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Monadic Σ1

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and Modal Logic with Quantified Binary Relations

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**Abstract**

We investigate the expressive power of a range of modal logics extended with second-order prenex quan- tification of binary and unary relations. Our principal result is that Σ1(*BML*=), i.e., Boolean modal logic extended with the identity modality and existential prenex quantification of binary and unary relations, translates into *monadic* Σ1. We also briefly discuss a variety of decidability results in multimodal logic

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implied by our result.

*Keywords:* Monadic Σ1, Boolean modal logic, expressive power, decidability.

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# Introduction

Modal correspondence theory concerns itself with the classification of formulae of modal logic according to whether they define elementary classes of Kripke frames. On the level of frames, modal logic can be regarded as a fragment of monadic Π1, also known as *∀MSO* . Hence correspondence theory studies a special fragment of *∀MSO* . When inspecting a modal formula from the point of view of frames, one universally quantifies the proposition symbols occurring in the formula. It is therefore rather natural to ask what happens if one also quantifies binary relation symbols occurring in (the standard translation of) a modal formula. This question is studied in [[10](#_bookmark22)], where the focus is on the expressive power of multimodal logic with universal prenex quantification of (not necessarily all of the) binary and unary relation symbols occurring in a formula. It is natural to ask whether there exists

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any class of multimodal frames definable in this logic, let us call it Π1(*ML*), but not definable in monadic second-order logic (*MSO* ). This can be regarded as a question of modal correspondence theory. In this case, however, the correspondence language is *MSO* rather than *FO* . In addition to [[10](#_bookmark22)], modal logic with quantification of binary relations is investigated for example in [[2](#_bookmark16),[11](#_bookmark23),[15](#_bookmark29)].

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In the current paper we study two systems of multimodal logic with existential second-order prenex quantification of binary and unary relation symbols. We warm up by showing that formulae of Σ1(*ML*), i.e., ordinary multimodal logic with ex- istential second-order prenex quantification of binary and unary relation symbols, translate into *EMSO* (*MLE* ), i.e., multimodal logic with the universal modality and existential second-order prenex quantification of *only unary* predicate symbols. The method of proof is based on the definition of the accessibility relation in a *largest ﬁltration* (see [[1](#_bookmark15)] for the definition). We then push the method and establish that Σ1(*BML*=), i.e., Boolean modal logic with the identity modality and existential second-order prenex quantification of binary and unary relation symbols, translates into monadic Σ1, also known as *EMSO* . Note that both of these results directly imply that Π1(*ML*) translates into *6MSO* , and therefore show that *MSO* would be a dull correspondence language for correspondence theory of Π1(*ML*).

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It could be argued that *{ч, ∪, ∩, ◦, ∗, ×, E, /*=*}* is, more or less, the core collec- tion of operations on binary relations used for defining extensions of modal logic for the purposes of applications. Here *ч*, *∪*, *∩*, *◦*, *∗*, and *×* denote the complement, union, intersection, composition, transitive reflexive closure, and converse opera- tions, respectively. *E* and */*= denote the constant operations universal modality and difference modality. Logics where these operations are used (possibly together with other operations) include for example *PDL* [[3](#_bookmark17),[7](#_bookmark21)], Boolean modal logic [[4](#_bookmark18),[12](#_bookmark24)], de- scription logics [[8](#_bookmark25),[14](#_bookmark28)], modal logic with the universal modality [[5](#_bookmark19)], and modal logic with the difference modality [[17](#_bookmark31)]. The fact that *BML*= subsumes a large number of typical extensions of modal logic is one of the motivations for our study.

We describe a possible application of our result concerning Σ1(*BML*=) (cf. The- orem [4.11](#_bookmark13)). Let *Ð* be a class of Kripke frames (*W, R*1), and consider the class *C* = *{* (*W, R*1*, R*2*, ...*) *| Ri ⊆ W × W,* (*W, R*1) *∈Ð }* of multimodal Kripke frames. Assume that the *6MSO* -theory of *Ð*, that is, the *6MSO* -theory of the class of *R*1- reducts of structures in *C*, is decidable. For example, we could assume that *C* is the class of countably infinite frames (*W, R*1*, R*2*...*), where *R*1 is a dense linear ordering without endpoints (see [[16](#_bookmark30)]). Assume we wish to know whether the satisfiability problem of multimodal logic (perhaps extended with, say, the difference modal- ity) with respect to *C* is decidable. By Theorem [4.11](#_bookmark13) below, we immediately see that, indeed, it is. Theorem [4.11](#_bookmark13) implies a whole range of decidability results for multimodal logic. We note that there exists a large body of knowledge concerning structures and classes of structures with a decidable *MSO*- (and therefore *6MSO* -) theory, see [[18](#_bookmark32)] for example.

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Another motivation for our study is related to descriptive complexity theory [[9](#_bookmark26)]. *FO*2 is the fragment of first-order logic, where the use of only two variables is allowed. Σ1(*FO*2) is the extension of this logic with existential second-order

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prenex quantification. In [[6](#_bookmark20)], Gra¨del and Rosen pose the question whether there exists any class of finite directed graphs definable in Σ1(*FO*2), but not definable in *EMSO* . Lutz, Sattler, and Wolter show in [[13](#_bookmark27)] that *BML*= extended with the converse operator is expressively complete for *FO*2. Therefore, in order to show that Σ1(*FO*2) translates into *EMSO* , one would have to modify our proof such that it takes into account the possibility of using the converse operation. We have succeeded neither in this, nor in finding a Σ1(*FO*2) definable class of directed graphs that is not definable in *EMSO* . However, we find modal logic a promising framework for working on this problem.

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# Preliminaries

In this section we introduce technical notions that occupy a central role in the rest of the discourse.

* 1. *Syntax and Semantics*

With a model we mean a model of predicate logic. A *pointed model* is a pair (*M, w*), where *M* is a model and *w ∈ Dom*(*M* ). We only consider models associated with a relational vocabulary containing unary and binary relation symbols. If *V* is a vocabulary, we let *V*1 and *V*2 denote the sets of unary and binary relation symbols in *V* , respectively. The following *BNF* determines the set *MP* (*V*2) of *modal parameters* over *V*2:

*M* ::= *id | R | чM |* (*M∩ M*)

Here *R ∈ V*2 and *id /∈ V* is a constant relation symbol. The following *BNF*

determines the set of formulae of *BML*= over vocabulary *V* :

*ϕ* ::= *P | чϕ |* (*ϕ ∧ ϕ*) *| ⟨M⟩ϕ*

Here *P ∈ V*1 and *M ∈ MP* (*V*2). Operators *⟨M⟩* are called *diamonds*. The *modal depth* of a formula *ϕ*, or *Md*(*ϕ*), is the maximum number of nested diamonds in *ϕ*.

Let *M* be a model. The *extension MM* of a modal parameter *M* over *M* is a binary relation over *Dom*(*M* ). The extension of *R ∈ V*2 over *M* is simply the interpretation *RM* of the symbol *R*. The extension *idM* of the symbol *id* is

*{*(*w, w*) *| w ∈ Dom*(*M* )*}*. Other modal parameters are interpreted recursively such that *чMM* = (*Dom*(*M* ) *× Dom*(*M* )) *\ MM* and (*M∩ N* )*M* = *MM ∩ NM* . The satisfaction relation H of *BML*= is defined as follows:

(*M, w*) H *P ⇔ w ∈ PM ,*

(*M, w*) H *чϕ ⇔* (*M, w*) */*H *ϕ,*

(*M, w*)*,* H *ϕ ∧ ψ ⇔* (*M, w*) H *ϕ* and (*M, w*) H *ψ,*

(*M, w*) H *⟨M⟩ ϕ ⇔ Eu ∈ Dom*(*M* ) such that (*w, u*) *∈ MM* and (*M, u*) H *ϕ.*

For each formula *ϕ*, we let *||ϕ||M* denote the set *{w ∈ Dom*(*M* ) *|* (*M, w*) H *ϕ}*. The set *||ϕ||M* is called the *extension* of *ϕ* over *M* . We write *ϕ* H *ψ* if (*M, w*) H *ϕ ⇒* (*M, w*) H *ψ* for all pointed models (*M, w*).

