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On Paraconsistent Extensions of *C*1

# Mauricio Osorio[1](#_bookmark0)

*Universidad de las Am´ericas - Puebla Sta. Catarina M´artir, Cholula, Puebla, M´exico*

# Jos´e Luis Carballido[2](#_bookmark1)

*Benem´erita Universidad Aut´onoma de Puebla Puebla, Puebla, M´exico*

# Claudia Zepeda[3](#_bookmark2)

*Benem´erita Universidad Aut´onoma de Puebla Puebla, Puebla, M´exico*

Abstract

We show that logic *C*1 cannot be extended to a paraconsistent logic in which the substitution theorem is valid. We show that *C*1 can be extended to larger paraconsistent logics by adding some desirable properties as axioms. We use three-valued logics to support our claims.

*Keywords:* multi-valued logics, substitution theorem, logic *C*1, paraconsistent logic.

# Introduction

Two main approaches are common to define a logic, the Hilbert axiomatic system and the use of multi-valued tables that define the connectives of the logic. In the first approach the validity of a formula is determined by a set of axioms and a family of inference rules, namely, if the formula can be derived from those axioms and the use of the inference rules, then the formula is valid, otherwise it is not valid. In general, there are many ways of choosing the family of axioms to define a logic and Modus Ponens is one of the most common inference rules appearing in the definition of logics. In the second approach, the tables used to define the logic are called truth

1 Email: [osoriomauri@gmail.com](mailto:osoriomauri@gmail.com)

2 Email: [jlcarballido7@gmail.com](mailto:jlcarballido7@gmail.com)

3 Email: [czepedac@gmail.com](mailto:czepedac@gmail.com)

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tables, each connective is regarded as a function taking values in a set of numbers (usually integers) that are specified from the beginning and are called truth values. Some of the values are chosen as designated values. Any formula that evaluates to one of the designated values regardless of the truth values taken by the atoms that appear in the formula, is considered valid. In this paper, we combine both approaches.

In logic, as in any other area of mathematics, when choosing a family of ax- ioms to define a logic, it is desirable to have independence of the axioms, that is, any formula chosen as an axiom should be independent from the other axioms. Multi-valued logics can be used for this purpose (see an example of this in [[14](#_bookmark35)]). This methodology sometimes can have limitations (see [[10](#_bookmark32)]), however it is useful to researchers interested in the study of logics, such as in our case.

One of the properties we are particularly interested in is paraconsistency. Fol- lowing B´eziau [[2](#_bookmark23)], a logic is paraconsistent if it has a negation ¬, which is paracon- sistent in the sense that the formula *a,* ¬*a* ▶ *b* is not valid, and at the same time has enough strong properties to be called a negation. Paraconsistent logics have im- portant applications, specifically [[7](#_bookmark29)] mentions three applications in different fields: Mathematics, Artificial Intelligence and Philosophy. In relation to the second one, the authors mention that in certain domains, such as the construction of expert

systems, the presence of inconsistencies is almost unavoidable (see for example [[9](#_bookmark31)]). An application that has not been fully recognized is the use of paraconsistent logics in non-monotonic reasoning. In this sense [[21](#_bookmark40),[20](#_bookmark41)] illustrate such novel applications. One example where intuition indicates that paraconsistent logics would be use-

ful for describing abstract structures is provided by Birkhoff and Von Neumann’s approach to quantum logic [[5](#_bookmark27)].

We emphasize the convenience of accepting local inconsistencies by mentioning Minsky’s comment [4](#_bookmark3) [[15](#_bookmark37)]: “*But I do not believe that consistency is necessary or even desirable in a developing intelligent system. No one is ever completely consis- tent. What is important is how one handles paradox or conflict, how one learns from mistakes, how one turns aside from suspected inconsistencies*”. We think that para- consistent logics could help to give an answer to this important issue addressed by Minsky. In fact, in [[16](#_bookmark38)] an interesting approach for Knowledge Representation (KR) was proposed. This approach can be supported by any paraconsistent logic stronger than or equal to *Cω*, the weakest paraconsistent logic introduced by Da Costa [[8](#_bookmark30)].

Therefore we must consider paraconsistent logics as a supplement to classi- cal logic that deviates from it only in some of its principles (mainly the non- contradiction principle) but that might be applied to contradictory or inconsistent systems like those caused by vagueness or empirical theories whose postulates or basic assumptions are contradictory [[5](#_bookmark27)].

