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On Fair Cost Facility Location Games with Non-singleton Players

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**Abstract**

In the fair cost facility location game, players control terminals and must open and connect each terminal to a facility, while paying connection costs and equally sharing the opening costs associated with the facilities it connects to. In most of the literature, it is assumed that each player control a single terminal. We explore a more general version of the game where each player may control multiple terminals. We prove that this game does not always possess pure Nash equilibria, and deciding whether an instance has equilibria is NP-Hard, even in metric instances. Furthermore, we present results regarding the efficiency of equilibria, showing that the price of stability of this game is equal to the price of anarchy, in both uncapacitated and capacitated settings.

*Keywords:* price of stability, facility location, algorithmic game theory

# Introduction

Facility location problems covers a broad range of optimization problems, with practical applications in many different areas such as public policy, urban planning, telecommunications and computer networking. In the general sense, the facility location problem can be stated as follows. Let *F* be a set of facilities, *T* a set of terminals, with opening costs *cf* for each facility *f ∈ F* and connection costs *dtf*

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for connecting terminal *t ∈ T* to facility *f ∈ F* . The problem is to find a subset of facilities to open and establish connections from terminals to this subset such that the sum of all costs are minimized.

Consider a scenario where multiple supermarket or stores share big warehouses that stockpile supplies for them. Each store is located some distance away from a warehouse and they pay the costs associated with maintaining its products in the warehouse and the transportation costs to its specific location. Such scenario can be seen as a facility location problem, where each store is a terminal and each warehouse location is a possible facility. In the classical optimization version, the view point of the warehouse company is given priority, and each terminal can be seen as being controlled by a central authority in order to minimize opening and connection costs globally. When we consider the fact that each supermarket is competing with each other, this approach of global optimization is not possible to be adopted anymore. In order to analyze such scenarios, we can use game theory.

A *non–cooperative game* is a decision scenario where an agent or player selfishly and without coordination other players, chooses a strategy in order to maximize its own utility (or minimize its own cost), which in turn depends on the strategies chosen by the other players. We say that a game is in a pure Nash equilibrium (PNE) if no player has any incentive to unilaterally change its own strategy. In order to compare the social cost of pure equilibria and the social optimum, we use the standard measures found in the literature: the price of anarchy (PoA) and the price of stability (PoS). The PoA of a game is defined as the ratio between the PNE with worst social cost and the social optimal cost, while the PoS is the ratio between the PNE with the best social cost and the social optimum.

In our example scenario, each store is traditionally viewed as a player choosing a facility (warehouses) to connect, such that its own cost is minimized. Each store would then share equally the costs associated with storing the goods in the ware- house they are connected to, as well as individually paying for the connection to such warehouse. This scenario exemplifies the singleton fair cost facility location game, where each store is controlled by a single player. This scenario however fails to accommodate the common case when multiple supermarket or stores are part of the same chain.

When stores are part of the same larger group, each store behaviour cannot be said to be independent and uncoordinated, however one whole group does not coordinate with other competitor stores or chains and therefore it makes sense to analyze this scenario by allowing players to control multiple stores or terminals.

In the fair cost facility location games we analyze, each player controls multiple terminals, and can move them simultaneously to minimize their own costs. For singleton games, terminals sharing opening costs evenly means the same as players sharing these costs evenly, since players and terminals are effectively the same in these scenarios. However when players control multiple terminals, the same is not true. In our paper we use the term fair cost to mean that in any strategy profile each facility has its opening cost shared evenly among all terminals that connect to it, i.e. if in a certain strategy profile a player *a* has two terminals connected to a

facility *f* and a different player *b* has a single terminal connected to it, player *a* will pay for two thirds of the opening cost of *f* , while player *b* pays only a third of the opening cost.

In this work we study how hard it is to find pure Nash equilibria in these fair cost facility location games, as well as how efficient these equilibria can be when compared with the social optimal cost. Finally, we extend our results to weighted and capacitated versions of these games.

# Related Work and Organization

There have been multiple works examining facility location problems from the out- look of game theory. The class of valid utility games [[12,](#_bookmark31) [14,](#_bookmark33) [18](#_bookmark37)] can be viewed as facility games when both facilities and players are controlled by singleton players. There is a whole area in game theory focused in cooperative games, and Goemans and Skutella [[8](#_bookmark27)] studied cooperative facility location games.

Facility location has also been extensively studied from the mechanism design perspective, with multiple relevant work in strategy-proof mechanisms for variants of facility location problems [[5,](#_bookmark23) [13,](#_bookmark32) [15](#_bookmark34)].

We study in this paper the non-cooperative facility location game where players control terminals, and each terminal *t* share the opening cost of its chosen facility *f* equally among all terminals connected to *f* . These games share great similarities to connection and network design games, and thus multiple results from these games are valid for fair cost facility location games.

For the singleton fair cost facility location game, the results from network de- sign [[1](#_bookmark20)] extend to facility location, with a bound of *k* for the PoA and *Hk* = *Θ*(log *k*) for the PoS [[17](#_bookmark36)]. For the metric version, where each connection obeys the triangle inequality, Hansen and Telelis [[9](#_bookmark28)] proved constant bounds both for the PoS and the strong PoA. Furthermore, when there are weights associated with terminals deter- mining how much of the opening cost is paid by each terminal, they show that there is always an *e*-approximate equilibrium and that the PoS can be in Θ(log *W* ), where *W* is the sum of all terminal weights. In [[3](#_bookmark22)], Chen and Roughgarden prove that there are instances of the weighted network design game where there are no possible PNE, however it is still an open question whether the same applies to weighted fair cost facility location games, even for the non-singleton version.

For facility location games where players do not posses any limitations on how to share opening costs, the price of anarchy and the price of stability have been proven to be in Θ(*k*) by Cardinal and Hoefer [[2,](#_bookmark21) [11](#_bookmark30)], where *k* is the number of players in the game. Furthermore, for non-singleton games it is NP-hard to decide whether an instance has a PNE [[2](#_bookmark21)].

The capacitated version of the game was considered by Rodrigues and Xavier [[16](#_bookmark35)]. They show that for the metric singleton fair cost facility location game, the PoA can be unbounded, while the PoS is *Hk*, where *k* is the number of players. Furthermore, they show that it is NP-hard to decide whether there is PNE in singleton capacitated facility location games with arbitrary opening cost sharing.

For a sequential version of facility location, they show that the metric version has bounded PoA and PoS.

In this paper we extend the results from Cardinal and Hoefer [[2](#_bookmark21)] to fair cost facility location, proving that it is NP-hard to decide whether there is a PNE in this game, even in the metric case, as long as players are allowed to control multiple terminals. For this, we prove that there are fair cost facility location game instances with no PNE. We base this proof in the instances without PNE in weighted network design games used by Chen and Roughgarden in [[3](#_bookmark22)]. Finally, we prove that the PoS of this game can be as inefficient as the PoA for both the uncapacitated metric case and the general capacitated case.