A formula *ϕ* of Σ1(*BML*=) of vocabulary *V* is a formula of type *ES*1*...ESnEP*1*...EPm*(*ψ*), where the variables *Si* are binary and *Pi* unary relation symbols, and *ψ* is a *BML*= formula of vocabulary *V ∪ {S*1*, ..., Sn, P*1*, ..., Pm}*. We define (*M, w*) H *ϕ* if there exists an expansion *M'* = (*M, SM' , ..., SM' ,PM' ,PM'* )

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1 *n* 1 *m*

of the model *M* such that (*M', w*) H *ψ*. We define the logic Π1(*BML*=) simi-

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larly, but with universal second-order quantifiers instead of existential ones. *ML* is the fragment of *BML*=, where the modal parameters are required to be atomic binary relation symbols in the vocabulary under discourse. *MLE* is the extension of *ML* with the universal diamond *⟨E⟩*, i.e., the diamond *⟨ч*(*id ∩ чid*)*⟩*. The logics Σ1(*ML*) and Σ1(*MLE* ) are defined in the natural way. *EMSO* (*MLE* ) is the frag-

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ment of Σ1(*MLE* ), where we only allow second-order quantifiers quantifying *unary*

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relation symbols.

We assume the reader is familiar with the systems *EMSO* (i.e., monadic Σ1) and

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*6MSO* (i.e., monadic Π1) of predicate logic. We write *M, u v |*= *ϕ*(*x, y*), if *M* satisfies

1 *x y*

the formula *ϕ*(*x, y*) of predicate logic under the assignment *x '→ u ∈ Dom*(*M* ), *y '→ v ∈ Dom*(*M* ). A modal formula *ϕ* and a formula *ψ*(*x*) of predicate logic are called *equivalent* if for all pointed models (*M, w*), (*M, w*) H *ϕ ⇔ M, w |*= *ψ*(*x*).

*x*

* 1. *Types*

Let *V* be a finite vocabulary. Let *U* be a set of size *|V*2*|* + 1 such that for all *T ∈ V*2, exactly one of *T* and *чT* is in *U* , and exactly one of *id* and *чid* is in *U* . Let *M∈ MP* (*V*2) be an intersection consisting of all the members of *U* . Note that if *|V*2*| /*= 0, there are several such intersections corresponding to *U* . Therefore, for each *U* , we always assume some standard ordering and bracketing of the related modal parameter *M*, so that there is a one-to-one correspondence between the sets *U* and the related modal parameters. We call such modal parameters *access types* (over *V* ). We let *ATP V* denote the set of access types over *V* .

Let *M* be an access type over *V* , and let *T ∈ V*2 *∪ {id}*. We write *T ≤M* if *T* occurs in *M*, i.e., *чT* does *not* occur in *M*. Let *U ⊆ V* be a finite vocabulary and *N* an access type over *U* . We say that *M* is *consistent* with *N* (or alternatively, *N* is consistent with *M*), if for all symbols *T ∈ U*2 *∪ {id}*, *T ≤M* iff *T ≤ N* .

Let (*M, w*) be a pointed model of vocabulary *V* . We define

0

*τ*

(*M,w*)

:=

*P ∈ V*1 *,*

(*M,w*) H *P*

*P ∧*

*Q ∈ V*1 *,*

(*M,w*) */*H *Q*

*чQ.*

Formula *τ* 0 is the *type* of modal depth 0 of (*M, w*). We choose formulae *τ* 0

(*M,w*)

(*M,w*)

such that if *τ* 0

(*M,w*)

0

(*N,v*)

and *τ*

= *τ*

*V*

are equivalent for some pointed models (*M, w*) and

(*N, v*), then actually *τ* 0

(*M,w*)

0

(*N,v*)

. We let *TP* 0

be the set containing exactly the

formulae *τ* such that for some pointed model (*M, w*) of vocabulary *V* , *τ* is the type

of modal depth 0 of (*M, w*). Clearly the set *TP* 0 is finite. Now assume we have

*V*

defined the formulae *τn* for all pointed models (*M, w*), and that the set *TPn* is

(*M,w*)

*V*

a finite set containing exactly all these formulae. We then define

*n*+1 (*M,w*)

*τ*

*n*

(*M,w*)

:= *τ*

*∧* *{ ⟨M⟩σ | M ∈ ATP V , σ ∈ TPn ,* (*M, w*) H *⟨M⟩σ}*

*V*

*∧* *{ ч⟨M⟩σ | M ∈ ATP V , σ ∈ TPn ,* (*M, w*) */*H *⟨M⟩σ}.*

*V*

Again we assume some standard ordering of the conjuncts and some standard brack-

eting, so that there is exactly one formula *τn*+1 . Also, we choose these formulae

(*M,w*)

and *τ*

such that if *τn*+1

(*M,w*)

*n*+1 (*N,v*)

are equivalent, then in fact *τn*+1

= *τn*+1 . Formula

*n*+1 (*M,w*)

(*M,w*)

(*N,v*)

*τ*

is the *type* of modal depth *n* + 1 of (*M, w*). We let *TPn*+1 be the set contain-

ing exactly the formulae *τ* such that for some pointed model (*M, w*) of vocabulary

*V* , *τ* is the type of modal depth *n* + 1 of (*M, w*). We see that *TPn*+1 is finite.

*V*

We then list a number of properties of types that are straightforward to prove. Let (*M, w*) be a pointed model of vocabulary *V* and let *U ⊆ V* be a finite vocabulary. Let *n ∈* N. Firstly, (*M, w*) satisfies exactly one type in *TPn* . Also, for all *τ ∈ TPn*

*V*

*U U*

and all *m ≤ n*, there exists exactly one type *σ ∈ TPm* such that *τ* H *σ*. Let *α ∈ TPn*

*U U*

and let *ψ* be an arbitrary formula of vocabulary *U* and of modal depth *m ≤ n*. Then

either *α* H *ψ* or *α* H *чψ*, and also, for all points *u, v ∈ Dom*(*M* ) *∩ ||α||M* , we have

(*M, u*) H *ψ* iff (*M, v*) H *ψ*. Finally, *ψ* is equivalent to *{τ ∈ TPn | τ* H *ψ}*.

*U*

1. Σ1(*ML*) *≤ EMSO* (*MLE* )

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In this subsection we show how to translate Σ1(*ML*) formulae to equivalent formulae of *EMSO* (*MLE* ). We begin by fixing a Σ1(*ML*) formula *ϕ* and show how to translate it to an equivalent formula *ϕ∗*(*x*) of *EMSO* . We then show that the first-order part of *ϕ∗*(*x*) translates to an equivalent *MLE* formula.

