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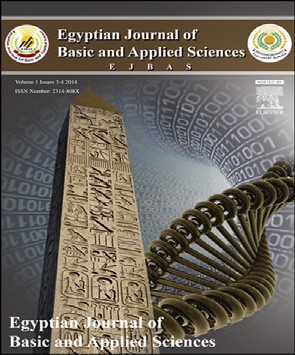
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Numerical computation of fractional BlackeScholes equation arising in financial market

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## a b s t r a c t

The aim of present paper is topresent a numerical algorithm for time-fractional BlackeScholes equation with boundary condition for a European option problem by using homotopy pertur- bation method and homotopy analysis method. The fractional derivative is described in the Caputo sense. The methods give an analytic solution in the form of a convergent series with easily computable components, requiring no linearization or small perturbation. The methods show improvements over existing analytical techniques. Two examples are given and show that the homotopy perturbation method and homotopy analysis method are very effective and convenient overcomes the difficulty of traditional methods. The numerical results show that the approaches are easy to implement and accurate when applied to time-fractional BlackeScholes equation.

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# Introduction

In 1973, Fischer Black and Myron Scholes [[1]](#_bookmark26) derived the famous theoretical valuation formula for options. The main conceptual idea of Black and Scholes lie in the construction of a riskless portfolio taking positions in bonds (cash), option and the underlying stock. Such an approach strengthens the use of

the no-arbitrage principle as well. Thus, the BlackeScholes formula is used as a model for valuing European (the option can be exercised only on a specified future date) or American (the option can be exercised at any time up to the date, the option expires) call and put options on a non-dividend paying stock [[2]](#_bookmark27). Derivation of a closed form solution to the Black- eScholes equation depends on the fundamental solution of the heat equation. Hence, it is important, at this point, to

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transform the BlackeScholes equation to the heat equation by change of variables. Having found the closed form solution to the heat equation, it is possible to transform it back to find the corresponding solution of the BlackeScholes PDE. Financial models were generally formulated in terms of stochastic dif- ferential equations. However, it was soon found that under certain restrictions these models could written as linear evolutionary PDEs with variable coefficients [[3]](#_bookmark28). Thus, the BlackeScholes model for the value of an option is described by the equation

2 2 2

*v* *x*; *T* = max *x* — *E*; 0 ; *x*2*R*+; *v* 0; *t* = 0; *t*2 0; *T* (4)

# Basic definitions of fractional calculus

In this section, we mention the following basic definitions of fractional calculus which are used further in the present paper.

Definition 1. The Riemann-Liouville fractional integral oper-

ator of order a > 0, of a function *f* (*t*)2*C* ; m ≥ —1 is defined as

v*v* + s *x* v *v* + *r* *t* *x* v*v* — *r* *t* *v* = 0; *x*; *t* 2*R*+ × 0; *T* (1) m

Z

v*t*

2

v*x*2

v*x*

[[15]](#_bookmark30):

where *v*(*x*,*t*) is the European call option price at asset price *x* and at time *t*, *K* is the exercise price, *T* is the maturity, *r*(*t*) is the risk free interest rate, and s(*x*,*t*) represents the volatility

1 *t*

*J*a*f* (*t*)=

G(a)

0

(*t* — t)a—1*f* (t)dt; (a > 0); (5)

function of underlying asset. Let us denote by *vc*(*x*,*t*) and *vp*(*x*,*t*)

the value of the European call and put options, respectively.

Then, the payoff functions are

*vc x*; *t* = max *x* — *E*; 0 ; *vp x*; *t* = max *E* — *x*; 0 ; (2) where *E* denotes the expiration price for the option and the

function *max*(*x*,0) gives the larger value between *x* and 0. During the past few decades, many researchers studied the existence of solutions of the Black-Scholes model using many methods in [[4](#_bookmark29)e[12]](#_bookmark29). Many important phenomena are well

described by fractional differential equations in electromag-

*J*0*f t* = *f t* . (6)

For the Riemann-Liouville fractional integral we have:

*J*a*t*g = G(g + 1) *t*a+g. (7)

G(g + a + 1)

Definition 2. The fractional derivative of *f*(*t*) in the Caputo

sense is defined as [[36]](#_bookmark40):

*t*

*D*a*f* (*t*)= *Jm*—a*Dnf* (*t*  1 (*t* — t)*m*—a—1*f* (*m*)(t)dt; (8)

)=

Z

*t*

netics, acoustics, viscoelasticity, electro chemistry and ma-

terial science. That is because of the fact that, a realistic

G(*n* — a)

0

modelling of a physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional calculus. The

for *m* — 1 < a ≤ *m*; *m*2*N*; *t* > 0.