In Section [3](#_bookmark1), we present the formal definitions needed to understand our paper and define our game. In Section [4](#_bookmark2), we provide instances with no PNE and prove the NP-hardness of PNE existence, as well as link these instances to weighted games. In Section [5](#_bookmark15), we prove that the price of stability is as poor as the price of anarchy in both the uncapacitated and capacitated versions of fair cost facility location games. Finally, in Section [6](#_bookmark19) we present our final remarks and discuss future work.

# Preliminaries

First we define a few key concepts from non–cooperative game theory.

In game theory, a *non–cooperative game* is a scenario where players choose strate- gies independently trying to either minimize their costs or maximize their utility. For each player *i*, there is a set *Ai* of actions that it can choose to play. A pure strategy *Si* consists of one action from *Ai*, while a mixed strategy corresponds to a probability distribution over *Ai*. In a pure game each player choses one action to play, while in a mixed game each player randomizes its action according to the probability distribution. In this paper we assume pure strategies games unless men- tioned otherwise.

A set of strategies *S* = (*S*1*, S*2*,..., Sn*) consisting of one strategy for each player, is denominated a *strategy proﬁle*. Let *S* = *A*1 *×A*2 *×.. .×An* be the set of all possible strategy profiles and let *c* : *S →* R*n* be a cost function that attributes a cost *ci*(*S*) for each player *i* given a strategy profile *S*. Define *S−i* = (*S*1*,..., Si−*1*, Si*+1*,..., Sn*) a strategy profile *S* without *i*’s strategy, so that we can write *S* = (*Si, S−i*). If all players other than *i* decide to play *S−i*, then player *i* is faced with the problem of determining a best response to *S−i*. A strategy *S∗* from a player *i* is a *best response* to *S−i*, if there is no other strategy which could yield a better outcome for the player, i.e.

*i*

*ci*(*S∗, S−i*) *≤ ci*(*Si, S−i*) *, ∀Si ∈ Ai.*

*i*

A strategy profile is in a *pure Nash equilibrium* (PNE) if no player can reduce its cost by choosing a different strategy, i.e. for each player, its strategy in the strategy profile is a best response.

The social welfare or social cost is a function mapping a strategy profile to a real number, indicating a measure of the total cost or payoff of a game. Two of the most important concepts for efficiency analysis are the *Price of Anarchy* (PoA) and

the *Price of Stability* (PoS). The PoA is the ratio between a Nash equilibrium with worst possible social cost and the strategy profile with optimal social cost, while the PoS is the ratio between the best possible Nash equilibrium to the social optimum. With these ideas formalized, we define the *Facility Location Game with Fair*

*Cost Sharing (FLG-FC)*.

**Definition 3.1** Let *G* = (*T ∪ F, T × F* ) be a bipartite graph, with vertex sets *F* of *n* facilities and *T* of *m* terminals. Each facility *f ∈ F* has an opening cost *cf* , and connection costs *dtf* for each terminal *t ∈ T* . Let *K* = *{*1*,..., k}* be the set of players.

Each player *i* controls a subset of terminals *Ti ⊆ T* (also forming a partition of *T* ), and each terminal must be connected to exactly one opened facility. When a player controls only a single terminal, it is denominated a *singleton* player. A player *i* chooses a strategy *Si ⊆ Ti × F* .

Let *S* = (*S*1*,..., Sk*) be a strategy profile. We abuse notation and use the expression *f ∈ S* to represent any facility *f* connected to a terminal in a strategy profile *S* and (*t, f* ) *∈ S* to represent any pair of terminal and facility that are connected in *S*, while *f ∈ Si* represents any facility *f* player *i* uses to connect one of its terminals in strategy *Si*. Each player tries to minimize its own payment

*pi*(*S*)=

Σ

(*t,f* )*∈Si*

*cf* +

*xf* (*S*)

Σ

(*t,f* )*∈Si*

*dtf ,*

where *xf* (*S*)= *|{*(*tj,f* ) *∈ Si |* 1 *≤ i ≤ k ∧* 1 *≤ j ≤ m}|* is the number of terminals connected to facility *f* in strategy profile *S*.

The *social welfare cost* for a strategy *S* is defined as the sum of all player payments, i.e.,

*C*(*S*)= Σ *pi*(*S*)= Σ *cf* + Σ *dtf .*

*i∈K*

*f∈S*

(*t,f* )*∈S*

In games with general connection costs, some connections (*t, f* ) should be avoided in any solution, because they do not exist for example. In this case we assume they have a prohibitively large constant cost *Ud*. For general costs, if a connection is not shown, it is assumed that it has a cost equal to *Ud*, unless men- tioned otherwise. For metric connection costs, where connection costs must obey the triangle inequality, it is assumed that any connection (*t, f* ) not shown has a cost equal to the shortest cost path from *t* to *f* in the undirected graph formed from the explicitly shown connections.

When dealing with capacitated FLG-FC, we extend this definition to include a capacity for each facility, as well as ways to enforce players to propose valid solutions.

**Definition 3.2** Let *G* = (*T ∪ F, T × F* ) be a bipartite graph, with vertex sets *F* of *n* facilities and *T* of *m* terminals. Each facility *f ∈ F* has an opening cost *cf* and a capacity *uf* indicating how many terminals can be connected to *f* at any given time. Furthermore, there are connection costs *dtf* for each pair (*t, f* ) where *t ∈ T* and *f ∈ F* .

Given a strategy profile *S*, for *Capacitated Facility Location Games with Fair Cost Sharing (CFLG-FC)*, the definitions of players *i* payment, *pi*(*S*)*,* and social cost, *C*(*S*), are the same as before for the uncapacitated game. However, to ensure that capacity restrictions are respected, if a player *i* in the solution *S* has one of its terminals connected to *f* where *xf* (*S*) *> uf* , then a prohibitively large constant cost *Uc* is added to the payment of player *i*, i.e., it pays *pi*(*S*)+ *Uc*.

# On the Existence of Pure Equilibria

Pure Nash equilibria is one of the most well known solution concepts in game theory. While it is not guaranteed to exist in all games, in several practical scenarios it reflects to a greater degree the behaviour of players. The singleton version of FLG- FC is a potential game, and therefore there always exists a PNE [[17](#_bookmark36)]. We show that this does not extend to every fair cost facility location game, by showing instances with no PNE when players control multiple terminals and the opening cost sharing happens in relation to terminals. We construct these instances loosely based on an example used to prove the non-existence of PNE in weighted network design games [[3](#_bookmark22)]. We first prove this result for weighted fair cost facility location games and then extend this result for unweighted metric games.