Thus, the research on paraconsistent logics is far from being over and, in this work we focus our attention on the paraconsistent logic *C*1, which has been studied in [[12](#_bookmark34)].

4 “*Minsky’s Frame paper* ” (1975) in its original form had an appendix entitled “*Criticism of the Logistic approach*”

As mentioned before the applications of paraconsistency can be done by using any paraconsistent logic stronger than *Cω*. There are many of such logics and their particular properties depend on which properties that are valid in Classical Logic they preserve. The reason to study *C*1 is not just because the applications of its paraconsistency, but also because of the theoretical value in terms of studying possible ways of approaching Classical Logic while preserving paraconsistency.

In this paper we present two results, first we show that there is no paraconsistent logic that extends *C*1 and for which the substitution theorem holds. Second we extend *C*1 to larger paraconsistent logics by adding different axioms to the family of axioms that define it. Each of such axioms is a formula valid in Classical logic, so that by adding them, the logics we obtain are paraconsistent but at the same time closer to classical logic. For each of these extensions we present the tables of one or more three-valued logics that guarantee that each axiom added is independent of the axioms of *C*1 and that the logic obtained is paraconsistent.

Our paper is structured as follows. In section [2](#_bookmark4), we summarize some basic concepts and definitions. In section [3](#_bookmark8), we show our results. Finally, in section [4](#_bookmark22), we present some conclusions.

* 1. *Contribution*

We can think of certain properties or theorems of classical logic that would enrich the scope of any paraconsistent logic. Some of this properties are: the De Morgan laws, the necessitation rule (if *α* is a theorem then ¬¬*α* is a theorem), the week explosion principle: ¬*α* → (¬¬*α* → *β*), the weak contrapositive: (¬*α* → ¬*β*) ↔ (¬¬*β* → ¬¬*α*), and double negation equivalence: *α* ↔ ¬¬*α*.

In this work we study the relation of logic *C*1 with some of this properties after

showing that *C*1 can not be extended to a paraconsistent logic where the substitution property holds.

# Background

There are two ways to define a logic: by giving a set of axioms and specifying a set of inference rules; and by the use of truth values and interpretations. In this section we summarize each of them and we present some basic concepts and definitions useful to understand this paper. From here on, when we refer to any logic, we understand

that the only primitive connectives are ∧, ∨, →, ¬ and the biconditional ↔ that is an abbreviation of (*A* → *B*) ∧ (*B* → *A*).

* 1. *Hilbert style*

In Hilbert style proof systems, also known as axiomatic systems, a logic is specified by giving a set of axioms and a set of inference rules. In these systems, it is common to use the notation ▶*X F* for provability of a logic formula *F* in the logic *X*. In that case we say that *F* is a theorem of *X*.

We say that a logic *X* is paraconsistent if the formula (*A* ∧ ¬*A*) → *B* is not a

Pos1: *A* → (*B* → *A*) Cω 1: *A* ∨ ¬*A*

Pos2: (*A* → (*B* → *C*)) → ((*A* → *B*) → (*A* → *C*)) Cω 2: ¬¬*A* → *A*

Pos3: *A* ∧ *B* → *A*

Pos4: *A* ∧ *B* → *B*

Pos5: *A* → (*B* → (*A* ∧ *B*))

Pos6: *A* → (*A* ∨ *B*)

Pos7: *B* → (*A* ∨ *B*)

Pos8: (*A* → *C*) → ((*B* → *C*) → (*A* ∨ *B* → *C*))

Table 1 Axiomatization of *C*ω .

theorem [5](#_bookmark6) . The relevance of logics for which the formula (*A* ∧ ¬*A*) → *B* is not a theorem, is that they are useful to define alternative semantics that can be applied in the study of non monotonic reasoning as we mentioned in the introduction.

A very important property satisfied by many logics is the substitution theorem which we present now.

Definition 2.1 A logic *X* satisfies the substitution property if: Γ ▶*X α* ↔ *β* then Γ ▶*X* Ψ[*α/p*] ↔ Ψ[*β/p*] for any formulas *α*, *β*, and Ψ and any atom *p* that appear in Ψ where Ψ[*α/p*] denotes the resulting formula that is left after every occurrence of *p* is substituted by the formula *α*.