For the Riemann-Liouville fractional integral and the

Caputo fractional derivative, we have the following relation

book by Oldham and Spanier [[13]](#_bookmark24) has played a key role in the

development of the subject. Some fundamental results related

*m*—1 *k*

*J*a*D*a*f* (*t*)= *f* (*t*)— *f* (*k*)(0 

X

*t*

(9)

*t t*

to solving fractional differential equations may be found in Miller and Rose [[14]](#_bookmark25), Podlubny [[15]](#_bookmark30), Kilbas et al. [[16]](#_bookmark31), Podlubny

*k*=0

+) .

*k*!

[[17]](#_bookmark32).

The objective of this paper is to extend the application of

Definition 3. The Mittag-Leffler is defined as [[37]](#_bookmark41):

homotopy perturbation method (HPM) and homotopy analysis ∞ *k*

; (a

*E* (*z*)= X *z* 2*C*; Re(a) > 0). (10)

method (HAM) to obtain analytic and approximate solution fractional BlackeScholes equations. The homotopy perturba- tion method is first proposed and applied by Chinese mathe- matician He [[18](#_bookmark33)e[22]](#_bookmark33). The method was successfully applied to

a

*k*=0

G(a*k* + 1)

spaceetime fractional advectionedispersion equation by Yil- dirim and Kocak [[23]](#_bookmark34), fractional ZakharoveKuznetsov equa- tions by Yildirim and Gulkanat [[24]](#_bookmark35), fractional modified Kdv equation by Abdulaziz et al. [[25]](#_bookmark36), Fractional Chemical Engi- neering equation by Khan et al. [[26]](#_bookmark37). One of the highly appli- cable analytical techniques is homotopy analysis method

(HAM), which was introduced and developed by Liao [[27](#_bookmark38)e[31]](#_bookmark38).

# Basic idea of homotopy perturbation method (HPM)

To illustrate the basic idea of the HPM for fractional differ- ential equations, we consider the following problem

*D*a *u* *x*; *t* = *v* *x*; *t* — *Lu* *x*; *t* — *Nu* *x*; *t* ; *m* — 1 < a ≤ *m*;

This method is applied to solve many nonlinear problems [[32](#_bookmark39)e[35]](#_bookmark39) and the references therein to handle a wide variety of scientific and engineering applications: linear and nonlinear

\**t*

*m*2*N*; *t* ≥ 0; *x*2*Rn*

Subject to the initial and boundary conditions

(11)

as well as homogeneous and inhomogeneous.

The fractional BlackeScholes equation can be written as

*u*(*i*)

(0; 0)= *ci*; *B*

*u*; v*u*; v*u*

v*xj* v*t*

= 0; *i* = 0; 1; 2;…; *m* — 1; *j* = 1; 2; 3;…; *n*

(12)

a s2*x*2 2

*Dt v* + 2 *Dxv* + *r*(*t*) *xDxv* — *r*(*t*)*v* = 0; 0 < a ≤ 1; (3)

where *L* is a linear operator, while *N* is a nonlinear operator, *v*

is a known analytical function and *D*a denotes the fractional

\**t*

equipped with the terminal and boundary condition derivative in the Caputo sense [[15]](#_bookmark30). *u* is assumed to be a causal

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derivative of *u*,*ci*, *i* = 0,1,2,…,*m*—1 are the specified initial function of time, i.e., vanishing for *t* < 0. Also *u*(*i*)(*x*,*t*) is the *i*th conditions and *B* is a boundary operator.

