l*, so together

*> j >*

we see that *tm < t < tm*, i.e. *цα*(*t*) is located somewhere on the sub-curve between

*j* *i*

*xm* and *xm*.

*j* *i*

Using the special choice of *i* and *j* and the density of the points *si*, we additionally

get Length(*smsm ... sm*) *≤ t·lm*+*ϵ*(*m*)+2*−n−*3, implying Length(*smsm ... smsm*) *≥*

1 2 *i > i i*+1 *m* 1

*lm — t · lm — ϵ*(*m*) *—* 2*−n−*3. So Length(*цα*([*tm*; 1])) *≥ lm — t · lm — ϵ*(*m*) *—* 2*−n−*3.

*< > i < >*

In a similar manner we get Length(*цα*([0; *tm*])) *≥ lm —* (1 *— t*) *· lm — ϵ*(*m*) *—* 2*−n−*3,

*j < >*

hence Length(*цα*([*tm*; *tm*])) *≤ l* + *lm —* 2*lm* + 2*ϵ*(*m*)+ 2*−n−*2. This results in

*j i > <*

Length(*цα*([*tm*; *tm*])) *≤* 2*−n−*1

*j i*

This implies that also the distance between *sm* and *цα*(*t*) is at most 2*−n*.

*j*

So for any *t* we are able to approximate *цα*(*t*) with any desired precision. This is sufficient to show that we are able to compute *цα* from *⟨φ, ψ⟩*.

[0;1]

If in the proof we just use the constructed Jordan *curve ц*0 (given via *δ* ) in-

*→*

stead of using the Jordan *area* (via the representation *Q→*), we see that we essentially

*r*

constructed a mapping from Jordan areas to their boundaries:

**Corollary 3.12** *There isa* (*Q*#*, δ*[0;1])*-computable multi-valued function Bƒ* : *Jc →*

*f → f*

*C*([0; 1]*,* R2) *such that Bƒ* (*S*) *is a Jordan curve ц deﬁning the boundary of S.*

As there are many Jordan curves defining the same Jordan area, this func- tion seems to be inherently multi-valued. There are three sources for this multi- valuedness, only two of which we could avoid in the proof of [3.11](#_bookmark8):

* Orientation of the curves: In main part of the proof, we restricted the orientation to always be clockwise.
* ‘Speed’ of the curves: Because of the finite length, we were able to ‘normal- ize’ the speed, i.e. for a total length *l* any initial segment of the curve fulfills Length(*ц*[0; *x*]) = *x · l*.
* Starting point of the curves: Here we were unable to find a computable single-

valued mapping *S '→ α* from *Jc* to R2 such that *α* is on the boundary of *S*.

*f*

Although we don’t believe that such a mapping exists, we cannot present a proof here.

# 4 Closing Remarks and Questions

Many open questions and problems arose during the preparation of the paper:

* There should be a multivalued(?) continuous mapping *ц →* Γ, i.e. there should be a constructive version of the Jordan-Scho¨nflies theorem.
* Can we say anything about the complexity of Γ? How large is its complexity for standard examples?
* Does the length of a rectifiable curve also help to compute its measure? Compare this to [[11](#_bookmark25)], where a rectifiable curve with nonrecursive measure was constructed. That curve had a non-computable length.

While the representation *Q→* is hard to generalize, it is straightforward to define a higher-dimensional analogue to *Q* and *Q*#:

## Definition 4.1

* + For any dimension **d** *∈* N, **d** *≥* 2, the **d**-dimensional unit sphere S**d** *⊆* R**d** can be defined as S**d** := *{x ∈* R**d** : *|x|* = 1*}*.
  + The **d**-dimensional Jordan areas *Jc* can be defined via the image of S**d** under a

**d**

continuous injective function.

* + A representation *Q* can be defined by *Q*(*p*) = *S* for a *S ∈ Jc* iff

**d d d**

*ƒ* := *δ*S**d** (*p*) is one-to-one on S**d** and *ƒ* (S**d**) = *∂S*

*→*

* + A representation *Q*# can defined by *Q*#(*p*) := *S* for a *S ∈ Jc* iff

**d d d**

*φ* : N *→ F* (D**d**) satisfies (*6n ∈* N) (*φ*(*n*) *∈ F* (D**d**) *Λ dH* (*S, φ*(*n*)) *≤* 2*−n*)

*n*

For complexity, instead of using **d**-dimensional unit sphere the **d**-dimensional unit cube could be of interest: Here we have a grid of dyadic points on each face of the cube, which should enable us to simplify several constructions.

The idea of using the length of the boundary in order to compute the boundary itself is presumably not useful in higher dimensions: Here we would like to use the area of the surface, but there could be very thin ‘needles’ (i.e. with a very small surface) that destroy the locality of the boundary.

Unfortunately, the Jordan-Scho¨nflies theorem cannot be generalized to higher dimensions, with Alexanders horned sphere being a counterexample.

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