As examples of axiomatic systems, we present two logics: the positive logic [[17](#_bookmark39)] and the *Cω* logic which is a paraconsistent logic defined by daCosta [[8](#_bookmark30)]. In Table [1](#_bookmark5) we present a list of axioms, the first eight of them define positive logic. *Cω* logic is defined by the axioms of positive logic plus axioms C*ω*1 and C*ω*2.

* 1. *Multi-valued logics*

An alternative way to define a logic is by the use of truth values and interpretations. Multi-valued logics generalize the idea of using truth tables to determine the validity of formulas in classical logic. The core of a multi-valued logic is its *domain* of values D, where some of such values are special and identified as *designated* or select

values. Logic connectives (e.g. ∧, ∨, →, ¬) are then introduced as operators over

D according to the particular definition of the logic, see [[14](#_bookmark35)].

An *interpretation* is a function *I* : L → D that maps atoms to elements in the domain. The application of *I* is then extended to arbitrary formulas by mapping first the atoms to values in D, and then evaluating the resulting expression in terms of the connectives of the logic (which are defined over D). It is understood in general that, if *I* is an interpretation defined on the arbitrary formulas of a given program *P* , then *I*(*P* ) is defined as the function *I* applied to the conjunction of all the formulas in *P* . A formula *F* is said to be a *tautology*, denoted usually by |= *F* if, for every possible interpretation, the formula *F* evaluates to a designated value. The simplest example of a multi-valued logic is classical logic where: D = {0*,* 1},1

5 For any logic X that contains Pos1 and Pos2 (axioms of positive logic defined in Table 1) among its axioms and Modus Ponens as its unique inference rule, the formula (*A* ∧ ¬*A*) → *B* is a theorem if and only if *A,* ¬*A* ▶X *B*.

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| ∧ | 0 | 1 | 2 |  | ∨ | 0 | 1 | 2 |  | → | 0 | 1 | 2 |  | *x* | ¬*x* |
| 0 | 0 | 0 | 0 |  | 0 | 0 | 2 | 2 |  | 0 | 2 | 2 | 2 |  | 0 | 2 |
| 1 | 0 | 2 | 2 |  | 1 | 2 | 2 | 2 |  | 1 | 0 | 2 | 2 |  | 1 | 1 |
| 2 | 0 | 2 | 2 |  | 2 | 2 | 2 | 2 |  | 2 | 0 | 2 | 2 |  | 2 | 0 |

Table 2 2

Truth tables of connectives ∧, ∨, →, and ¬ in P .

is the unique designated value, and the connectives are defined through the usual basic truth tables.

Note that in a multi-valued logic, so that it can truly be a *logic*, the implication connective has to satisfy the following property: for any given value *x* ∈ D, and for every designated value *y* ∈ D such that *y* → *x* is designated, then *x* must also be a designated value. This restriction enforces the validity of Modus Ponens in the logic.

As an example of a multi-valued logic, we present the well known paraconsistent logic P2 [[4](#_bookmark25)] (also called Cive).

The truth values of logic P2 are in the domain *D* = {0*,* 1*,* 2} where 1 and 2 are the designated values. The ∧, ∨, →, and ¬ connectives are defined according to the truth tables given in Table [2](#_bookmark7).

* 1. *C*1 *logic*

*C*1 is a paraconsistent logic defined by B´eziau [[12](#_bookmark34)]. It is defined by the axioms of

*Cω* plus the following axioms:

¬1 : *B*◦ → ((*A* → *B*) → ((*A* → ¬*B*) → ¬*A*))

¬2 : *A*◦ ∧ *B*◦ → (*A* ∧ *B*)◦ ∧ (*A* ∨ *B*)◦ ∧ (*A* → *B*)◦

where *B*◦ = ¬(*B* ∧ ¬*B*). We observe that formula ¬1 makes valid reductio ad absurdum whenever *B* satisfies the principle of non contradiction.

* + - De Morgan law ¬(*A* ∧ *B*) → (¬*A* ∨ ¬*B*) isa theorem in *C*1 [[11](#_bookmark33)].
    - ¬*A* ∨ *A*◦ is a theorem in *C*1 [[11](#_bookmark33)].
    - *A*◦◦ is a theorem in *C*1 [[6](#_bookmark28)].
    - If we replace the axiom ¬2 in *C*1 by the following stronger axiom, called ¬3, then we obtain logic *C*+ [[12](#_bookmark34)], a paraconsistent logic that extends *C*1:

1

¬3 : *A*◦ ∨ *B*◦ → (*A* ∧ *B*)◦ ∧ (*A* ∨ *B*)◦ ∧ (*A* → *B*)◦

* + - We define the *strong negation* as ¬∗ = ¬*α* ∧ *α*◦. The connectives →, ∧, ∨, and

¬∗ satisfy all schemas and inference rules of classical propositional calculus [[6](#_bookmark28)]. Then, we can say the classical propositional calculus is contained in *C*1, and at the same time *C*1 is a subcalculus of it.