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Let *ϕ* = *Q*(*ψ*), where *Q* is a vector of existential second-order quantifiers and *ψ*

a formula of *ML*. We let *V ψ* and *V ψ* denote the sets of unary and binary relation

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symbols, respectively, that occur in *ψ*. We define *V ψ* = *V ψ ∪ V ψ*. We let *Qψ* and

1 2 1

*Qψ* denote the sets of unary and binary relation symbols, respectively, that occur

2

in *Q*. We define *Qψ* = *Qψ ∪ Qψ*. We let *SUBψ* denote the set of subformulae of *ψ*.

1 2

We then fix a fresh (i.e., not occuring in *ϕ*) symbol *Pα* for each *α ∈ SUBψ*. Before

fixing the translated formula *ϕ∗*(*x*), we define a number of auxiliary formulae. We begin by defining the following formulae for all *P, чα,* (*β ∧ γ*)*, ⟨R⟩ρ, ⟨S⟩σ ∈ SUBψ*,

where *P ∈ V ψ*, *R ∈ V ψ \ Qψ* and *S ∈ Qψ*:

1 2 2 2

where

*ψP* := *6x* *PP* (*x*) *↔ P* (*x*) *,*

*ψ¬α* := *6x* *P¬α*(*x*) *↔ чPα*(*x*) *,*

*ψ*(*β∧γ*) := *6x* *P*(*β∧γ*)(*x*) *↔* (*Pβ*(*x*) *∧ Pγ*(*x*)) *, ψ⟨R⟩ρ* := *6x* *P⟨R⟩ρ*(*x*) *↔ Ey*(*R*(*x, y*) *∧ Pρ*(*y*)) *,*

*ψ⟨S⟩σ* := *6x* *P⟨S⟩σ* (*x*) *↔ Ey*(*AccessS* (*x, y*) *∧ Pσ*(*y*)) *,*

*AccessS* (*x, y*) :=

*(S⟩χ ∈ SUBfi*

*Pχ*(*y*) *→ P⟨S⟩χ*(*x*) *.*

Finally, we define

*δψ* :=

*α ∈ SUBfi*

*ψα,* and *ϕ∗*(*x*) := *Q∗*(*δψ ∧ Pψ*(*x*))*,*

where *Q∗* is a vector of existential quantifiers quantifying all the predicate symbols

*P ∈ Qψ* and also all the symbols *Pα*, where *α ∈ SUBψ*.

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We then establish that (*M, w*) H *ϕ* implies *M, w |*= *ϕ∗*(*x*). We therefore assume that (*M, w*) H *ϕ*, whence there exists some expansion *M*2 of *M* by interpretations of the binary and unary symbols in *Qψ* such that (*M*2*, w*) H *ψ*. We then define an expansion *M*1 of *M* by interpretations of the unary predicate symbols occurring in *Q∗*. We let *PM*1 = *PM*2 for all *P ∈ Qψ*. For all *Pα*, where *α ∈ SUBψ*, we define

*x*

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*PM*1 = *||α||M*2 .

*α*

**Lemma 3.1** *Let ⟨S⟩σ ∈ SUBψ, where S ∈ Qψ. Let v ∈ Dom*(*M* )*. We have*

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(*M*2*, v*) H *⟨S⟩σ iff M*1*, v |*= *Ey*(*AccessS*(*x, y*) *∧ Pσ*(*y*))*.*

*x*

**Proof.** Assume that (*M*2*, v*) H *⟨S⟩σ*. Therefore (*v, u*) *∈ SM*2 for some *u ∈ ||σ||M*2 =

*PM*1 . In order to establish that *M*1*, v |*= *Ey*(*AccessS*(*x, y*) *∧ Pσ*(*y*)), it therefore

*σ* *x*

suffices to show that for all *⟨S⟩χ ∈ SUBψ*, if *u ∈ PM*1 , then *v ∈ PM*1 . Therefore

*χ ⟨S⟩χ*

assume that *u ∈ PM*1 for some *⟨S⟩χ ∈ SUBψ*. As *||χ||M*2 = *PM*1 , we conclude that

*χ χ*

*u ∈ ||χ||M*2 . As (*v, u*) *∈ SM*2 , we have (*M*2*, v*) H *⟨S⟩χ*. As *||⟨S⟩χ||M*2 = *PM*1 , we

*⟨S⟩χ*

have *v ∈ PM*1 , as desired.

*⟨S⟩χ*

Assume then that *M*1*, v*

*x*

*|*= *Ey*(*AccessS*(*x, y*) *∧ Pσ*(*y*)). Thus *M*1*, v u |*=

*x y*

*AccessS*(*x, y*) for some *u ∈ PM*1 = *||σ||M*2 . By the definition of the formula

*σ*

*AccessS*(*x, y*), we see immediately that *v ∈ PM*1 . As *||⟨S⟩σ||M*2 = *PM*1 , we

*⟨S⟩σ ⟨S⟩σ*

conclude that *v ∈ ||⟨S⟩σ||M*2 . Thus (*M*2*, v*) H *⟨S⟩σ*, as desired.

**Lemma 3.2** (*M, w*) H *ϕ implies M, w |*= *ϕ∗*(*x*)*.*

*x*

**Proof.** We prove the claim by establishing that *M*1*, w |*= *δψ ∧Pψ*(*x*). As (*M*2*, w*) H

*x*

*ψ* and *||ψ||M*2 = *PM*1 , we have *M*1*, w |*= *Pψ*(*x*). Hence we only need to establish

*ψ* *x*

that *M*1 *|*= *δψ*. The non-trivial part of this is showing that *M*1 *|*= *ψ⟨S⟩σ* for an

arbitrary *⟨S⟩σ ∈ SUBψ*, where *S ∈ Qψ*. This follows directly from Lemma [3.1](#_bookmark1), as

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*M*1

*P*

*⟨S⟩σ*

= *||⟨S⟩σ||M*2 .

We then show that *M, w |*= *ϕ∗*(*x*) implies (*M, w*) H *ϕ*. Therefore we assume

*x*

that *M, w |*= *ϕ∗*(*x*), whence there exists an expansion *M'* of *M* by interpretations

*x* 1

of the unary predicate symbols occurring in *Q∗* such that *M*1*, w |*= *δψ ∧ Pψ*(*x*).

*x*

We then define an expansion *M'* of *M* by interpretations of the binary and unary

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symbols occurring in *Q*. For all *P ∈ Qψ*, we let *PM'* = *PM'* . For all *S ∈ Qψ*, we

let (*v, u*) *∈ SM'*

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if and only if *M' , v u |*= *Access*

2 1 2

(*x, y*).

1 *x y S*

**Lemma 3.3** *Let α ∈ SUBψ and v ∈ Dom*(*M* )*. We have* (*M' , v*) H *α if and only*

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*if M' , v |*= *Pα*(*x*)*.*

1 *x*

**Proof.** We prove the claim by induction on the structure of *α*. As *M' |*= *δψ*, the

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claim holds trivially for all atomic formulae *P ∈ V ψ*. Also, the cases where *α* is of type *чβ*, (*β ∧ γ*) or *⟨R⟩β*, where *R ∈ V ψ \ Qψ*, are straightforward to deal with, as

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2 2

*M' |*= *δψ*.

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We assume that (*M' , v*) H *⟨S⟩σ*, where *S ∈ Qψ* and *⟨S⟩σ ∈ SUBψ*. Thus

2 2

( *M' M' v u*

*v, u*) *∈ S* 2 for some *u ∈ ||σ||* 2 . Hence *M*1*, x y |*= *AccessS* (*x, y*) by our definition

of *SM'* . By the induction hypothesis, we have *PM'* = *||σ||M'* . Therefore *u ∈ PM'* ,

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2 *σ* 2 *σ* 1

whence *M' , v |*= *Ey*(*Access* (*x, y*) *∧ Pσ*(*y*)). Therefore, as *M' |*= *ψ⟨S⟩σ* , we conclude

1 *x S* 1

*'*

that *v ∈ PM* .