* 1. *Equilibria Existence in Weighted Games*

In the classic facility location game, each terminal is assumed to demand or require the same amount of goods from a facility, and therefore the “fair” way to share costs is to evenly divide opening costs of an opened facility between terminals that connect to it. However this is not always the case. In several practical scenarios some terminals might require more from a facility, and an egalitarian sharing might not reflect fairness.

In the *weighted fair cost facility location game*, each terminal *t* has an associated positive integer weight *wt ≥* 1, and each player *i* pays in a strategy profile *S*,

*pi*(*S*)=

Σ

(*t,f* )*∈Si*

*dtf*

+

(*t,f* )*∈Si*

Σ

*wtcf , Wf,S*

where *W**f,S* is the sum of the weights of all terminals connected to *f* in the strategy profile *S*.

Here we provide a partial answer to an open question regarding whether there are instances with no PNE for weighted FLG-FC. We provide an instance with no PNE when players are allowed to control more than a single terminal. It remains open whether there are singleton weighted instances with no equilibria.

**Theorem 4.1** *There exists a 3-player metric instance for the weighted FLG-FC game with only six terminals where there is no PNE.*

## Proof.

Consider the instance in Figure [1](#_bookmark4), denominated here as *I*. Let *w >* 1 be a parameter of this instance, and *ε* be a constant much smaller than 1 . Let player *A* control terminals *t*2 and *t*5, player *B* control terminals *t*1 and *t*3, and player *C* control terminals *t*4 and *t*6. Allow every terminal controlled by player A to have the same weight *wA* = *w*2, every terminal that *B* controls to have unitary weight *wB* = 1 and every terminal that player *C* controls to have weight *wC* = *w*.

*w*3

*cf*5 =1



*c* = *w*3 *− ε*

*f*1

*w*2 +*w*+1

*c* = *w*3 + *ε*

*f*2

*w*2 +*w*+1

*f*1

*dt*1 *,f*1 = 1+ 3*ε*

*dt*1 *,f*2 =1

*f*2

*Pa*

*t*2

*t*4

*Pc*

*c*

*c*

= *w*3+*w* +

*f*3

= *w*3+*w*2 *−*

*f*4 *w*2 +*w*+1

*ε*( 2*w* +1 )

*w*2 +*w*+1

2

*f*3

*f*4

*ε*( 2*w*+1 )

2*w*+2

2*w*2 +2

*t*5

*f*5

*t*6

*t*3

*Pb*

*t*1

Fig. 1. Game instance of the weighted FLG-FC without PNE. All edges except *dt*1 *,f*1 have cost equal to zero.

For player *Pa*, there are only two feasible strategies: either all terminals connect to *f*3, or *t*2 connects to *f*1 and *t*5 connects to *f*5. The same happens for player *Pc*: either all terminals connect to *f*4, or *t*4 connects to *f*2 and *t*6 connects to *f*5. For player *Pb* terminal *t*1 has to choose between *f*1 and *f*2, while terminal *t*3 will always connect to *f*5 in any PNE. The proof is based on players *Pa* and *Pc* having different facility preferences when presented with mirrored choices from player *Pb*. To achieve this, we use the fact that player *Pa* has squared times the weight than *Pc* does, as well as carefully constructed opening costs.

Note that all direct connection costs are equal for *Pa* and for *Pc*, and thus do not interfere in their choices. They only ensure that it is not beneficial for neither *Pa* nor *Pc* to connect their terminals to facilities without a direct connection available. For *Pb*, the only influence that connection costs have is in choosing whether to connect to *f*1 or *f*2.

Since the terminals from player *Pa* have weight *w*2, player *A* will always pay for the majority of the cost of any facility it helps open. With this in mind, we can sort the five possible scenarios for player *Pa* by the cost incurred from each in increasing order: (i) player *Pa* shares *f*1 with *Pb* and *f*5 with all players, (ii) it connects to *f*1 alone and shares *f*5 with all players, (iii) it connects to *f*3 alone, (iv) it shares *f*1 with *Pb* and shares *f*5 with *Pb* only and (v) it connects to *f*1 alone and shares *f*5 with *Pb* only. The following inequalities show why this is the case:

*w*2 *w*2

2 *cf*1 + 2

*cf*5 *≤* (1)

*w* +1

*w* + *w* +1

*w*2

*cf*1 +

*cf*5 *≤* (2)

*w* + *w* +1

2

*w*3 + *w*2

*w*2 + *w* +1 *− ε*

2*w*2 + 1

2*w*2 +2

= *cf*3 *≤* (3)

*w*2 *w*3

*w*2 + *w* +1 *− ε*

*w*2 +1

*w*2 *w*2 *w*2 *w*2

*w*

2

*w*

2

*w*

2

(4)

+ *w*2 +1 =

*cf*1 +

+1

*cf*5 *≤ cf*1 +

+1

*cf*5 *.*

+1

From this we gather that, for player *Pa*, the preferred scenario is where player *Pc* connects to *f*5, even if player *Pb* does not connect to *f*1. For player *Pc*, we do the same, observing that now the facility which dictates its strategy is *f*2, as even if player *Pa* does not connect to *f*5, if player *Pb* does connect to *f*2, the cheapest for *Pc* is to connect to *f*2 and *f*5. Player *Pc* can, going from least to most expensive,

(i) share *f*2 with *Pb* and *f*5 with all players, (ii) share *f*2 and *f*5 with *Pb* only, (iii) connect to *f*4 alone, (iv) connect to *f*2 alone and share *f*5 with all players and finally

(v) connect alone to *f*2 and share *f*5 with *Pb* only. The inequalities that show this is the case for player *Pc* are:

*w w*

*cf*2 + 2

*cf*5 *≤* (5)

*w* +1

*w*

*cf*2 +

*w* + *w* +1

*w*

*cf*5 =

*w*3 + *w w*

2 + *ε*

*≤* (6)

*w* +1

*w* +1

*w* + *w* +1 *w* +1

*w*3 + *w*

*w*2 + *w* +1 + *ε*

*w*3

2*w* + 1

2*w* +2

*w*

= *cf*4 *≤* (7)

*w*

*w*2 + *w* +1 + *ε* + *w*2 + *w* +1 = *cf*2 +

*w*

*cf*5 *≤* (8)

*w* + *w* +1

2

*cf*2 +

*cf*5 *.* (9)

*w* +1

Now we prove the theorem by contradiction. Suppose that there exists a PNE for this instance. Terminal *t*1 can be connected to either *f*1 or *f*2. First assume *t*1 is connected to *f*1. Then, player *Pc* has no incentive to connect to either *f*2 or *f*5, as shown by ([7](#_bookmark9)) and ([8](#_bookmark10)), and connects all its terminals to *f*4. Since player *Pc* does not connect to *f*5, player *Pa* also does not have enough incentive to connect to *f*1 and *f*5, as shown by ([3](#_bookmark5)) and ([4](#_bookmark7)), and connects all its terminals to *f*3. With this, player *Pb* is paying alone for *f*1, and since *f*2 is cheaper to connect, it is not in a PNE, and thus *t*1 cannot connect to *f*1 in any PNE.