As we know, paraconsistent logics have been used as a tool in knowledge represen- tation due to the fact that they allow local inconsistencies without being explosive,

i.e., they accept formulas like *α* and ¬*α* as theorems without becoming explosive in the sense that every formula becomes a theorem, that is, the formula *α* ∧ ¬*α* → *β* is not valid in the paraconsistent logic.

According to some authors [[1](#_bookmark24)], a paraconsistent logic should retain as much of classical logic, but must allow non-trivial inconsistent theories. Also, it should not validate any inference which is not valid in classical logic. Then, it should be contained in classical logic.

According to some of the facts mentioned before, we obtain an increasing se- quence of paraconsistent logics

*Cω* ⊂ *C*1 ⊂ *C*+ ⊂ *Cive*

1

where Cive is defined by the axioms of *Cω* plus the next three formulas as axioms:

¬1 : *B*◦ → ((*A* → *B*) → ((*A* → ¬*B*) → ¬*A*))

*A* → ¬¬*A*

(*A* ∧ *B*)◦ ∧ (*A* ∨ *B*)◦ ∧ (*A* → *B*)◦

Cive is an axiomatization of *P*2, the tautologies of *P*2 are the theorems of cive and vice-versa [[13](#_bookmark36)].

# Main contribution

An interesting theoretical question that arises in the study of logics is whether a given logic satisfies the substitution theorem [[22](#_bookmark42)]. It is well known that there are several paraconsistent logics for which that theorem is not valid [[3](#_bookmark26)]. We show that logic *C*1 can not be extended to a paraconsistent logic in which the substitution theorem is valid. Then we will explore the relation between *C*1 and the De Morgan laws. Since one of our purposes is to extend *C*1 to other paraconsistent logics, in what follows, whenever we provide a three-valued logic, it will be such that the formula (*A* ∧ ¬*A*) → *B* is not a tautology, and also will have the following

property: if *A* and *A* → *B* evaluate both to designated values then *B* must also evaluate to a designated value. These two conditions guarantee that the extensions

we are defining are paraconsistent and sound with respect to this three-valued logic. Besides we will need to define two three-valued logics for each axiom we add, in one of them the new axiom must be a tautology and in the other one must be not a tautology, this is to guarantee that the new axiom is independent of the family of axioms that define *C*1.

The first result of our paper is that *C*1 can not be extended to a paraconsistent logic in which the substitution theorem holds. In order to do this, we provide a definition and a theorem.

Definition 3.1 A logic *X* satisfies the weak substitution property if: Γ ▶*X A* ↔ *B*

then Γ ▶*X* ¬*A* ↔ ¬*B*.

Next theorem is presented in [[18](#_bookmark43)] in a slightly different form. Their proofs are similar.

Theorem 3.2 *Any logic stronger than* C*ω satisﬁes the weak substitution property iff satisﬁes the substitution property.*

Theorem 3.3 *Any extension of logic C*1 *to a logic where the substitution theorem holds, is not paraconsistent.*

Proof.

It is easy to see that *a,* ¬*a* ▶ (*a* ∧ ¬*a*) ↔ *a*.

By substitution theorem we have *a,* ¬*a* ▶ ¬(*a* ∧ ¬*a*) ↔ ¬*a*, but *a,* ¬*a* ▶ ¬*a*,

therefore *a,* ¬*a* ▶ ¬(*a* ∧ ¬*a*).

We also know by an instance of axiom ¬1 that

▶ ¬(*a* ∧ ¬*a*) → ((¬*b* → *a*) → ((¬*b* → ¬*a*) → ¬¬*b*))), then by using modus ponens in the previous line *a,* ¬*a* ▶ ((¬*b* → *a*) → ((¬*b* → ¬*a*) → ¬¬*b*))),

now we use Pos1 to obtain *a,* ¬*a* ▶ (¬*b* → *a*),

and modus ponens to obtain ▶ (¬*b* → ¬*a*) → ¬¬*b*, we also have *a,* ¬*a* ▶ (¬*b* → *a*).

then *a,* ¬*a* ▶ ¬¬*b*, using C*ω*2, ¬¬*b* ▶ *b*.