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*⟨S⟩σ*

For the converse, assume that *M' , v*

*|*= *P⟨S⟩σ* (*x*). As *M' |*= *ψ⟨S⟩σ* , we conclude

1 *x* 1

*'*

that *M' , v |*= *Ey*(*Access* (*x, y*) *∧ P* (*y*)). Therefore there exists some *u ∈ PM*

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1 *x S σ σ*

such that *M' , v u |*= *Access* (*x, y*). We now have (*v, u*) *∈ SM'* by our definition of

1. *x y*

*M' M'*

2

*S*

*M' M'*

*S* 2 . As *u ∈ Pσ* 1 and as *||σ||* 2 = *Pσ* 1 by the induction hypothesis, we therefore

conclude that (*M' , v*) H *⟨S⟩σ*, as desired.

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By Lemma [3.3](#_bookmark3) we immediately see that *M, w |*= *ϕ∗*(*x*) implies (*M, w*) H *ϕ*.

*x*

This, combined with Lemma [3.2](#_bookmark2), justifies the following conclusion:

**Theorem 3.4** *Each formula of* Σ1(*ML*) *is expressible in EMSO.*

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It is easy to see that *ϕ∗*(*x*) is expressible in *EMSO* (*MLE* ): Let *S ∈ Qψ* and let *A* be the subset of *SUBψ* of formulae of type *⟨S⟩α*. Formula *Ey*(*AccessS* (*x, y*) *∧Pσ*(*y*)) is equivalent to the following formula of *MLE* :

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*B ⊆ A*

*(S⟩χ ∈ B*

*P⟨S⟩χ ∧ ⟨E⟩* *Pσ ∧*

*(S⟩χ ∈ B*

*Pχ ∧*

*(S⟩χ ∈ A\B*

*чPχ*

Thus we see that each formula *ψα*, where *α ∈ SUBψ*, can be expressed in *MLE* . We may therefore draw the following conclusion:

**Theorem 3.5** *Each formula of* Σ1(*ML*) *is expressible in EMSO* (*MLE* )*.*

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1. Σ1(*BML*=) *≤ EMSO*

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In this section we prove that each formula of Σ1(*BML*=) can be translated to an

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equivalent formula of *EMSO* .

* 1. *A translation from* Σ1(*BML*=) *into EMSO*

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In this subsection we define a translation of Σ1(*BML*=) formulae to equivalent for- mulae of *EMSO* . For this aim, we fix a Σ1(*BML*=) formula *ϕ* and show how it is translated. Let *ϕ* = *Q*(*ψ*), where *Q* is vector of existential second-order quan- tifiers and *ψ* a formula of *BML*=. For technical reasons, we assume w.l.o.g. that *Md*(*ψ*) *≥* 2. We let *V ψ* and *V ψ* denote the sets of unary and binary relation sym-

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bols, respectively, that occur in *ψ*. We define *V ψ* = *V ψ ∪ V ψ*. We let *Qψ* and *Qψ*

1 2 1 2

denote the sets of unary and binary relation symbols, respectively, that occur in *Q*.

We define *Qψ* = *Qψ ∪ Qψ*. We let *ATP ψ* denote the set of access types over *V ψ*.

1 2

Let *n ∈* N. We let *TPn*

*ψ*

denote the set of types of modal depth *n* over *V ψ*, and

define *TPψ* = *i≤Md*(*ψ*) *TPi* .

*ψ*

We then define fresh (i.e., not occuring in *ϕ*) unary predicate symbols. We fix a unique symbol *Pτ* for each *τ ∈ TPψ*. We also fix a symbol *P*(*α, M, β*) for all

*α ∈ TPMd*(*ψ*), *M∈ ATP ψ*, *β ∈ TPMd*(*ψ*)*−*1.

*ψ ψ*

The translation *ϕ∗*(*x*) of *ϕ* will be the formula

*EP*

*P ∈ Qψ*

1

*EPτ*

*τ ∈ TPψ*

*EP*(*α, M, β*)

*α ∈ TPMd*(*fi*) *,*

*fi*

*n ∈ ATP fi,*

*β ∈ TPMd*(*fi*)*−*1

*fi*

*ψ∗*(*x*) *,*

where *ψ∗*(*x*) is a first-order formula in one free variable, *x*. We let *Q∗* denote the above vector of monadic existential second-order quantifiers.

The main idea in the translation is that we can use the symbols *Pτ* in order to encode information about the extensions of types *τ* over models of vocabulary *V ψ*.

The symbols *P*(*α, M, β*), in turn, will help us fix formulae that encode information about the extensions of access types *M ∈ ATP ψ*. Before fixing the translation

*ϕ∗*(*x*) of *ϕ*, we define a number of auxiliary formulae. The first formula we define ensures that for all *n ∈ {*0*, ..., Md*(*ψ*)*}*, the extensions of the predicate symbols *Pτ* , where *τ ∈ TPn* , always cover all of the domain of a model and never overlap. We

*ψ*

define

*ψuniq* := *6x*

0 *≤ i ≤ Md*(*ψ*)

*τ ∈ TPi*

*fi*

*Pτ* (*x*) *∧*

*σ ∈ TPi , σ /*= *τ*

*fi*

*чPσ*(*x*) *.*

The next formula asserts that each predicate symbol *Pβ*, where *β ∈ TPMd*(*ψ*)*−*1,

*ψ*

must be interpreted such that for all symbols *Pτ* , where *Md*(*τ* ) *< Md*(*β*), the extension of *Pβ* is either wholly included in the extension of *Pτ* , or does not overlap with it. We let

*ψpack* := *6x6y*

*β ∈ TPMd*(*fi*)*−*1

*fi*

(*Pβ*(*x*) *∧ Pβ*(*y*)) *→*

(*Pτ* (*x*) *↔ Pτ* (*y*)) *.*

*τ ∈ TP<Md*(*fi*)*−*1

*fi*

We then define formulae that encode information about access types connecting points in models of vocabulary *V ψ*:

*AccessM*(*x, y*) :=

*α ∈ TPMd*(*fi*) *, β ∈ TPMd*(*fi*)*−*1

*fi*

*fi*

*Pα*(*x*) *∧ Pβ*(*y*) *∧ P*(*α, M, β*)(*y*) *.*

Next we define formulae for all *τ ∈ TPn* , 0 *≤ n ≤ Md*(*ψ*) that recursively force the

*ψ*

interpretations of *Pτ* to match the extensions of *τ* over models of vocabulary *V ψ*. First, let *τ ∈ TP* 0 . We define

*ψ*

*χτ* (*x*) :=

*P ∈ V fi, τ* H *P*

1

*P* (*x*) *∧*

*Q ∈ V fi, τ /*H *Q*

1

*чQ*(*x*)*.*

Now let *τ ∈ TPn*+1, where 0 *≤ n ≤ Md*(*ψ*) *−* 1. We define

*ψ*

*χ*+(*x*) :=

*τ*

*n ∈ ATP* (*ψ*)*, σ ∈ TPn ,*

*fi*

*τ* H *(n⟩σ*

*Ey*(*AccessM*(*x, y*) *∧ Pσ*(*y*))*,*

*χ−*(*x*) :=

*τ*

*чEy*(*AccessM*(*x, y*) *∧ Pσ*(*y*))*,*

and

*n ∈ ATP fi, σ ∈ TPn ,*

*τ* H *¬(n⟩σ*

*fi*

*χτ* (*x*) := *Pτ'* (*x*) *∧ χ*+(*x*) *∧ χ−*(*x*)*,*

*τ τ*

where *τ'* is the unique type in *TPn* such that *τ* H *τ'*.