We construct the following homotopy

1 — *p* *D*a *u* *x*; *t* + *p* *D*a *u* *x*; *t* + *Lu* *x*; *t* + *Nu* *x*; *t* — *v* *x*; *t* = 0;

\**t*

\**t*

where *NF* is a nonlinear fractional operator, *x* and *t* denote the independent variables and *u* is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of the HAM, we first construct the so-called zeroth-order deformation equa-

tion as

*p*2 0; 1

(13)

(1 — *q*)*LF*[f(*x*; *t*; *q*)— *u*0(*x*; *t*)] = Z*qH*(*x*; *t*)*NF*[*u*(*x*; *t*)]; (22)

which is equivalent to

*D*a *u x*; *t* + *p*(*Lu*(*x*; *t*)+ *Nu*(*x*; *t*)— *v*(*x*; *t*)) = 0; *p*2 0; 1 (14) unity. In case *p* = 0, Eq. [(14)](#_bookmark4) becomes The homotopy parameter *p* always changes from zero to

*D u* *x*; *t* = 0; (15)

\**t*

a

\**t*

when *p* = 1, Eq. [(14)](#_bookmark4) turns out to be the original fractional differential equation. The homotopy parameter *p* is used to

expand the solution in the following form

*u* *x*; *t* = *u*0 *x*; *t* + *pu*1 *x*; *t* + *p*2*u*2 *x*; *t* + *p*3*u*3 *x*; *t* + … . (16)

where *q*2[0; 1] is the embedding parameter, Z s 0 is an

auxiliary parameter, *LF* is an auxiliary linear operator, 4(*x*,*t* ;*q*)

is an unknown function,*u*0(*x*,*t*) is an initial guess of *u*(*x*,*t*) and

the embedding parameter *q* = 0 and *q* = 1, it holds *H*(*x*,*t*) denotes a nonzero auxiliary function. Obviously, when

f(*x*; *t*; 0)= *u*0(*x*; *t*); f(*x*; *t*; 1)= *u*(*x*; *t*); (23)

respectively. Thus as *q* increases from 0 to 1, the solution

4(*x*,*t* ;*q*) varies from the initial guess *u*0(*x*,*t*) to the solution *u*(*x*,*t*). Expanding 4(*x*,*t* ;*q*) in Taylor series with respect to *q*, we have

X

∞

For nonlinear problems, we set *Nu*(*x*,*t*) = *S*(*x*,*t*). Substituting Eq. [(16)](#_bookmark6) into Eq. [(14)](#_bookmark4) and equating the terms with identical

power of *p*, we obtain a sequence of equations of the form

f(*x*; *t*; *q*) = *u*0(*x*; *t*) + *um*(*x*; *t*)*qm*; (24)

*m*=1

where

*p*0 *D*a *u*

0

*x*; *t*

= 0;

*um*(*x*; *t*)=

1 v*m*f(*x*; *t*; *q*)

*m*

. (25)

: \**t*

*p*1 : *D*a *u*1 *x*; *t*

*m*! v*q*

*q*=0

*p*2 : *D*a *u x*; *t* = —*Lu x*; *t* — *S u x*; *t* ; *u x*; *t* ;

\**t*

\**t*

2 1 1 0 1

= —*Lu*0 *x*; *t*

— *S*0 *u*0 *x*; *t*

+ *v x*; *t* ;

*pj* : *D*a *uj x*; *t* = —*Luj*—1 *x*; *t* — *Sj*—1 *u*0 *x*; *t* ; *u*1 *x*; *t* ;

\**t*

*u*2 *x*; *t* ; …; *uj*—1 *x*; *t* ; *j* = 2; 3; 4; … (17)

If the auxiliary linear operator, the initial guess, the

chosen, the series [(24)](#_bookmark7) converges at *q* = 1, then we have auxiliary parameter Z, and the auxiliary function are properly

∞

X

The functions *S*0,*S*1,*S*2,… satisfy the following equation

*S* *u* *x*; *t* + *pu* *x*; *t* + *p*2*u* *x*; *t* + *p*3*u* *x*; *t* + …

0

1

2

3

*u*(*x*; *t*)= *u*0(*x*; *t*)+ *um*(*x*; *t*); (26)

*m*=1

= *S*0(*u*0(*x*; *t*)) + *pS*1(*u*0(*x*; *t*); *u*1(*x*; *t*))

+ *p*2*S*2(*u*0(*x*; *t*); *u*1(*x*; *t*); *u*2(*x*; *t*)) + … .