Finally, we have *a,* ¬*a* ▶ *b*.

This last line shows that paraconsistency does not hold. 2 2

* 1. *De Morgan laws and C*1

As we said before, the De Morgan laws are relevant properties which are desirable in any logic. *Cω*, *P*2 and *Z* fail to satisfy at least one of this laws. We explore these laws in the context of *C*1. First, we prove that De Morgan law ¬*A* ∨¬*B* → ¬(*A* ∧ *B*) is not provable in *C*1.

Theorem 3.4 *The De Morgan law* ¬*A* ∨ ¬*B* → ¬(*A* ∧ *B*) *does not hold in C*1*.*

Proof. If we put *B* as ¬*A* in the formula ¬*A* ∨ ¬*B* → ¬(*A* ∧ *B*) we obtain ¬*A* ∨

¬¬*A* → ¬(*A* ∧ ¬*A*) and then by using axiom C*ω*1 and Modus Ponens we obtain

¬(*A* ∧ ¬*A*) = *A*◦, and from this and theorem 2.1.5 in [[6](#_bookmark28)] we obtain classical logic. This contradicts the paraconsistency of *C*1. 2 2

As a consequence, we have three easy statements. The first one is a well known fact.

Corollary 3.5 *C*1 *has the following properties:*

1. *C*1 *is not Classical logic.*
2. *If we add* ¬*A* ∨ ¬*B* → ¬(*A* ∧ *B*) *as an axiom of C*1*, the resulting logic is Classical logic.*
3. *If we add* ¬*A* ∨ ¬*B* → ¬(*A* ∧ *B*) *as an axiom of C*1*, paraconsistency does not hold any longer.*

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| ∧ | 0 | 1 | 2 |  | ∨ | 0 | 1 | 2 | → | 0 | 1 | 2 |  | *x* | ¬*x* |
| 0 | 0 | 0 | 0 |  | 0 | 0 | 2 | 2 | 0 | 2 | 2 | 2 |  | 0 | 2 |
| 1 | 0 | 2 | 2 |  | 1 | 2 | 2 | 1 | 1 | 0 | 2 | 2 |  | 1 | 1 |
| 2 | 0 | 2 | 2 |  | 2 | 2 | 2 | 2 | 2 | 0 | 2 | 2 |  | 2 | 0 |
|  |  |  |  |  |  |  |  | Table 3 |  |  |  |  |  |  |  |
| ∧ | 0 | 1 | 2 |  | ∨ | 0 | 1 | 2 | → | 0 | 1 | 2 |  | *x* | ¬*x* |
| 0 | 0 | 0 | 0 |  | 0 | 0 | 2 | 2 | 0 | 2 | 2 | 2 |  | 0 | 2 |
| 1 | 0 | 2 | 2 |  | 1 | 2 | 2 | 2 | 1 | 0 | 2 | 2 |  | 1 | 1 |
| 2 | 0 | 2 | 2 |  | 2 | 2 | 2 | 2 | 2 | 0 | 2 | 2 |  | 2 | 0 |
|  |  |  |  |  |  |  |  | Table 4 |  |  |  |  |  |  |  |
| ∧ | 0 | 1 | 2 |  | ∨ | 0 | 1 | 2 | → | 0 | 1 | 2 |  | *x* | ¬*x* |
| 0 | 0 | 0 | 0 |  | 0 | 0 | 1 | 2 | 0 | 2 | 2 | 2 |  | 0 | 2 |
| 1 | 0 | 2 | 2 |  | 1 | 1 | 1 | 1 | 1 | 0 | 2 | 2 |  | 1 | 2 |
| 2 | 0 | 2 | 2 |  | 2 | 2 | 2 | 2 | 2 | 0 | 2 | 2 |  | 2 | 0 |
|  |  |  |  |  |  |  |  | Table 5 |  |  |  |  |  |  |  |

* 1. *Extending C*1 *with De Morgan axioms.*

We start by extending *C*1 with some of the De Morgan laws that are consistent with paraconsistency in the context of the axioms that define *C*1.