*ψ*

Let *A ⊆ ATP ψ*, *A /*= *∅*. Let *α ∈ TPMd*(*ψ*) and *β ∈ TPMd*(*ψ*)*−*1. The next

*ψ*

*ψ*

formula encodes information about the *sets* of access types connecting points in

extensions of *α* to points in extensions of *β* in models of vocabulary *V ψ*. We define

*ψ*(*α, A, β*)(*x*) := *Pα*(*x*) *∧*

*n ∈ A*

*Ey*(*AccessM*(*x, y*) *∧ Pβ*(*y*))*.*

The next two formulae ensure that information about the access types realized in models of vocabulary *V ψ* is consistent with the interpretation of the access types *R ∈ ATP V ψ\Qψ* , i.e., the access types describing non-quantified binary relations. We define a linear ordering on *ATP ψ*. Let *A*(*k*) denote the *kth* member of a set

*A ⊆ ATP ψ* with respect to this ordering, and let *χA*(*k*)(*x, yk*) denote a first-order

formula stating that *x* and *yk* are connected according to the unique access type in

*ATP V ψ\Qψ* consistent with the access type *A*(*k*) *∈ A*. We define

*ψcons* := *6x*

*A ⊆ ATP fi, A /*= *$, α ∈ TPMd*(*fi*) *,*

*fi*

*β ∈ TPMd*(*fi*)*−*1

*ψ*(*α, A, β*)(*x*) *→*

*fi*

*Ey*1*, ..., y|A|*

*i, j ∈ {*1 *,..., |A|}, i /*= *j*

*yi /*= *yj ∧*

*k ∈ {*1 *,..., |A|}*

(*χA*(*k*)(*x, yk*) *∧ Pβ*(*yk*)) *.*

For each *R ∈ ATP V ψ\Qψ* , we let *c*(*R*) denote the set *A ⊆ ATP ψ* of access types that are consistent with *R*. We define

*'*

*ψ*

*cons*

:= *6x*

*Y ∈ ATP V fi\Qfi , β ∈ TPMd*(*fi*)*−*1

*fi*

*Ey*(*ψR*(*x, y*) *∧ Pβ*(*y*)) *→*

*Ey*(*AccessM*(*x, y*) *∧ Pβ*(*y*)) *,*

*n ∈ c*(*Y*)

where *ψR*(*x, y*) denotes a first-order formula stating that *x* and *y* are connected according to the access type *R*.

Finally, we define

*δψ* := *ψuniq ∧ ψpack ∧ ψcons ∧ ψ'*

*∧*

*cons*

*τ ∈ TPfi*

*6x* *Pτ* (*x*) *↔ χτ* (*x*)

and

*ϕ∗*(*x*) := *Q∗* *δψ ∧*

*α ∈ TPMd*(*fi*) *, α* H *ψ*

*fi*

*Pα*(*x*) *.*

We then fix an arbitrary pointed model (*M, w*) of vocabulary *V ψ \ Qψ*. In the next subsection we establish that (*M, w*) H *ϕ* if and only if *M, w |*= *ϕ∗*(*x*).

*x*

* 1. Σ1(*BML*=) *≤ EMSO*

1

We first show that (*M, w*) H *ϕ* implies *M, w*

*x*

*|*= *ϕ∗*(*x*). Thus we assume that

(*M, w*) H *ϕ*. Therefore there exists some expansion *M*2 of *M* by interpretations of the binary and unary symbols in *Qψ* such that (*M*2*, w*) H *ψ*.

We then define an expansion *M*1 of *M* by interpreting the unary variable symbols

*ψ*

in *Qψ*, and also the symbols *Pτ* and *P*

1

(*α,M,β*)

for all *τ ∈ TPψ*, *α ∈ TPMd*(*ψ*),

*M ∈ ATP ψ*, *β ∈ TPMd*(*ψ*)*−*1. For all *P ∈ Qψ*, we define *PM*1 = *PM*2 . For

*ψ* 1

all *τ ∈ TPψ*, we let *PM*1 = *||τ||M*2 . We choose exactly one point from each set

*τ*

*||α||M*2 *⊆ Dom*(*M* ), where *α ∈ TPMd*(*ψ*) and *||α||M*2 */*= *∅*. We call such a point the *selector* of *||α||M*2 and denote it by *vα*. We use selectors in order to fix extensions of the predicate symbols *P*(*α, M, β*). For each *α ∈ TPMd*(*ψ*), *M ∈ ATP ψ*, and

*ψ*

*ψ*

*β ∈ TPMd*(*ψ*)*−*1, where *||α||M*2 */*= *∅*, we define

*ψ*

*M*1

*P*

(*α, M, β*)

= *{u ∈ Dom*(*M* ) *|* (*vα, u*) *∈ MM*2 *, u ∈ PM*1 *}.*

If *||α||M*2 = *∅*, we define *PM*1 = *∅*.

*β*

(*α, M, β*)

Next we prove a number of auxiliary lemmata, and then establish that *M*1*, w |*=

*x*

*ψ∗*(*x*). The first two lemmata follow directly from the above definitions.

**Lemma 4.1** *Let α ∈ TPMd*(*ψ*) *and M∈ ATP ψ. Let u ∈ Dom*(*M* )*. Then* (*vα, u*) *∈* *MM*2 *if and only if M*1*, vα u |*= *AccessM*(*x, y*)*.*

*ψ*

*x y*

**Lemma 4.2** *Let α ∈ TPMd*(*ψ*) *and M ∈ ATP ψ. Let v ∈ PM*1 *. Then, for all*

*ψ*

*α*

*u ∈ Dom*(*M* )*, M*1*, v u |*= *AccessM*(*x, y*) *if and only if M*1*, vα u |*= *AccessM*(*x, y*)*.*

*x y x y*

We then show that the formula *AccessM*(*x, y*) captures all the relevant infor- mation about the action of the diamond *⟨M⟩* on *M*2:

**Lemma 4.3** *Let τ ∈ TP<Md*(*ψ*) *and M ∈ ATP ψ. Let v ∈ Dom*(*M* )*. Then*

*ψ*

(*M*2*, v*) H *⟨M⟩τ if and only if M*1*, v |*= *Ey*(*AccessM*(*x, y*) *∧ Pτ* (*y*))*.*

*x*

**Proof.** Assume that (*M*2*, v*) H *⟨M⟩τ* . Let *α* be the type in *TPMd*(*ψ*) such that

*ψ*

*v ∈ ||α||M*2 . As (*M*2*, v*) H *⟨M⟩τ* , also (*M*2*, vα*) H *⟨M⟩τ* . Therefore there ex-

ists some *u ∈ ||τ||M*2 such that (*vα, u*) *∈ MM*2 . We conclude that *M*1*, vα u |*=

*x y*

*AccessM*(*x, y*) by Lemma [4.1](#_bookmark4), and therefore *M*1*, v u |*= *AccessM*(*x, y*) by Lemma

*x y*

* 1. As *u ∈ ||τ||M*2 = *PM*1 , we have *M*1*, v u |*= *AccessM*(*x, y*) *∧ Pτ* (*y*), whence

*τ* *x y*

*M*1*, v |*= *Ey*(*AccessM*(*x, y*) *∧ Pτ* (*y*)), as desired.