Now assume *t*1 is connected to *f*2. Player *Pc* now will connect to *f*2 with *t*4, with *t*6 connecting to *f*5, as the inequalities in ([6](#_bookmark8)) and ([7](#_bookmark9)) show. Since player *Pc* connects to *f*5, now player *Pa* will also opt to connect to *f*5 and therefore will also open *f*1, as shown by ([2](#_bookmark6)) and ([3](#_bookmark5)). Since *f*1 has player *Pa* connected to it, player

*Pb* now has enough incentive to connect *t*1 to *f*1 instead of *f*2, and therefore our assumption is not true.

Since connecting *t*1 to neither *f*1 nor *f*2 results in a PNE, there is a contradiction with our claim that there exists a PNE for this instance, and thus we have proven that there is no pure equilibrium. *2*

* 1. *Equilibria Existence in FLG-FC*

Now we extend the results from Theorem [4.1](#_bookmark3) for the unweighted case. To accomplish this we first make the key observation that for any weighted fair cost facility location game instance *G* with *m* terminals, there is a fair cost facility location game instance *Gj* with *W* terminals with equivalent PNE, where *W* is the total sum of weights *w*1 + *···* + *wm*. This is the case since for each terminal *t* controlled by a player *i* with weight *wt >* 1, we can add in *Gj* terminals *tj*, for *j ∈* [2*, wt*], all controlled by *i* with the same connections and costs as *t*. Since they are equal in every way in regards to cost calculation, player *i* has no reason to split strategies when choosing which facility to connect *t* and the added terminals *tj*.

**Theorem 4.2** *There exists a metric 3-player instance for the FLG-FC game where there is no PNE.*

**Proof.** Consider the instance *I* depicted in Figure [2](#_bookmark12). Let *w >* 1 bea parameter on this graph, and *ε* be a constant much smaller than 1 , such that player *Pa* controls

*w*3

terminals *t*1*,..., tw*2

1 *w*2 2

2 2 and *t*5*,..., t*5 , for a total of 2*w* terminals. Player *Pb* controls

two terminals, *t*1 and *t*3, while player *Pc* controls terminals *t*1*,..., tw* and *t*1*,..., tw*,

4 4 6 6

for a total of 2*w* terminals. All connection costs are unitary, with the exception of

*dt*1*,f*1 , which has cost 1 + 3*ε*. All opening costs are as shown in Figure [2.](#_bookmark12)

*dt*1 *,f*1 = 1+ 3*ε*



*c* =

*f*1

*w*3

*w*2+*w*+1

*— ε*

*f*1

*t*1

*f*2

*c* =

*f*2

*w*3

*w*2+*w*+1

+ *ε*

*Pa*

*t*1

2

2

*tw*

2

*Pb*

*t*3

*t*1

4

*tw*

4

*Pc*

*c* =

*f*3

*w*2+*w*+1 2*w*2+2

*w*3+*w*2 *− ε*( 2*w*2+1 )

*cf* = *w* +*w* + *ε*( 2*w*+1 )

3

*f*3

*f*

4

4

*w*2+*w*+1 2*w*+2

*t*1

5

*t*5

*w*2

*f*5

*t*1

6

*tw*

6

*w*2

*cf*5 =1

*w*

*dt*1 *,f*2 =1

Fig. 2. Game instance of the FLG-FC without a PNE. All edges except *dt*1 *,f*1 have cost equal to one. Any edge (*t, f* ) not drawn has cost equal to the shortest path cost from *t* to *f* .

*{*

*{*

We prove the theorem by showing that the instance seen in Theorem [4.1](#_bookmark3), here denominated *I*0, can be transformed to instance *I*, seen in Figure [2](#_bookmark12) without changes to the players overall strategies, and thus the proof for Theorem [4.1](#_bookmark3) applies for in-

5

stance *I*. Start by changing *I*0 to *Ij* so that we add terminals *tj*

2

and *tj*

connected

to the same facilities (and the corresponding connection costs) as *t*2 and *t*5, respec-

2

tively, while changing the weight *wA* of all terminals of player *A* to *w* . In any pure

2

equilibria for this instance, if we connect any terminal to facility *f*3, all terminals from *A* will connect to it, since *A* will pay the full cost of *f*3. Thus, if *t*2 connects to

*f*3, terminal *tj*

2

will also connect to *f*3, and if *t*2 connects to *f*1, *tj*

will do the same.

The same is true for *tj* in relation to *t*5. Since this is the case, if player *A* connects

2

5

to *f*

or *f* , it will still pay exactly the same share *w*2

of the opening cost of

1 5 *W* (*f,S*)

facility *f* as in *I*, where *W* (*f, S*) is the sum of weights of all terminals connected to

*f* in the strategy profile *S*. Instance *Ij* thus incurs the same decisions from players as instance *I*0. The same can be done to player *C* and its terminals *t*4 and *t*6.

Therefore, in order to transform *I*0 into instance *I* while preserving the same possible equilibria, it suffices to incrementally add terminals with the same connec- tions for players *A* and *C*, while dividing the weight of these players by the number of added terminals until the number of terminals A controls is 2*w*2 (*w*2 of “*t*2” terminals and *w*2 of “*t*5”) and they all have weight *wA* = 1, while player *C* will control 2*w* terminals (*w* of “*t*4” terminals and *w* of “*t*6”), all with weight *wC* = 1. Thus, Theorem [4.1](#_bookmark3) applies to the unweighted metric instance *I*, and there exists a 3-player weighted instance *I* without any equilibria. *2*

Using a version of this instance where *w* = 2 as a gadget, we can prove that deciding whether an instance of FLG-FC has a PNE is NP-Hard.

**Theorem 4.3** *It is NP-hard to determine if an instance of the Metric FLG-FC has a PNE or not.*

**Proof.** First notice that it is NP-hard to verify whether a given solution *S* to an instance of the FLG-FC is a PNE or not when players control multiple facilities, since for each player we need to solve an optimization facility location problem on the instance restricted to this player alone. To verify that a strategy of a player is a best response, we have to fix all other players strategies, and check if the solution restricted to this player is minimum, which is an NP-hard problem.

In order to prove NP-hardness for the existence of a PNE, we make a reduction from the 3-SAT problem to the problem of deciding whether an instance of the metric FLG-FC game has a PNE or not.