Applying the inverse operator*J*a

*t*

(18)

on both sides of the

which must be one of the solutions of the original nonlinear

fractional differential equations. According to the definition [(26)](#_bookmark8), the governing equation can be deduced from the zero- order deformation [(22)](#_bookmark3). Define the vectors

equation [(15)](#_bookmark5) and considering the initial and boundary con-

ditions, the various components of the series solution are

→*u m* = {*u*0(*x*; *t*); *u*1(*x*; *t*); …; *um*

(*x*; *t*)}. (27)

given by

*n*—1 *ti*

X

*u* (*x*; *t*)= *c*

;

0

*i*=0

*ii*!

mation equation:

Differentiating the zeroth-order deformation Eq. [(22)](#_bookmark3) m-

finally setting *q* = 0, we get the following *m*th-order defor- times with respect to *q* and then dividing them by m! and

*u*1 *x*; *t* = —*J*a *Lu*0 *x*; *t* — *J*a*S*0 *u*0 *x*; *t* + *J*a*v* *x*; *t* ;

*t*

*t*

*t*

→

*uj* *x*; *t* = —*J*a *Luj* 1 *x*; *t* — *J*a*Sj* 1 *u*0 *x*; *t* ; *u*1 *x*; *t* ;

*t*

—

*t*

—

*LF*[*um*(*x*; *t*)— c*mum*—1(*x*; *t*)] = Z*H*(*x*; *t*)঩*m*( *u m*—1); (28)

*u*2 *x*; *t* ; …; *uj*—1 *x*; *t* ; *j* = 2; 3; 4; ... . (19)

Hence, the HPM solution *u*(*x*,*t*) is given by

where

঩*m*(→*u* )=

*m*—1

1

v*m*—1*NF* f *x*; *t*; *q*

; (29)

∞

*u*(*x*; *t*)=

X

*i*=0

*ui*(*x*; *t*) (20)

and

(*m* — 1)!

v*qm*—1

*q*=0

c*m*

# Basic idea of homotopy analysis method (HAM)

0; *m* ≤ 1;

1; *m* > 1.

=

(30)

In order to show the basic idea of HAM, consider the following nonlinear fractional differential equation

*NF*[*u*(*x*; *t*)] = 0; (21)

# Numerical examples of European option pricing equation

In order to illustrate the advantages and the accuracy of the HPM and HAM for solving fractional Black-Scholes equation

180 [e gypti an j o ur nal o f b a sic and a p p l i ed sci e n c e s 1 ( 2014) 1 77](http://dx.doi.org/10.1016/j.ejbas.2014.10.003) e[183](http://dx.doi.org/10.1016/j.ejbas.2014.10.003)

with boundary condition for a European option problem, we have applied the methods for two different examples.

Example 1. We consider the fractional BlackeScholes equa-

*v*(*x*; *t*)= max(*ex*; 0)(1 — *E*a(— *kt*a)) + max(*ex* — 1; 0)*E*a(— *kt*a).

This is the exact solution of the given problem of [(31)](#_bookmark9).

(36)

tion as [[12]](#_bookmark23)

To solve the Eq. [(31)](#_bookmark9) by HAM, we choose the auxiliary op- erators as follows

va*v* v2*v* v*v*

*LF* f *x*; *t*; *q* = *D* f *x*; *t*; *q* ;

= + (*k* — 1)  — *kv*; 0 < a ≤ 1; (31)

v*t*a v*x*2 v*x*

with initial condition *v*(*x*,0) = *max* (*ex*—1, 0). Notice that this eters *k* = 2 *r*/s2, where *k* represents the balance between the system of equations contains just two dimensionless param- rate of interests and the variability of stock returns and the

dimensionless time to expiry ½s2*T*, even though there are four dimensional parameters, *E*, *T*, s2 and *r*, in the original state- ments of the problem.

According to the HPM [[18](#_bookmark33)e[22]](#_bookmark33), we construct the following homotopy

*D*a*v* — *p* *D*2*v* + *k* — 1 *Dxv* — *kv* = 0; (32)

*t*

*x*

a

*t*

with the property *LF*[*c*1] = 0.