Theorem 3.6 *It is possible to add to C*1 *any of the two following axioms or both of them at the same time to obtain three different paraconsistent extensions of C*1*.*

*D1:*¬(*A* ∨ *B*) → (¬*A* ∧ ¬*B*)*,*

*D2:*(¬*A* ∧ ¬*B*) → ¬(*A* ∨ *B*)*.*

Proof. We split the proof in four steps:

1. Table [3](#_bookmark9) shows a three-valued logic with 1 and 2 as designated values, where the axioms of *C*1 are tautologies and none of the axioms D1 and D2 is a tautology.
2. Table [4](#_bookmark10) shows a three-valued logic with 1 and 2 as designated values, where the axioms of *C*1 are tautologies and the axiom D1 is a tautology but axiom D2 is not a tautology.
3. Table [5](#_bookmark11) shows a three-valued logic with 1 and 2 as designated values, where the axioms of *C*1 are tautologies and the axiom D2 is also a tautology but axiom D1 is not a tautology.
4. Table [6](#_bookmark12) shows a three-valued logic with 1 and 2 as designated values, where

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| ∧ | 0 | 1 | 2 |  | ∨ | 0 | 1 | 2 | → | 0 | 1 | 2 |  | *x* | ¬*x* |
| 0 | 0 | 0 | 0 |  | 0 | 0 | 1 | 2 | 0 | 2 | 2 | 2 |  | 0 | 2 |
| 1 | 0 | 2 | 2 |  | 1 | 1 | 1 | 2 | 1 | 0 | 2 | 2 |  | 1 | 2 |
| 2 | 0 | 2 | 2 |  | 2 | 2 | 2 | 2 | 2 | 0 | 2 | 2 |  | 2 | 0 |
|  |  |  |  |  |  |  |  | Table 6 |  |  |  |  |  |  |  |
| ∧ | 0 | 1 | 2 |  | ∨ | 0 | 1 | 2 | → | 0 | 1 | 2 |  | *x* | ¬*x* |
| 0 | 0 | 0 | 0 |  | 0 | 0 | 2 | 2 | 0 | 0 | 2 | 2 |  | 0 | 2 |
| 1 | 0 | 2 | 2 |  | 1 | 2 | 2 | 2 | 1 | 2 | 2 | 2 |  | 1 | 2 |
| 2 | 0 | 2 | 2 |  | 2 | 2 | 2 | 2 | 2 | 0 | 2 | 2 |  | 2 | 0 |
|  |  |  |  |  |  |  |  | Table 7 |  |  |  |  |  |  |  |

the axioms of *C*1 are tautologies and the axioms D1 and D2 are tautologies.

2

* 1. *Other ways of extending C*1

Now we search for more paraconsistent extensions of *C*1 by adding as axioms to *C*1 some formulas that express interesting properties and that are valid in a more gen- eral form in classical logic. These formulas are A1:¬¬*A* → (¬*A* → *B*), A2:¬¬(*A* → *B*) → (¬¬*A* → ¬¬*B*) and A3:¬¬(*A* ∧ *B*) ↔ (¬¬*A* ∧¬¬*B*). In particular A1 (week explosion principle) is equivalent to the formula (¬*A* → ¬*B*) ↔ (¬¬*B* → ¬¬*A*) (weak contrapositive) in any paraconsistent logic that extends *Cω*, as is the case of

*Z*, *G*j and *P* − *four* [[19](#_bookmark44),[2](#_bookmark23)].

3

Now we add axioms A1 and A2 to *C*1.

Theorem 3.7 *C*1 *can be extended by adding formula A1 or formula A2 or both as axioms to obtain three different paraconsistent extensions of C*1*.*

Proof.

We show three truth tables for the connectives of three-valued logics with 1 and 2 as designated values.

1. In the first one, table [7](#_bookmark13), all axioms of *C*1 as well as A1 are tautologies, but A2

is not a tautology.

1. In the second one, table [8](#_bookmark14), all axioms of *C*1 as well as A2 are tautologies, but

A1 is not a tautology.