Let *I* be an instance from 3-SAT, with clauses *C*1*,.* *, Cq*, where each clause *Cj*

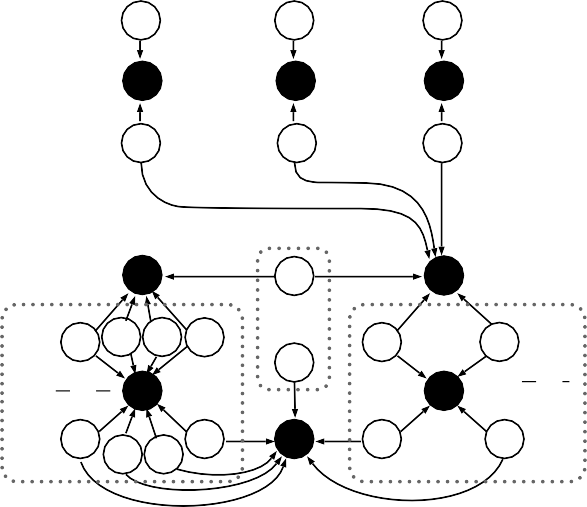
consists of a triple of literals from the set of decision variables *x*1*,..., xp* in clausal normal form, so that each literal in a clause can assume form *xi* or *xi* for a decision variable *xi*. Then, we form the fair cost facility location instance *G* as follows: for each decision variable *xi*, we createa terminal *ti*, controlled by a single player, and facilities *fxi* and *fxi* and link *ti* to them, as Figure [3a](#_bookmark14) shows. The opening cost of each facility *fxi* and *fxi* is equal to the number of terminals that can directly connect to them times 1 + *ε*. We use the term directly here to discern from the connections not shown, which are assumed to cost the shortest path from a terminal to the facility.

We create for each clause *Cj* two gadgets: one with no equilibria unless stabilized by the second gadget, which in turn links to the decision gadget, as exemplified in Figure [3b](#_bookmark14). For the bottom gadget, note that it is exactly the same example as the one used in Theorem [4.2](#_bookmark11), with parameter *w* = 2. We have three players in

Decision gadget for *x*1

Decision gadget for *x*2

Decision gadget for *x*3



*tj*

*a*1

*tj*

*a*2

*tj*

*a*3

*dtj ,fj* = *ε*

*a*1 *a*1

*dtj ,fj* = *ε*

*c j* =2

*a*2 *a*2

*dtj ,fj* = *ε*

*fa*

*f*

1

*j a*1

*c j* =2

*fa*

*f*

2

*j a*2

*fj*

*a*3

*c j* =2

*a*3 *a*3

*fa*3

*tj*

*b*1

*tj*

*b*2

*tj*

*b*3

*dtj ,fj* =2

*b*1 2

*dtj ,fj* =2

*b*2 2

*d j j* =2

*tb*3 *,f*2

*dtj ,fj* = 1+ 3*ε*

1 1

*dtj ,fj* =1

1 2

*c j* = 8 *− ε*

*f*1 7

*f*1

*j*

*t*1

*j*

*f*2

*j*

*c j* = 8 + *ε*

*f*2 7

*Pj*

*a tj*

2*,*1

*tj*

2*,*2 2*,*3

*tj*

*tj*

*Pj*

*b*

2*,*4

*tj*

*tj c*

*Pj*

*tj*

4*,*1

4*,*2

3

*c*

*f*3

= 12 *−* 9 *ε*

*c j* = + *ε*

10 5

7 10

*f*3

*j*

*f*4

*j f*4 7 6

*tj*

5*,*1 *tj tj*

*tj*

5*,*4

*fj*

*tj*

*tj*

5*,*2 5*,*3

5

6*,*1

6*,*2

*cfj* =1

5

*fx*1

*fx*2

*fx*3

*cfx*

*ti*

= 4(1 + *ε*)

*i*

*fxi*

*cf*

*i*

*x*

*fxi*

= 3(1 + *ε*)

Clause gadgets containing *xi*

Clause gadgets containing *xi*

1. A decision gadget for *xi* in an in-

stance with three clauses containing

*xi* and two containing *xi*.

1. Clause gadget for *Cj* = (*x*1 *∨ x*2 *∨ x*3).

Fig. 3. A decision (a) and a clause (b) gadget. Drawn edges without explicit costs have connection cost one, any edge (*t, f* ) not drawn has cost equal to the shortest path cost from *t* to *f* .

this gadget for each clause *C* , where player *Pj* controls terminals *tj*

*,..., tj*

and

*j a* 2*,*1

2*,*4

*tj ,..., tj* , player *Pj* controls terminals *tj* and *tj* and player *Pj* control terminals

5*,*1

5*,*4 *b*

1. 3 *c*

*j*

*t*

4*,*1

*j*

4*,*2

*, t*

*j*

6*,*1

*, t*

*j*

6*,*2

*, t*

. The opening and connection costs of this gadget follow the same

costs presented in Theorem [4.2,](#_bookmark11) when parameter *w* equals to two.

For the link gadget, each player controls a single terminal, and for each literal

*xi* or *xi* in *Cj* there is a terminal *tj*

*ai*

with possible direct connection to both the

facility in the decision gadget for *xi* (either *fxi* or *fxi* ), with connection cost one, and central facility *fj* , with connection cost *ε* and opening cost 2. Furthermore, we create another terminal *tbi* which can connect directly to either the central facility

*ai*

*fj* , with connection cost one, or to facility *fj* with connection cost 2. Now we add

*ai* 2

connections from every terminal *t* to every facility *f* obeying the triangle inequality

by setting cost *dtf* to the shortest cost path from *t* to *f* in the undirected network formed by the connections between terminals and facilities.

Suppose there is a truth assignment for *I*. Then we can build the following PNE for instance *G*: if decision variable *xi* = 1, assign *ti* to *fxi* , and otherwise assign *ti*

to *fx* . Now for each clause gadget *Cj* where literal *xi* appears connect *tj* to *fx*

*i*

if *xi* = 1, or to *fj*

if *xi* = 0. Furthermore, connect *tj*

*ai* *i*

to *fj* if *xi* = 1, or to *fj* if

*ai bi* 2 *ai*

*xi* = 0. Do the same for clauses where literal *xi* appears.

Finally, connect the rest of the terminals in *Cj*

*a*

as follows: player *Pj*

connects

its terminals to *fj* and *fj*, player *Pj* connects to *fj* and *fj* and player *Pj* connects

1. 5 *b*

to *fj* and *fj*.