Using the above definition, we first construct the zeroth-

order deformation equations as

(1 — *q*)*LF*[f(*x*; *t*; *q*)— c*mv*0(*x*; *t*)] = Z*qNF*[f(*x*; *t*; *q*)]. (37) Obviously, when the embedding parameter *q* = 0 and *q* = 1,

it holds

f(*x*; *t*; 0)= *v*0(*x*; *t*); f(*x*; *t*; 1)= *v*(*x*; *t*). (38) Differentiating the zeroth-order deformation Eq. [(37)](#_bookmark11) m-

times with respect to *q* and then dividing them by m! and

parameter *p*2[0; 1]. Now applying the classical perturbation where the homotopy parameter *p* is considered as a small technique, we can assume that the solution of Eq. [(32)](#_bookmark10) can be

expressed as power series in *p* as given below

*v* = *v*0 + *pv*1 + *p*2*v*2 + *p*3*v*2 + … (33)

finally setting *q* = 0, we get the following *m*th-order defor- mation equations

*LF*[*vm*(*x*; *t*)— c*mvm*—1(*x*; *t*)] = Z঩*m*(→*u m*—1); (39) where

Substituting [(33)](#_bookmark12) into [(32)](#_bookmark10) and equating the coefficients of

→ va *vm*—1

v2 *vm*—1

v*vm*—1

like powers of *p*, we get the following set of differential equations that

*t*

঩*m*( *v m*—1)=

v*t*a —

v*x*2 — (*k* — 1)

+ *kvm*—1. (40)

v*x*

*p*0 : *D*a*v*0 *x*; *t* = 0; (34)

*t*

On applying the operator *J*a both the sides of Eq. [(39)](#_bookmark13), we get

*v*

*m*

(*x*; *t*)= c*mvm*—1

(*x*; *t*)+ Z঩

*m*

(→*u*

*m*—1

); (41)

*pn* : *D*a*vn x*; *t* = *D*2*vn*—1 *x*; *t* + *k* — 1 *Dxvn*—1 *x*; *t*

*t* *x*

— *kvn*—1 *x*; *t* ; *n* ≥ 1.

(35)

subsequently solving the *m*th-order deformation equations one has

The above equations can be easily solved by applying the

operator *J*a to [(34,35)](#_bookmark14) giving the various components *vn*(*x*,*t*) as

*v*0(*x*;*t*)= max(*ex* — 1;0);

a a

*t*

*v*0(*x*; *t*)= *v*(*x*; 0)= max(*ex* — 1; 0);

*v*1(*x*;*t*)= Zmax(*ex*;0)

(— *kt* )

G(a+ 1)

— Zmax(*ex* — 1;0)

(— *kt* )

G(a+ 1);

(— *kt*a)

(— *kt*a)

*x* (— *kt*a)

*x* (— *kt*a)

*v*1(*x*; *t*)= —max(*ex*; 0)

G(a + 1)

+ max(*ex* — 1; 0)

G(a + 1) ;

*v*2(*x*;*t*)= Z(1 +Z)max(*e* ;0)G(a+ 1)— Z(1 +Z)max(*e* — 1;0)G(a+ 1)

( — *kt*a)2

*v*2(*x*; *t*)= —max(*ex*; 0)

G(2a + 1)

«

( — *kt*a)2

+ max(*ex* — 1; 0) ;

G(2a + 1)

—Z2max(*ex*;0)

and so on.

(— *kt*a)2

G(2a+ 1)

+Z2max(*ex* — 1;0)

(— *kt*a)2

;

G(2a+ 1)

Finally, we approximate the analytical solution *v*(*x*,*t*) by the

X

X

!

series *v*(*x*; *t*)= lim P*N*—1*vn*(*x*; *t*); then we get

Therefore, the HAM series solution is

(— *kt*a)

*x*

*N*/∞

*n*=0

(— *kt*a)

(— *kt*a)2 !

*v*(*x*; *t*)= max(*ex* — 1; 0)+ Zmax(*ex*; 0)

(— *kt*a)

G(a + 1)

(— *kt*a)

*v*(*x*; *t*)= max(*ex*; 0)— max(*ex*; 0)

1 + G(a + 1)

+ G(2a + 1) + /

— Zmax(*ex* — 1; 0)

G(a + 1)

+ Z(1 + Z)max(*ex*; 0)

G(a + 1)

+ max(*ex* — 1; 0)

(— *kt*a)

1 +

G(a + 1)

(— *kt*a)2

+ G(2a + 1) + /

(— *kt*a)

— Z(1 + Z)max(*e* — 1; 0)

G(a + 1)

(— *kt*a)2

(— *kt*a)2

or

∞ a *n*

— Z2max(*ex*; 0)

G(2a + 1)

+ Z2max(*ex* — 1; 0)

G(2a + 1)

+ …

(42)

*v*(*x*; *t*)= max(*ex*; 0)— max(*ex*; 0) (— *kt* )

*n*=0 G(*n*a + 1)

∞ (— *kt*a)*n*

+ max(*ex* — 1; 0) ;

*n*=0 G(*n*a + 1)

and in closed form

Taking Z = —1, we get the same solution as obtained by

HPM.