1. In the third one, table [9](#_bookmark15), all the axioms for *C*1 as well as A1 and A2 are tautologies.

This shows that adding axiom A1 or axiom A2 or both to *C*1, we obtain three different paraconsitent logics stronger than *C*1. 2 2

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| ∧ | 0 | 1 | 2 |  | ∨ | 0 | 1 | 2 | → | 0 | 1 | 2 |  | *x* | ¬*x* |
| 0 | 0 | 0 | 0 |  | 0 | 0 | 1 | 2 | 0 | 1 | 1 | 2 |  | 0 | 1 |
| 1 | 0 | 1 | 1 |  | 1 | 2 | 1 | 1 | 1 | 0 | 1 | 2 |  | 1 | 0 |
| 2 | 0 | 1 | 1 |  | 2 | 2 | 2 | 2 | 2 | 0 | 2 | 2 |  | 2 | 2 |
|  |  |  |  |  |  |  |  | Table 8 |  |  |  |  |  |  |  |
| ∧ | 0 | 1 | 2 |  | ∨ | 0 | 1 | 2 | → | 0 | 1 | 2 |  | *x* | ¬*x* |
| 0 | 0 | 0 | 0 |  | 0 | 0 | 2 | 2 | 0 | 2 | 2 | 2 |  | 0 | 2 |
| 1 | 0 | 2 | 2 |  | 1 | 2 | 2 | 2 | 1 | 0 | 2 | 2 |  | 1 | 2 |
| 2 | 0 | 2 | 2 |  | 2 | 2 | 2 | 2 | 2 | 0 | 1 | 2 |  | 2 | 0 |
|  |  |  |  |  |  |  |  | Table 9 |  |  |  |  |  |  |  |
| ∧ | 0 | 1 | 2 |  | ∨ | 0 | 1 | 2 | → | 0 | 1 | 2 |  | *x* | ¬*x* |
| 0 | 0 | 0 | 0 |  | 0 | 0 | 2 | 2 | 0 | 2 | 2 | 2 |  | 0 | 2 |
| 1 | 0 | 2 | 2 |  | 1 | 2 | 2 | 2 | 1 | 0 | 2 | 2 |  | 1 | 1 |
| 2 | 0 | 2 | 2 |  | 2 | 2 | 2 | 2 | 2 | 0 | 2 | 2 |  | 2 | 0 |
|  |  |  |  |  |  |  |  | Table 10 |  |  |  |  |  |  |  |
| ∧ | 0 | 1 | 2 |  | ∨ | 0 | 1 | 2 | → | 0 | 1 | 2 |  | *x* | ¬*x* |
| 0 | 0 | 0 | 0 |  | 0 | 0 | 2 | 2 | 0 | 2 | 2 | 2 |  | 0 | 2 |
| 1 | 0 | 2 | 2 |  | 1 | 2 | 2 | 2 | 1 | 0 | 2 | 2 |  | 1 | 2 |
| 2 | 0 | 2 | 2 |  | 2 | 2 | 2 | 2 | 2 | 0 | 2 | 2 |  | 2 | 0 |
|  |  |  |  |  |  |  |  | Table 11 |  |  |  |  |  |  |  |

Corollary 3.8 *None of the three extensions presented in the previous theorem is*

*P*2*.*

Proof. In *P*2 the formula *A* → ¬¬*A* is a theorem which together with formula A1

leads to *A* → (¬*A* → *B*) which contradicts the property of paraconsistency. 2 2

Theorem 3.9 *C*1 *can be extended to a paraconsistent logic by adding axiom A3.*

Proof. We show two truth tables for the connectives of three-valued logics with 1 and 2 as designated values.

1. In the first one, table [10](#_bookmark16), all axioms of *C*1 and also A3 are tautologies.
2. In the second one, table [11](#_bookmark17), all axioms of *C*1 are tautologies but A3 is not.

This shows that by adding axiom A3 to *C*1, we obtain a paraconsistent extension of *C*1. 2 2