1. 5 *c*

2 5

First, note that if *xi* = 1, then *fxi* is open and connected with all terminals that can directly connect to it, meaning that each terminal pays 1 + *ε* opening

cost. At the same time, in each clause where *xi* appears, *fj*

*ai*

is not opened, and

thus the cheapest alternative for terminal *tj*

*ai*

*ai*

costs 2 + *ε*, the same amount it pays

for connecting to *fxi* . Alternatively, if *xi* = 0, facility *fxi*

connected to its cheapest possible facility.

is not open and *tj* is

Since *I* has an satisfying assignment, at least one terminal *tj*

*bi*

will be connected

to *fj*, paying at most 2 + 2 + *ε* and stabilizing the game induced by players *Pj,Pj*

1. 7 4 *a b*

and *Pj*. Furthermore, when *xi* = 1, terminal *tj* is in equilibrium connected to *fj*,

*c*

*bi*

2

since the alternative facility *fj* would require *tj* to pay 3 in opening and connection

*ai*

costs. Alternatively, if *xi* = 0, terminal *tj*

*bi*

*bi*

is also in equilibrium by connecting to *fj* ,

*ai*

since in this case it shares the opening cost with *tj* , paying a total of 2. Therefore, we have a PNE for *G* given a satisfying assignment for *I*.

*ai*

Now suppose there exists a PNE in *G*. For any clause *Cj*, at least one terminal

*tj* must be connected to *fj*, since otherwise players *Pj,Pj* and *Pj* would not be in

*bi* 2

*a b c*

equilibrium. Then, for each terminal *tj* connected to *fj*, terminal *tj*

must connect

*bi* 2 *ai*

to *fx*

(or *fx* , if representing literal *xi*). This is the case because if *tj*

were to

*i i ai*

connect to *fj* , terminal *tj* would not be in equilibrium connecting to *fj*.

*ai bi* 2

Note that for terminal *tj*

*ai*

in clause *Cj* connected to facility *fxi* , even when

all terminals with a direct possible connection apart from *ti* are connected to it,

terminal *tj* still is not in equilibrium, since it would be cheaper for it to open *fj*

*ai*

*ai*

paying *ε* connection cost and 2 opening cost than paying 1 as connection cost and

*l* (1 + *ε*) opening cost, where *l* is the number of terminals that can connect directly to *fxi* . Terminal *ti* in any PNE therefore must connect to either *fxi* or *fxi* .

*l—*1

This shows that in any PNE in *G*, at least one literal of each clause is set to true in the corresponding assignment in *I*. Furthermore, it also shows that there can be no decision variable *xi* with *xi* = 1 and *xi* = 1. If a decision variable *xi* is set as both *xi* = 0 and *xi* = 0, then it must be irrelevant to the truth assignment of *I*, since *G* is in equilibrium, and therefore we can assign either *xi* or *xi* to 1. Thus, any PNE in *G* is consistent and can be used to build a satisfying assignment for

1. *2*

Note that while we use several players in our proof, it is possible to reduce the total number of players to only six.

**Corollary 4.4** *It is NP-hard to determine if a metric FLG-FC has a PNE, even for games with 6 players.*

**Proof.** First note that in the instance *G* in Theorem [4.3](#_bookmark13), there are three players which control multiple terminals for each clause *Cj*, as well as six singleton players. Furthermore, for each decision gadget there is one additional player. We form a new instance *Gj* with the same terminals, facilities and costs as *G*, but only six players. We remark that each clause gadget is “de facto” isolated from each other in *G*, since the connection cost necessary for a terminal in a clause gadget *Cj* to connect to a facility in either a different clause gadget *Cj*+1 or in a decision gadget not directly

connected to *Cj* is greater than opening any possible facility it can directly connect. Thus, we can form players *Pa*, *Pb* and *Pc* in instance *Gj* that control terminals controlled by players *Pj,Pj* and *Pj* in any clause *C* . Furthermore, we can make

*a b c j*

players *Pa*1*, Pa*2*, Pa*3 and *Pb*1*, Pb*2*, Pb*3 in *Gj* to control every terminal *tj*

*, tj*

*, tj*

and

*j j j*

*a*1 *a*2 *a*3

*tb*1*, tb*2*, tb*3 in any clause gadget *Cj*.

Similarly, we can join all players from the decision gadgets into a single player *Px* in *Gj*, which controls any terminal *ti* in any decision gadget *xi*. Now the total number of players in this instance is reduced to ten. To achieve the number of six players, we remark that player *Pa*1 has no direct connection to facilities that *Pb*2 can directly connect, and thus can be safely merged into a single player *Pa*1*b*2. Similarly, player *Pa*2 can be safely merged with *Pb*3 and *Pa*3 can be merged with *Pb*1, resulting in players *Pa*2*b*3 and *Pa*3*b*1 in *Gj*. Finally, note that player *Pa* has no direct connection to *fj* in any clause gadget *Cj*, and therefore can safely merge with either *Pa*1*b*2, *Pa*2*b*3 or *Pa*3*b*1. Thus, we have one player controlling all decision gadgets and five players controlling the clause gadgets, resulting in a total of six players in instance *Gj* while maintaining the same features and possible PNE as instance *G*. *2*

2

# Efficiency of Equilibria in Non-singleton Fair Cost Fa- cility Location Games

In this section we turn our analysis towards loss of efficiency due to player behaviour. The most well known measures of efficiency for non-cooperative games are the PoA and the PoS. The price of anarchy in several games can be high and sometimes unrealistic, due to comparing only the worst possible equilibrium in terms of social cost or welfare to the optimal social welfare or cost. For facility location games, we can show that the price of anarchy is at least *k*, where *k* is the number of players of the game. To see that this is the case, see Figure [4](#_bookmark16). In this example, the strategy where all players are connected to the facility to the right is a PNE, and has social cost *k*, while in the optimal social solution all players are connected to the facility to the left, with total cost equal to 1 (connection costs are zero).

*cf*1 =1 *cf*2 = *k*



*f*1

*t*1

*t*2 *t*3

*f*2

*tk*

Fig. 4. A game instance of the FLG-FC with PoA equal to *k*.

In [[1](#_bookmark20)], Anshelevich et al. argue that the concept of PoS, formalized in the same paper, better captures the efficiency loss in network design games and consequently singleton fair cost facility location games. They show a bound for the PoS of

*Hk* = 1 + 1 + *···* + 1

= *Θ*(log *k*) for these network design games, where *k* is the

2 *k*

number of players in the game, which can be extended for singleton fair cost facility

location games [[17](#_bookmark36)].

In this paper we show that the same is not true for fair cost facility location games where players can control multiple terminals, even when connection costs obey the triangle inequality. We prove that there are instances where the PoS is *Θ*(*k*), where *k* is the number of players, the same as the PoA.