Case 1. Consider the vanilla call option with parameter [[5]](#_bookmark21)

s = 0.2, *r* = 0.04, a = 1, t = 0.5 years then *k* = 2.

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In this case, we get the solution of Eq. [(31)](#_bookmark9)

*v* *x*, *t* = max *ex*, 0 1 — *e*—2*t* + max *ex* — 1, 0 *e*—2*t*. (43) This is the given exact solution of the standard Black-

Substituting [(33)](#_bookmark12) into [(46)](#_bookmark17) and equating the coefficients of like powers of *p*, we get following set of differential equations

*p*0 : *D*a*v* *x*, *t* = 0, (47)

*t*

0

*pn* : *D*a*v* (*x*, *t*)= — 0.08(2 + sin *x*)2*x*2*D*2*v* (*x*, *t*)

eScholes equation.

Case 2. In this case, we consider the vanilla call option with parameter [[5]](#_bookmark21)

*t n x n*—1

+ 0.06*xDxvn*—1(*x*, *t*)— 0.06*vn*—1(*x*, *t*) , *n* ≥ 1.

(48)

s = 0.2, *r* = 0.01, a = 1, t = 1 year then *k* = 5.

We get the solution of Eq. [(31)](#_bookmark9)

*v* *x*, *t* = max *ex*, 0 1 — *e*—5*t* + max *ex* — 1, 0 *e*—5*t*. (44) This is the exact solution for the given case.

The numerical results for the fractional BlackeScholes Eq.

[(31)](#_bookmark9) obtained by using the HPM and HAM for various values of

Consequently, the above system of nonlinear equations

can be easily solved by applying the operator *J*a to [(47,48)](#_bookmark15) giv- ing the various components *vn*(*x*,*t*), thus enabling the series solution to be entirely determined. The first few components of the homotopy perturbation solution for Eq. [(45)](#_bookmark16) are derived as follows:

*t*

*v*0 *x*, *t* = *v* *x*, 0 = max *x* — 25*e*—0.06, 0 ,

a

a

*x* and *t*, when a = 1 and *k* = 2 are shown by [Fig. 1](#_bookmark18). It can be

*v*1(*x*, *t*

(0.06*t* )

)=— + max *x* — 25*e* , 0 ,

*x*(0.06*t* ) —0.06

observed from [Fig. 1](#_bookmark18) that n(*x*,*t*) increases with the increase in

both *x* and *t*.

G(a + 1)

a 2

*x*(0.06*t* ) —0.06

G(a + 1)

a 2

*v*2(*x*, *t*)

(0.06*t* )

=— + max *x* — 25*e* , 0 ,

Examples 2. In this example, we consider the following generalized BlackeScholes equation as [[6]](#_bookmark22)

«

and so on.

G(2a + 1)

G(2a + 1)

va*v* v*t*a

+ 0.08(2 + sin *x*)2*x*2

v2*v* v*x*2

+ 0.06*x* v*v*

v*x*

— 0.06*v* = 0, 0 < a ≤ 1,

Finally, we approximate the analytical solution *v*(*x*,*t*) by the

series *v*(*x*, *t*)= lim *vN*(*x*, *t*), where *vN*(*x*, *t*)= P*N*—1*vn*(*x*, *t*), then

with initial condition *v*(*x*,0) = *max* (*x*—25 *e*—0.06, 0).

(45)

we get

*N*/∞

*n*=0

We construct the homotopy which satisfies the relation

*D*a*v* + *p* 0.08(2 + sin *x*)2*x*2*D*2*v* + 0.06*xDxv* — 0.06*v* = 0, *p*2 0, 1 .