*x*

Assume *M*1*, v |*= *Ey*(*AccessM*(*x, y*) *∧ Pτ* (*y*)). We conclude, using Lemma [4.2](#_bookmark5),

*x*

that *M*1*, vα u |*= *AccessM*(*x, y*)*∧Pτ* (*y*) for some *u ∈ PM*1 . Therefore (*vα, u*) *∈ MM*2

*x y τ*

by Lemma [4.1](#_bookmark4). As *PM*1 = *||τ||M*2 , we conclude that (*M*2*, vα*) H *⟨M⟩τ* , and therefore (*M*2*, v*) H *⟨M⟩τ* .

*τ*

Interpretations of the formulae *χτ* (*x*) and the predicate symbols *Pτ* coincide:

**Lemma 4.4** *Let v ∈ Dom*(*M* ) *and τ ∈ TPψ. Then M*1*, v |*= *Pτ* (*x*) *iff M*1*, v |*=

*x*

*x*

*χτ* (*x*)*.*

**Proof.** As *||P ||M*2 = *PM*1 for all *P ∈ V ψ*, the claim follows directly for all *τ ∈ TP* 0 .

1 *ψ*

Therefore we may assume that *τ ∈ TP≥*1. Throughout the proof, we let *τ'* denote

*ψ*

the unique type in *TPMd*(*τ*)*−*1 such that *τ* H *τ'*.

*ψ*

Assume that *M*1*, v |*= *Pτ* (*x*). As *PM*1 = *||τ||M*2 , we have (*M*2*, v*) H *τ* . As *τ* H *τ'*,

*x*

we have (*M , v*) H *τ'*. Since *PM*1

*τ*

= *||τ '||M*2 , we conclude that *M , v*

*|*= *P '* (*x*).

1. *τ'* 1 *x τ*

We still need to establish that *M*1*, v |*= *χ*+(*x*) *∧ χ−*(*x*). Therefore assume that

*x τ τ*

*τ* H *⟨M⟩σ*, where *M ∈ ATP ψ* and *σ ∈ TPMd*(*τ*)*−*1. As (*M*2*, v*) H *τ* , we have

*ψ*

(*M*2*, v*) H *⟨M⟩σ*. Therefore *M*1*, v |*= *Ey*(*AccessM*(*x, y*) *∧ Pσ*(*y*)) by Lemma [4.3](#_bookmark6).

*x*

Similarly, if *τ* H *ч⟨M⟩σ*, we conclude that *M*1*, v |*= *чEy*(*AccessM*(*x, y*) *∧ Pσ*(*y*)) by

*x*

Lemma [4.3](#_bookmark6). Thus *M*1*, v |*= *χ*+(*x*) *∧ χ−*(*x*), as desired.

*x τ τ*

Assume then that *M*1*, v |*= *χτ* (*x*). In order to show that *M*1*, v*

*x*

*x*

*|*= *Pτ* (*x*), we

will establish that (*M*2*, v*) H *τ* . This suffices, as *PM*1 = *||τ||M*2 . We immediately

*τ*

*M*1

see that (*M , v*) H *τ'*, as *M , v |*= *P '* (*x*) and *P*

= *||τ '||M*2 . Assume then that

2 1 *x τ τ'*

*τ* H *⟨M⟩σ*, where *M ∈ ATP ψ* and *σ ∈ TPMd*(*τ*)*−*1. As *M*1*, v |*= *χ*+(*x*), we have

*ψ x τ*

*M*1*, v |*= *Ey*(*AccessM*(*x, y*) *∧ Pσ*(*y*)), and therefore (*M*2*, v*) H *⟨M⟩σ* by Lemma

*x*

* 1. Similarly, if *τ* H *ч⟨M⟩σ*, then, as *M*1*, v |*= *χ−*(*x*), we conclude that *M*1*, v |*=

*x τ* *x*

*чEy*(*AccessM*(*x, y*) *∧ Pσ*(*y*)), and therefore (*M*2*, v*) H *ч⟨M⟩σ* by Lemma [4.3](#_bookmark6). Thus (*M*2*, v*) H *τ* , and hence *M*1*, v |*= *Pτ* (*x*), as desired.

*x*

We then conclude the first direction of the proof:

**Lemma 4.5** *If* (*M, w*) H *ϕ, then M, w |*= *ϕ∗*(*x*)*.*

*x*

**Proof.** We establish the claim by showing that *M*1*, w |*= *ψ∗*(*x*).

*x*

Let *ψ'* denote a disjunction of all types *α ∈ TPMd*(*ψ*) such that *α* H *ψ*. As

*ψ*

*ψ* and *ψ'* are equivalent, we have (*M*2*, w*) H *ψ'*. Therefore (*M*2*, w*) H *α* for some

*α*

*α ∈ TPMd*(*ψ*)

*ψ*

occurring in the disjunction. Hence, as *||α||M*2 = *PM*1 , we have

*M*1*, w |*= *Pα*(*x*).

*x*

We then show that *M*1 *|*= *ψcons*. Let *v ∈ Dom*(*M* ) and assume *M*1*, v |*=

*x*

*ψ* (*x*) for some *α ∈ TPMd*(*ψ*), *A ⊆ ATP ψ*, *β ∈ TPMd*(*ψ*)*−*1. Recall that we

(*α, A, β*) *ψ ψ*

may write *A* = *{A*(1)*, ..., A*(*|A|*)*}*, where *A*(*k*) refers to the *kth* member of the set *A*

w.r.t. the ordering of *ATP ψ* we fixed. As *M*1*, v |*= *ψ*(*α, A, β*)(*x*), we see by Lemma

*x*

* 1. that (*M*2*, v*) H *⟨A*(*k*)*⟩β* for all *k ∈ {*1*, ..., |A|}*. Thus there must exist distinct

points *u*1*, ..., u|A| ∈ ||β||M*2 = *PM*1 such that (*v, uk*) *∈ A*(*k*)*M*2 for each *k*. Let *Rk* be the type in *ATP V ψ\Qψ* consistent with *A*(*k*). Recall that *χA*(*k*)(*x, yk*) is a first-order

*β*

formula stating that *x* is connected to *yk* by *Rk*. We have (*v, uk*) *∈ RM*2 = *RM*1

*k k*

for each *k*, and thus *M*1*, v uk |*= *χA*(*k*)(*x, yk*) *∧ Pβ*(*yk*) for each *k*.

*x y*

We then show that *M*1 *|*= *ψ'* . Let *v ∈ Dom*(*M* ) and assume *M*1*, v u |*=

*cons*

*x y*

*ψ*

*ψR*(*x, y*) for some *u ∈ PM*1

*β*

with *β ∈ TPMd*(*ψ*)*−*1 and some *R ∈ ATP*

*ψ\Q*

*ψ* . Let

*M* be the access type in *ATP ψ* such that (*v, u*) *∈ MM*2 . Thus (*M*2*, v*) H *⟨M⟩β*, whence *M*1*, v |*= *Ey*(*AccessM*(*x, y*) *∧Pβ*(*y*)) by Lemma [4.3](#_bookmark6). Clearly *M* is consistent

*x*

*V*

with *R*. Therefore *M*1 *|*= *ψ'* .

*cons*

We have *M*1 *|*= *ψuniq ∧ ψpack* directly by properties of types. Therefore, in order to conclude the proof, we only need to establish that for all *τ ∈ TPMd*(*ψ*) and all *v ∈ Dom*(*M* ), *M*1*, v |*= *Pτ* (*x*) *↔ χτ* (*x*). This follows directly from Lemma [4.4](#_bookmark7).