**Theorem 5.1** *There are instances of the metric FLG-FC which admit PNE but have price of stability in Θ*(*k*)*, where k is the number of players.*

## Proof.

Consider the instance shown in Figure [5](#_bookmark17). Let *ε* be a constant number much

smaller than 1 . Each player *Pi*, for *i ∈* [1*, k*], controls terminal *tj*

. Player *Pa*

8 *i*

controls terminals *t*1 to *t*4 and *t*1 to *t*4, player *Pb* controls terminals *t*1 and *t*3, while

2 2 5 5

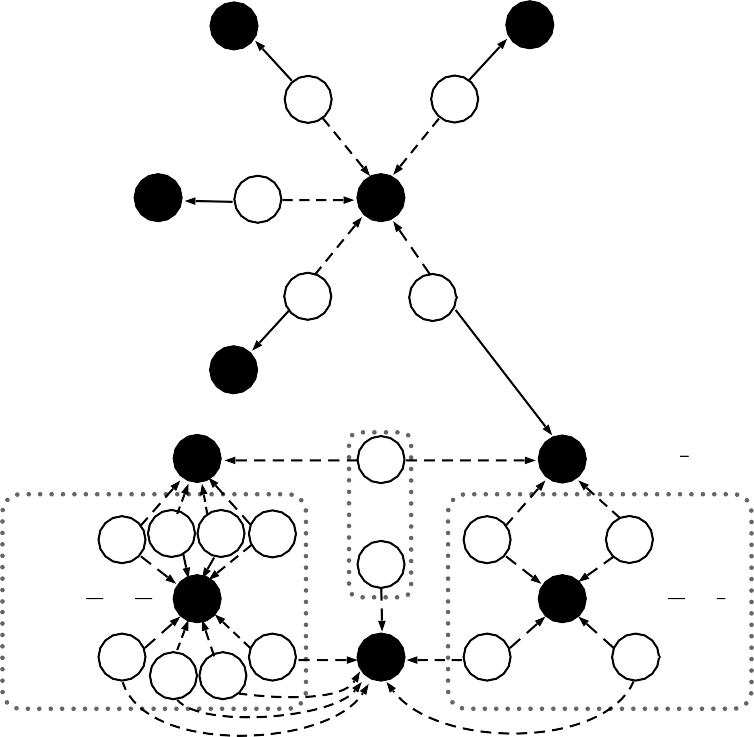
player *Pc* controls terminals *t*1*, t*2*, t*1 and *t*2. The full black lines have connection

4 4 6 6

cost one, while the dashed lines have connection cost *ε*, with the exception of *dt*1*,f* 1

*b b*

with cost 3*ε*2 + *ε*.



*cf′* = *ε*

4

*f′*

4

*f′*

*k cf′* = *ε*

*P*4

*Pk*

*k*

*t′*

4

*t′*

*k*

*P*3

*cf′* = *ε*

3

*f′*

3

*t′*

3

*fs*

*cf* =1

*s*

*P*2

*t′*

2

*P*1

*t′*

1

*cf′* = *ε*

2

*f′*

2

*d*

*t ,f*

1

1

= 3*ε* + *ε*

2

*dt*1 *,f*2 = *ε*

*cf* = *ε*( 8 *− ε*)

1

7

*f*1

*t*1

*f*2

*cf* = *ε*( 8 + *ε*)

2

7

*Pa*

*t*2

1

*t*2

2

*t*3

2

*t*2

4

*Pb*

*t*3

*t*1

*Pc*

4

*t*2

4

*cf* = *ε*( 12 *−* 9 *ε*) *f*3

3

7 10

*f*4 *cf* = *ε*( 10 + 5 *ε*)

4

7 6

*t*1

5

4

*t*2

*t*5

*f*

*t*6

1

*t*6

2

5

*t*3

5

5

*cf*5 = *ε*

Fig. 5. An instance of metric FLG-FC with PoS in *Θ*(*k*). Dashed lines have connection cost *ε* (except for

*dt ,f* , which has cost 3*ε*2 + *ε*), full black lines have cost 1. Labels next to terminals indicate which player

1 1

controls the terminal.

Notice that the subgame composed only by the players *P*2 to *Pk* and their connec- tions, on the left side of Figure [5](#_bookmark17), has two pure equilibria: either everyone connects to the central facility *fs* or each player *Pi* connects its terminal to facility *fj*. The

*i*

subgame composed by players *Pa*, *Pb* and *Pc* and their connections, on the other hand, is equal to the instance used in Theorem [4.2](#_bookmark11) when *w* = 2 with the costs scaled by *ε*, and therefore it has no equilibria unless the game is altered to allow some other player to stabilize it.

Player *P*1 is the one in this instance that can stabilize these two subgames. Recall from Theorem [4.2](#_bookmark11) that, as long as some terminal is connected to *f*2, player *Pc* will connect to *f*2 and *f*5. Since there are multiple terminals in *f*5, player *Pa* will want to connect to *f*1 and *f*5 as well. This then means that player *Pb* will want to connect to *f*2 and *f*5. Therefore in any PNE, *P*1 must connect to *f*2. For this to be the best possible move for *P*1, no player can open the central facility *fs*, and therefore the only possible PNE is the one where each *Pi* connects to *fj*, while *Pa*, *Pb* and *Pc* play as seen above. Note that players *Pa*, *Pb* and *Pc* can be merged with any players *Pi, Pj* and *Pr*, for all *i /*= *j /*= *r ∈* [2*, k*], since they have disjoint strategy sets. Therefore, we can assume that there are *k* players in this instance.

*i*

The cost of the optimal strategy is the following. In the left side, players *P*1 to *Pk* connect to the central facility *fs*, for a cost of 1 + *kε*. In the right side, players *Pb* and *Pc* connect to *f*2 and *f*5, while player *Pa* connects to *f*1 and *f*5, paying also a connection cost of *ε* for each terminal, for a total cost of

1+ *kε* + *ε* 8 *— ε* +1+ 8 + *ε* + 14 =1 + *kε* + 16 *ε* + 15*ε* =1 + *ε* *k* + 15+ 16 *.*

7 7 7 7

In the unique PNE, players *P*2 to *Pk* connect to their single facilities *fj* to *fj*

2

*k*

and player *P*1 connects to *f*2, while players *Pa*, *Pb* and *Pc* connect to facilities *f*2, *f*5 and *f*1 (with *Pb* using connection *dt ,f* which costs 3*ε*2 + *ε*). Thus, the total cost for the PNE is

1 1

*k* + (*k —* 1)*ε* + *ε* 8

7

8

*— ε* +1+

7

+ *ε* + 3*ε* + 14

= *k* + *kε — ε* +

16 *ε* + 15*ε* + 3*ε*2

7

= *k* + *ε* *k* + 14+ 16 + 3*ε* *.*

7

The PoS of this instance is therefore

*k* + *ε*(*k* + 14+ 16 + 3*ε*)

7 = *Θ*(*k*) *.*

1+ *ε*(*k* + 15+ 16 )

7

Since the PoA of FLG-FC is *Θ*(*k*) (and therefore the PoS is *O*(*k*)), this instance makes the bound for PoS for FLG-FC asymptotically tight, and thus the PoS for metric FLG-FC is *Θ*(*k*). *2*

* 1. *Price of Stability in Capacitated Facility Location Games*

Until now we have only considered scenarios where there are no limits on the number of terminals that a facility can supply. However this is not always the case in practice. In [[16](#_bookmark35)], Rodrigues and Xavier show that even metric facility location games have unbounded PoA, unless sequentiality is considered. Furthermore, they

show that metric singleton capacitated fair cost facility location games have a PoS of at least *Θ*(log *k*), in comparison to the constant bounds of the uncapacitated metric version [[9](#_bookmark28)].