*t* *x*

h i h i

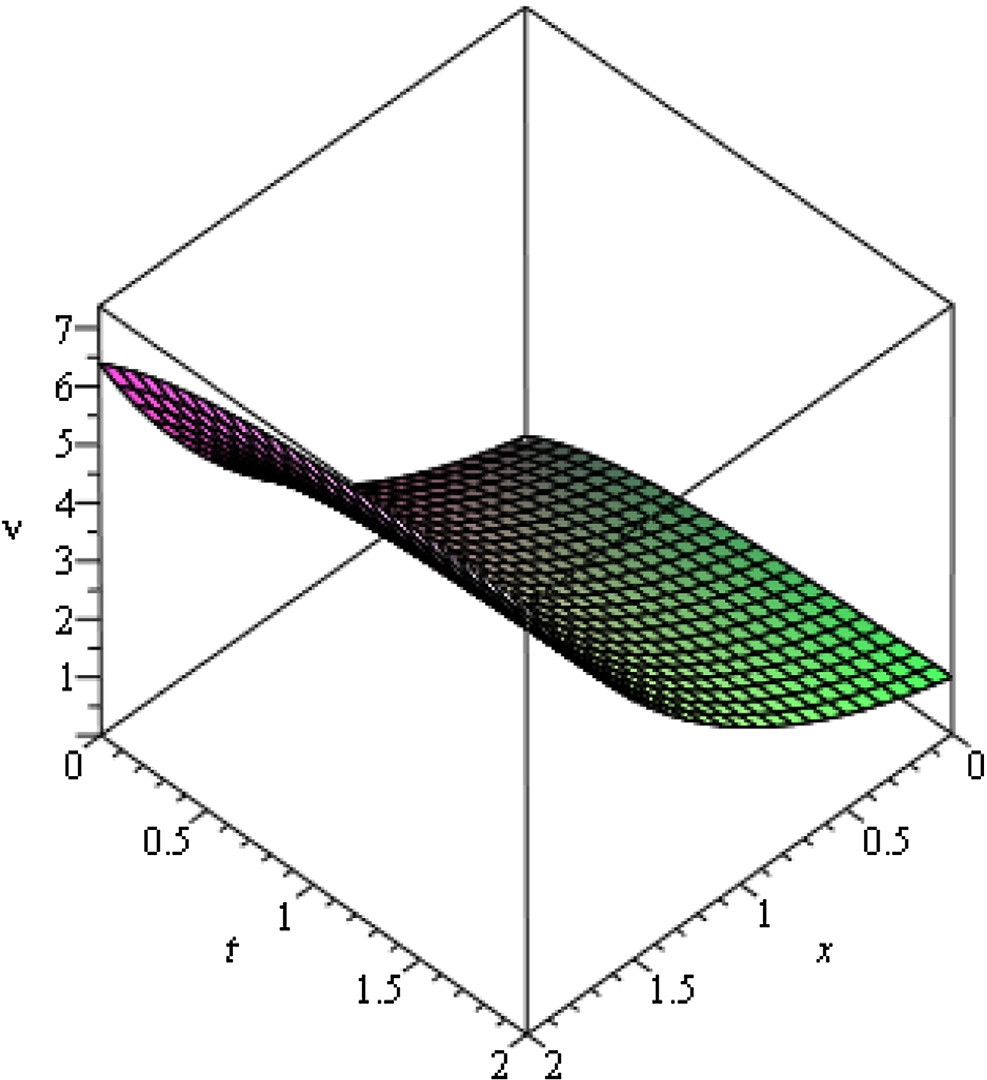
(46)

*v* *x*, *t* = *x* 1 — *E*a 0.06*t*a + max *x* — 25*e*—0.06, 0 *E*a 0.06*t*a . (49)

which is analytical solution of the fractional Black-Scholes Eq.

[(45)](#_bookmark16).

To solve the Eq. [(45)](#_bookmark16) by HAM, we choose the auxiliary op- erators as follows

*LF* f *x*, *t*; *q* = *D*a f *x*, *t*; *q* ,

*t*

with the property *LF* [*c*1] = 0.

Using the above definition, we first construct the zeroth-

order deformation equations as

(1 — *q*)*LF*[f(*x*, *t*; *q*)— c*mv*0(*x*, *t*)] = Z*qNF*[f(*x*, *t*; *q*)]. (50) Obviously, when the embedding parameter *q* = 0 and *q* = 1,

it holds

f(*x*, *t*; 0)= *v*0(*x*, *t*), f(*x*, *t*; 1)= *v*(*x*, *t*). (51) Differentiating the zeroth-order deformation Eq. [(50)](#_bookmark19) m-

finally setting *q* = 0, we get the following *m*th-order defor- times with respect to *q* and then dividing them by m! and mation equations

*LF*[*vm*(*x*, *t*)— c*mvm*—1(*x*, *t*)] = Z঩*m*(→*u m*—1), (52) where

→ va *vm*—1

2 v2 *vm*—1

v*vm*—1

঩*m*( *u m*—1)=

v*t*a

+ 0.08(2 + sin *x*) *x*2

v*x*2

+ 0.06*x*

v*x*

Fig. 1 e The surface shows the n(x,t) for [(31)](#_bookmark9) with respect to

*t*

— 0.06*vm*—1.