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| ∧ | 0 | 1 | 2 |  | ∨ | 0 | 1 | 2 | → | 0 | 1 | 2 |  | *x* | ¬*x* |
| 0 | 0 | 0 | 0 |  | 0 | 0 | 1 | 2 | 0 | 2 | 2 | 2 |  | 0 | 2 |
| 1 | 0 | 2 | 2 |  | 1 | 1 | 1 | 2 | 1 | 0 | 2 | 2 |  | 1 | 2 |
| 2 | 0 | 2 | 2 |  | 2 | 2 | 2 | 2 | 2 | 0 | 2 | 2 |  | 2 | 0 |
|  |  |  |  |  |  |  |  | Table 12 |  |  |  |  |  |  |  |
| ∧ | 0 | 1 | 2 |  | ∨ | 0 | 1 | 2 | → | 0 | 1 | 2 |  | *x* | ¬*x* |
| 0 | 0 | 0 | 0 |  | 0 | 0 | 1 | 2 | 0 | 1 | 1 | 1 |  | 0 | 1 |
| 1 | 0 | 1 | 1 |  | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |  | 1 | 0 |
| 2 | 0 | 1 | 1 |  | 2 | 2 | 1 | 2 | 2 | 0 | 2 | 2 |  | 2 | 2 |
|  |  |  |  |  |  |  |  | Table 13 |  |  |  |  |  |  |  |
| ∧ | 0 | 1 | 2 |  | ∨ | 0 | 1 | 2 | → | 0 | 1 | 2 |  | *x* | ¬*x* |
| 0 | 0 | 0 | 0 |  | 0 | 0 | 1 | 2 | 0 | 2 | 2 | 2 |  | 0 | 2 |
| 1 | 0 | 2 | 2 |  | 1 | 1 | 1 | 2 | 1 | 0 | 2 | 2 |  | 1 | 2 |
| 2 | 0 | 2 | 2 |  | 2 | 2 | 2 | 2 | 2 | 0 | 1 | 2 |  | 2 | 0 |
|  |  |  |  |  |  |  |  | Table 14 |  |  |  |  |  |  |  |

Theorem 3.10 *The three systems consisting of the axioms of*

* *C*1 *plus formula D1, D2 and A1,*
* *C*1 *plus formula D1, D2 and A2, and*
* *C*1 *plus formula D1, D2, A1 and A2*

*are different paraconsistent extensions of C*1*.*

Proof. We present three truth tables for the connectives of three-valued logics with 1 and 2 as designated values.

1. In the first one, table [12](#_bookmark18), all axioms of *C*1 and also D1, D2 and A1 are tautologies but A2 is not.
2. In the second one, table [13](#_bookmark19), all axioms of *C*1 and also D1, D2 and A2 are tautologies but A1 is not.
3. In the third one, table [14](#_bookmark20), all axioms of *C*1 and also D1, D2, A1 and A2 are tautologies.

This shows that we have three different paraconsistent extensions of *C*1. 2 2

Proposition 3.11 *There is no three-valued logic for which the axiomatic system consisting of the axioms of C*1 *plus D1, D2, A1, A2 and A3 is sound.*

The next results are easy to prove.

Proposition 3.12 *Other comments we can make about the extensions of C*1 *in relation to the formula A* → ¬¬*A are:*

* *The logic that results from adding the formula A* → ¬¬*A as an axiom to the system C*1*, has formulas A2 and A3 as theorems.*
* *There is not three valued logic for which*

1. *all axioms of C*1*, formula A2 and formula A3 are tautologies and*
2. *the formula A* → ¬¬*A is not a tautology.*

Figure [1](#_bookmark21) shows two ways of extending *C*1, one leads to Cive and the other one contains logics that have D2 as a theorem. We recall that D2 is not a theorem in Cive [[19](#_bookmark44)].

C1



C1+ C1+D1+D2



Cive C1+D1+D2+A1 C1+D1+D2+A2 C1+D1+D2+A3



C1+D1+D2+A1+A2

Fig. 1. Two ways of extending *C*1

Proposition 3.13 *D1, A2 and A3 are theorems in Cive, A1 is not.*

# Conclusion

It is well known that Classical logic is very useful in applications related to Artifi- cial Intelligence. In this paper we analyse different paraconsistent logics and their proximity with to Classical Logic. As we know, the substitution theorem is a prop- erty that meets Classical Logic and it is desirable that a logic that approximates Classical Logic satisfies it. In this paper we show that any paraconsistent logic that extends paraconsistent logic *C*1 does not satisfy the substitution theorem. On the other hand, we also know that the De Morgan laws are relevant properties which are valid in Classical Logic, then we shown different paraconsistent extensions of *C*1 where some of the De Morgan laws are valid.

At the end of this paper, we also present paraconsistent extensions of *C*1 obtained by adding as axioms to *C*1 some formulas that express interesting properties and that are valid in a more general form in classical logic, such as the week explosion

principle, the weak contrapositive property, etc. We consider that having different paraconsistent extensions, each of them close to classical logic, would allow Artificial Intelligence developers to select the one that according to its properties is best suitable for the desired application.

We use three valued logics to support our claims. We have also presented a diagram to show different ways of extending *C*1 to other paraconsistent logics. Be- sides the problem of discovering new paraconsistent logics, there is the problem of deciding which of them is a maximal paraconsistent logic. Exploring different para- consistent logics and solving the problem of maximality for each of them is part of our future work.

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