*ψ*

*x*

We then show that *M, w |*= *ϕ∗*(*x*) implies (*M, w*) H *ϕ*. Thus we assume that

*x*

*M, w |*= *ϕ∗*(*x*). Therefore there exists an expansion *M'* of *M* by interpretations of

*x* 1

the unary symbols *Pτ* and *P*(*α, M, β*) for all *τ ∈ TPψ*, *α ∈ TPMd*(*ψ*), *M ∈ ATP ψ*,

*ψ*

*β ∈ TPMd*(*ψ*)*−*1, and also the symbols *P ∈ Qψ*, such that *M' , w |*= *ψ∗*(*x*).

*ψ* 1 1 *x*

We then define an expansion of *M* by interpreting all the relation symbols, unary

and binary, in *Qψ*. We call the resulting expansion *M'* . For all *P ∈ Qψ*, we define

*M' M' M'*

2 1

*Md*(*ψ*)*−*1

*P* 2 = *P*

1 . Let *v ∈ Pα* 1 and *β ∈ TPψ* . Let *A ⊆ ATP ψ* be the set of access

*'*

types *M* such that *M' , v u |*= *Access* (*x, y*) for some *u ∈ PM* . As *M'*

1

satisfies

1. *x y M β* 1

*'*

the formula *ψ* , we see that there exists a bijection *f* from *A* to a set *B ⊆ PM*

1

*cons*

such that for all *M ∈ A*, we have (*v, f* (*M*)) *∈ RM'* , where *R*

1

*β*

is the access type

*M M*

in *ATP* consistent with *M*. Let *S ∈ Qψ*. We define (*v, f* (*M*)) *∈ SM'* if

*V ψ\Qψ*

2 2

*M' M'*

*S ≤ M*. We then consider the points in *Pβ* 1 *\ B*. Thus let *u ∈ Pβ* 1 *\ B*. Let *T* be

the access type in *ATP*

such that (*v, u*) *∈T M'* . As *M'* satisfies *ψ'*

, we see

*V ψ\Qψ*

1 1 *cons*

that there exists some *M∈ ATP*

consistent with *T* and some *u' ∈ PM'* such that

*M' , v u'*

1

*|*= *Access*

*ψ*

(*x, y*). We define, for all *S ∈ Qψ*, (*v, u*) *∈ SM'*

*β*

if *S ≤ M*. We

1 *x y M* 2 2

go through each *α ∈ TPMd*(*ψ*) and *β ∈ TPMd*(*ψ*)*−*1, and construct the extensions

*ψ ψ*

*SM' ψ '*

2 of the symbols *S ∈ Q*2 in the described way. As *M*1 satisfies *ψuniq*, each pair in *Dom*(*M* ) *× Dom*(*M* ) becomes associated with exactly one access type in *ATP ψ*.

Therefore *M'* is well defined.

2

We first prove a number of auxiliary lemmata, and then establish that (*M' , w*) H

2

*ψ*. The following lemma is a direct consequence of the way we define the extensions

*SM' ψ*

2 of the relation symbols *S ∈ Q*2 .

**Lemma 4.6** *Let β ∈ TPMd*(*ψ*)*−*1 *and M ∈ ATP ψ. Let v ∈ Dom*(*M* )*. Then*

*ψ*

*' '*

(*v, u*) *∈ MM' for some u ∈ PM iff M' , v u' |*= *Access* (*x, y*) *for some u' ∈ PM .*

1 1

2

*x y M β*

*β* 1

We then show that the diamond *⟨M⟩* captures relevant information about the relation that the formula *AccessM*(*x, y*) defines over *M'* :

1

*'*

**Lemma 4.7** *Let τ ∈ TP<Md*(*ψ*) *and M∈ ATP . Assume that ||τ||M'* = *PM*

*and*

1

2

*ψ ψ τ*

*let v ∈ Dom*(*M* )*. Then* (*M' , v*) H *⟨M⟩τ iff M' , v |*= *Ey*(*AccessM*(*x, y*) *∧ Pτ* (*y*))*.*

1. 1 *x*

*'*

**Proof.** Assume (*M' , v*) H *⟨M⟩τ* . Thus (*v, u*) *∈ MM'* for some *u ∈ ||τ||M'* = *PM* .

2 2 2 *τ* 1

As *M'*

*'*

*|*= *ψ* , there is a unique *β ∈ TPMd*(*ψ*)*−*1 such that *u ∈ PM* . Therefore

1

1

*M' , v u'*

*uniq*

*|*= *Access*

*ψ*

(*x, y*) for some *u' ∈ PM'*

1

*β*

by Lemma [4.6](#_bookmark9). Since *M'*

*|*= *ψ*

1 *x y M β*

1 *pack*

and as *u ∈ PM' ∩ PM'* and *u' ∈ PM'* , we have *u' ∈ PM'* . Therefore *M' , v |*=

1 1 1

*τ β β*

1

*τ* 1 *x*

*Ey*(*AccessM*(*x, y*) *∧ Pτ* (*y*)).

For the converse, assume *M' , v |*= *Ey*(*AccessM*(*x, y*) *∧ Pτ* (*y*)). Thus *M' , v u |*=

1 *x* 1 *x y*

*M' M' '*

*Access**M*(*x, y*) for some *u ∈ Pτ* 1 = *||τ||* 2 . As *M*1 *|*= *ψuniq*, there is a unique

*Md*(*ψ*)*−*1 *M'*

*' M'*

*β ∈ TPψ* such that *u ∈ Pβ* 1 . By Lemma [4.6](#_bookmark9), we therefore have (*v, u* ) *∈M* 2

for some *u' ∈ PM'* . Since *M' |*= *ψ*

and as *u ∈ PM' ∩ PM'* and *u' ∈ PM'* , we

1

*β* 1 *pack*

1 1 1

*τ β β*

have *u' ∈ PM'* . As *PM'* = *||τ||M'* , we conclude that (*M' , v*) *|*= *⟨M⟩τ* .

*τ* 1 *τ* 1 2 2

Next we show that extensions of the types *τ ∈ TPψ* and interpretations of the predicate symbols *Pτ* coincide, and then conclude the section.

**Lemma 4.8** *Let τ ∈ TPψ and v ∈ Dom*(*M* )*. Then* (*M' , v*) H *τ iff M' , v |*= *Pτ* (*x*)*.*

2 1 *x*

**Proof.** We prove the claim by induction on the modal depth of *τ* . If *τ ∈ TP* 0 ,

*ψ*

then, as *M' |*= *6x*(*Pτ* (*x*) *↔ χτ* (*x*)), the claim follows immediately.

1

Assume that (*M' , v*) H *τ* for some *τ ∈ TPn*+1 with *n ≥* 0. We will show that

2 *ψ*

*M' , v |*= *Pτ'* (*x*) *∧ χ*+(*x*) *∧ χ−*(*x*), where *τ'* is the type of modal depth *n* such that

1 *x τ τ*

*τ* H *τ'*. This directly implies that *M' , v |*= *Pτ* (*x*), since *M' |*= *6x*(*Pτ* (*x*) *↔ χτ* (*x*)).