(53)

x and t, when a ¼ 1 and k ¼ 2.

On applying the operator *J*a both the sides of Eq. [(52)](#_bookmark20), we get

182 [e gypti an j o ur nal o f b a sic and a p p l i ed sci e n c e s 1 ( 2014) 1 77](http://dx.doi.org/10.1016/j.ejbas.2014.10.003) e[183](http://dx.doi.org/10.1016/j.ejbas.2014.10.003)

*vm*(*x*, *t*)= c*mvm*—1(*x*, *t*)+ Z঩*m*(→*u*

*m*—1), (54)

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subsequently solving the *m*th-order deformation equations

one has

*v*0 *x*, *t* = max *x* — 25*e*—0.06, 0 ,

a

a

[equations. Comput Math Appl 2008;56:813](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref5)e[21](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref5).

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*v* (*x*, *t*)= Z*x* (0.06*t* ) — Zmax *x* — 25*e*—0.06, 0 (0.06*t* ) ,

1

G(a+ 1)

G(a+ 1)

[European option pricing with transaction costs nonlinear](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref7)

[equation. Math Comput Model 2009;50:910](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref7)e[20](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref7).

*v* (*x*, *t*)= Z(1 + Z)*x* (0.06*t* ) — Z(1 + Z)max *x* — 25*e*—0.06, 0 (0.06*t* )

a

a

2

G(a+ 1)

a 2

G(a+ 1)

a 2

[Dirichlet problem for a stationary nonlinear Black Scholes](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref8)

[equation. Nonlinear Anal 2009;71:4624](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref8)e[31](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref8).

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(0.06*t* ) (0.06*t* )

2 2 —0.06

—Z *x* + Z max *x* — 25*e* , 0 , «

1. [Amster P, Averbuj CG, Mariani MC. Solutions to a stationary](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref9)

and so on.

G(2a+ 1)

G(2a+ 1)

[nonlinear Black](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref9)e[Scholes type equation. J Math Anal Appl](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref9) [2002;276:231](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref9)e[8](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref9).

Therefore, the HAM series solution is

*v*(*x*, *t*)= max *x* — 25*e* , 0 + Z*x*  (0.06*t* )

—0.06 a

G(a + 1)

—0.06

a

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[two nonlinear Black](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref10)e[Scholes type equations. Appl Numer](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref10) [Math 2003;47:275](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref10)e[80](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref10).

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a [nonlinear Black](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref11)e[Scholes equations. Comput Math Appl](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref11)

(0.06*t* ) (0.06*t* )

— Zmax *x* — 25*e* , 0 + Z(1 + Z)*x*

G(a + 1)

G(a + 1)

a 2

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(0.06*t* ) (0.06*t* )

—0.06 2

a

— Z(1 + Z)max *x* — 25*e* , 0 — Z *x*

[Scholes equation. J Stat Comput Simul 2010;80:1349](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref12)e[54](http://refhub.elsevier.com/S2314-808X(14)00043-8/sref12).

G(a + 1)

2 —0.06 (0.06*t* )

a 2

+ Z max *x* — 25*e* , 0 + /.

G(2a + 1)

G(2a + 1)

(55)

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Taking Z = —1, we get the same solution as obtained by

HPM.

# Concluding remarks

In this study, the HPM and HAM are implemented to solve the time-fractional BlackeScholes equation with boundary con- dition for a European option problem. They provide the solu- tions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. The solution ob- tained with the help of HAM is more general as compared to HPM solution. We can easily recover all results HPM by

assuming Z = —1. The results show that the HPM and HAM are

powerful and efficient techniques in finding exact and

approximate solutions for fractional differential equations. In conclusion, the HPM and HAM may be considered as a nice refinement in existing numerical techniques and might find the wide applications.

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