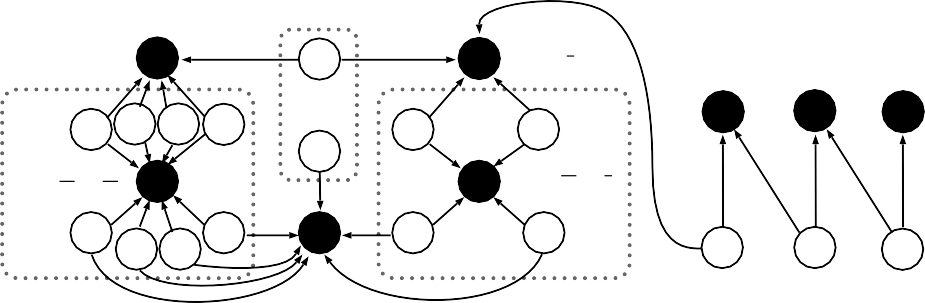
In this section we prove that for capacitated games where players are allowed to control more than a single terminal, there are instances with unbounded PoS. Note that, contrary to most results in this paper, we consider for this result that connection costs do not obey the triangle inequality, and thus any connections not shown in our construction have prohibitively large (and unbounded) cost *Ud*.

**Theorem 5.2** *There are 3-player instances of the capacitated FLG-FC that admit a PNE but have PoS that is unbounded.*

## Proof.

Assume that we are restricted to instances where there is a PNE and players are allowed to control multiple terminals. We combine the instance described in Theorem [4.2](#_bookmark11) into an instance with unbounded PoA [[16](#_bookmark35)] to force that the only possible PNE in the game opens facilities with unbounded opening costs.

Consider the game in Figure [6](#_bookmark18). Note that the subgraph induced by the terminals that players *Pa, Pb, Pc* control is the same as in Theorem [4.2](#_bookmark11) when parameter *w* = 2. Thus, the only way for them to be in a PNE is if terminal *t**a* connects to *f*2. For this to happen in an equilibrium, terminal *tb* must not connect to *fa*. If this was the case, then it would be cheaper for *ta* to also connect to *fa*, as it would pay only 1*/*4 which is less than the amount it pays to connect to *f*2. Since *f*2 must be occupied by *ta* in a PNE, terminal *tc* in any equilibrium will connect to *fc*, paying the connection cost of *U* which can be arbitrarily large, and thus the only possible equilibrium in this game is unbounded. Consequently, the PoS is also unbounded.



*c* = *— ε*

*f*

8

*dt*1 *,f*1 = 3*ε*

*dt*1 *,f*2 =0

1 7

*f*1

*t*1

*f*2

*c* = + *ε*

*f*

8

*ufa* =2 *u* =1 *uf* =1

2

7

*c*

*fa*

= 1

2

*cf* = 1 *cf* = *U*

*fb*

*c*

*b* 2 *c*

*Pa*

*t*2

1

*t*2 *t*3

2 2

*t*2

4

*Pb*

*t*3

*t*1

*Pc*

*fa*

*fb*

*fc*

4

*t*2

4

*c*

*f*3 7 10

= 12 *—* 9 *ε*

*f*3

*f*

4

*cf* = 10 + 5 *ε*

4 7 6

*t*1

5

*t*2 *t*3

*t*5

4

*f*5

*t*6

1

*t*6

2

5 5

*cf*5 =1

*ta*

*Pa*

*tb*

*Pb*

*tc*

*Pc*

Fig. 6. An instance of capacitated FLG-FC with unbounded PoS. All connections have zero cost (except

*dt*1 *,f* 1 , which has cost 3*ε*). Any facility *f* without a capacity restriction indicated is unlimited, i.e., *uf ≥* 8.

*b b*

Labels next to terminals indicate which player controls the terminal.

*2*

# 6 Conclusions and Future work

In this paper we have studied a general version of the fair cost facility location game. In this model, the opened facilities have their opening costs shared evenly among

all terminals that connect to them, and players are allowed to control multiple terminals. We prove that there are instances for these games with no pure Nash equilibrium, and that deciding whether an instance of the game has a PNE or not is NP-hard, even when connection costs are metric. Furthermore, we provide a negative partial answer to an open question whether weighted fair cost facility location games always posses PNE, by showing the connections of these weighted games to scenarios where players control multiple terminals. Finally, we prove negative efficiency bounds related to the price of stability in both uncapacitated and capacitated games, showing that the price of stability is as inefficient as the price of anarchy in this generalized version of fair cost facility location game.

There are several possible directions on future work that relates to our paper. For weighted games, it remains an open problem to find whether there is an instance with no PNE for the singleton case. While our instance with no PNE has only a few terminals, the effect that cooperation between terminals can have on the PNE is crucial for there to be no PNE in this instance. It is not clear whether there is a possible way to adapt this instance for the singleton case. When we consider strong equilibria [[6](#_bookmark25)] for the singleton case, where terminals might cooperate to choose a better strategy when it is beneficial to every player in a coalition, a few similarities appear with this general version of fair cost facility location game. It might be possible to extend some results from strong equilibria for singleton fair cost facility location games to our setting, such as the existence of *e*-approximate strong equilibria in singleton fair cost facility location games [[10](#_bookmark29)]. Alternatively, it might as well be possible to adapt some of our results to strong PNE in more restricted settings.

In [[2](#_bookmark21)], Cardinal and Hoefer prove a two parameter (*α, β*) constant approximation to facility location games with arbitrary cost sharing rules, where each player can reduce its cost by unilateral deviation by at most a factor of *α*, while the social cost of this *α*-approximated PNE is at most *β* times from the optimal social cost. An interesting question is whether it is possible to find a similar two parameter approximation for our setting, or if the limited fair cost aspect of our scenario causes one of the parameters to be too large.

Finally, in many practical scenarios there is a central authority with some ca- pacity for interference in the game which has a goal of minimizing the social cost globally. Thus it is important to model in what circumstances this intervention can improve efficiency. The use of stackelberg strategies [[7](#_bookmark26)], where a central authority control a few terminals that can play before the other players, might guarantee that a PNE is always possible to find, as well as using tolls [[4](#_bookmark24)] in either connections or facilities to ensure only “good” equilibria are chosen by players.

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