1. *x* 1

As *τ* H *τ'*, we have (*M' , v*) H *τ'*. Therefore *M*1*, v |*= *Pτ'* (*x*) by the induction

1. *x*

hypothesis. In order to establish that *M' , v |*= *χ*+(*x*) *∧ χ−*(*x*), let *τ* H *⟨M⟩σ*, where

1 *x τ τ*

*n '*

*M' M'*

*σ ∈ TPψ* and *M ∈ ATP ψ*. Therefore (*M*2*, v*) H *⟨M⟩σ*. Since *||σ||* 2 = *Pσ* 1 by

the induction hypothesis, we conclude that *M' , v |*= *Ey*(*AccessM*(*x, y*) *∧ Pσ*(*y*)) by

1 *x*

Lemma [4.7](#_bookmark10). Similarly, if *τ* H *ч⟨M⟩σ*, then *M' , v |*= *чEy*(*AccessM*(*x, y*) *∧ Pσ*(*y*))

1 *x*

by Lemma [4.7](#_bookmark10) and the induction hypothesis. Thus *M' , v |*= *χ*+(*x*) *∧ χ*+(*x*).

1. *x τ τ*

Assume then that *M' , v |*= *Pτ* (*x*), where *τ ∈ TPn*+1. Now, since *M' |*= *δψ*, we

1 *x ψ* 1

conclude that *M' , v |*= *χτ* (*x*). Therefore *M' , v |*= *Pτ'* (*x*), where *τ'* is the type of

1 *x* 1 *x*

modal depth *n* such that *τ* H *τ'*. Thus (*M*2*, v*) H *τ'* by the induction hypothesis.

Assume then that *τ* H *⟨M⟩σ*, where *σ ∈ TPn* and *M∈ ATP ψ*. As *M' , v |*= *χτ* (*x*),

*ψ* 1 *x*

we have *M' , v |*= *χ*+(*x*), and therefore *M' , v |*= *Ey*(*AccessM*(*x, y*) *∧ Pσ*(*y*)). Hence,

1 *x τ* 1 *x*

as *||σ||M'*

2

= *PM*

*'*

*σ* 1 by the induction hypothesis, we conclude that (*M' , v*) H *⟨M⟩σ*

2

by Lemma [4.7](#_bookmark10). Similarly, if *τ* H *ч⟨M⟩σ*, we conclude that (*M' , v*) H *ч⟨M⟩σ* using Lemma [4.7](#_bookmark10) and the induction hypothesis. We have therefore established that (*M' , v*) H *τ* , as required.

2

2

**Lemma 4.9** *If M, w |*= *ϕ∗*(*x*)*, then* (*M, w*) H *ϕ.*

*x*

**Proof.** We prove the claim by showing that (*M' , w*) H *ψ*. As *M' , w |*= *ψ∗*(*x*),

1. 1 *x*

we have *M' , w |*= *Pα*(*x*) for some type *α ∈ TPMd*(*ψ*) such that *α* H *ψ*. Therefore

1 *x*

(*M' , w*) H *α* by Lemma [4.8](#_bookmark11). As *α* H *ψ*, we have (*M' , w*) H *ψ*, as desired.

2 2

The following theorem is a direct consequence of Lemmata [4.5](#_bookmark8) and [4.9](#_bookmark12):

**Theorem 4.10** *Each formula of* Σ1(*BML*=) *translates to an equivalent formula*

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*of EMSO.*

Theorem [4.10](#_bookmark14) implies a range of decidability results:

**Theorem 4.11** *Let V and U ⊆ V be sets of indices. Let Ð be a class of Kripke frames* (*W, {Rj}j∈U* )*. Consider the class C* = *{* (*W, {Ri}i∈V* ) *|* (*W, {Rj}j∈U* ) *∈Ð } of Kripke frames. Now, if the 6MSO-theory of Ð is decidable, then the satisﬁability problem for BML*= *w.r.t. C is decidable.*

**Proof.** Given a formula *ϕ* of *BML*=, we existentially quantify all the relation sym- bols occurring *ϕ*, except for those in *{Rj}j∈U* . We end up with a Σ1(*BML*=) formula, which we then effectively translate to an equivalent *EMSO* formula *ϕ∗*(*x*), applying our result. We then modify this formula to an *EMSO sentence χ* equiva- lent to *Ex*(*ϕ∗*(*x*)). Using the decision procedure for the *6MSO* -theory of *Ð*, we then check whether the sentence *χ* is satisfiable w.r.t. *Ð*. If yes, then *ϕ* is satisfiable

1

w.r.t. *C*, and if not, then *ϕ* is not satisfiable w.r.t. *C*.

# 5 Conclusions

We have investigated the expressive power of modal logics with prenex quantification of binary relations. We have shown that Σ1(*BML*=) translates into *EMSO* , and also that Σ1(*ML*) translates into *EMSO* (*MLE* ). We have briefly discussed how these results can be used in order to prove decidability results for (extensions of) multimodal logic.

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It remains to be seen whether our investigations provide a stepping stone towards settling the question about the existence of any class of directed graphs definable in Σ1(*FO*2) but not definable in *EMSO* .

1

# References

1. Blackburn P., M. de Rijke and Y. Venema, “Modal Logic,” Cambridge University Press, 2001.
2. Costello T. and A. Patterson, *Quantifiers and Operations on Modalities and Contexts*, in Proc. of KR’98, 1998, pages 270-281.
3. Fischer M. and R. Ladner, *Propositional Dynamic Logic of Regular Programs*, Journal of Computer and System Sciences **18** (1979), pages 194-211.
4. Gargov G., S. Passy and T. Tinchev, *Modal environment for boolean speculations*, in D. Skordev, editor, Mathematical Logic and its Applications, 1987, pages 253-263.
5. Goranko V. and S. Passy, *Using the Universal Modality: Gains and Questions*, Journal of Logic and Computation **2** (1992), pages 5-30.
6. Gr¨adel E. and E. Rosen, *Two-Variable Descriptions of Regularity*, in Proc. of LICS 1999, pages 14-23.
7. Harel D., D. Kozen and J. Tiyrun, “Dynamic Logic,” MIT Press, 2000.
8. Hustadt U. and R. A. Schmidt, *Issues of Decidability for Description Logics in the Framework of* *Resolution*, in Proc. of FTP 1998, pages 191-205.
9. Immerman N., “Descriptive Complexity,” Graduate Texts in Computer Science, Springer-Verlag, 1999.
10. Lehtinen S., “Generalizing the Goldblatt-Thomason Theorem and Modal Definability,“ Ph.D. thesis, Acta Universitatis Tamperensis, University of Tampere, 2008.
11. Leivant D., *Propositional Dynamic Logic with Program Quantifiers*, Electronic Notes in Theoretical Computer Science **218** (2008), pages 231-240.
12. Lutz C. and U. Sattler, *The Complexity of Reasoning with Boolean Modal Logics*, In Proc. of AiML 2000, pages 329-348.
13. Lutz C., U. Sattler and F. Wolter, *Modal Logic and the Two-Variable Fragment*, Proc. of CSL 2001, pages 247-261.
14. Massacci F., *Decision Procedures for Expressive Description Logics with Intersection, Composition, Converse of Roles and Role Identity*, in Proc. of IJCAI 2001, pages 193-198.
15. Pesonen R., “Propositional State Transition Logics,“ Thesis in Minor Subject, University of Tampere, 2008.
16. Rabin M. O., *Decidability of second-order theories and automata on infinite trees*, Transactions of the American Mathematical Society **141** (1969), pages 1-35.
17. de Rijke M., *The Modal Logic of Inequality*, Journal of Symbolic Logic **57** (1992), pages 566-584.
18. Thomas W., *Constructing Infinite Graphs with a Decidable MSO-Theory*, in Proc. of MFCS 2003, pages 113-124.