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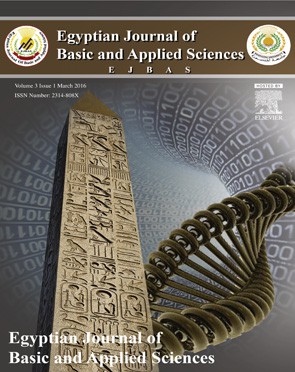
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**Full Length Article**

**On new critical point theorems without the Palais–Smale condition**



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In this paper we prove new theorems on critical point theory based on the weak Ekeland’s variational principle.

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Weak Ekeland’s variational principle

Almost critical point Critical point

# Introduction

## The weak Ekeland variational principle is an important tool in critical point theory and nonlinear analysis, and in this paper

Lemma 1 (Weak Ekeland variational principle). *Let (E, d) be a complete metric space and let * : *E*  ℝ *be a lower semicontinuous functional, bounded from below. Then for every ε > 0, there exists a point u*\*  *E such that*

## we will use this principle to establish some new results in criti- cal point theory.

** *u*\*  ** *v*  ** *d**u*\*, *v*,

*v*  *E such that v*  *u*\*.

# Preliminaries

## We need the following weak Ekeland variational principle which can be found for example in Ref. [1](#_bookmark4).

Definition 1. *We say that a functional*   *C*1 *E*, R *has a se-* quence of almost critical points if there exists a sequence *vn* *n in E* such that *φ′(vn) → 0 in E*\* *as n → ∞.*

Lemma 2 (Minimization principle). *(* [*[2]*](#_bookmark5)*) Let E be a Banach space and * : *E*  ℝ *a functional, bounded from below and Gâteaux dif-*

*ferentiable. Then, there exists a minimizing sequence* *vn* *n of almost* critical points of *φ in the sense that*

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lim** *vn*   inf** *v*

*and*

lim***vn*   0.

Theorem 2. *Let E be a Banach space and let the functional*

*n* 

*v**E n* 

*J*  *C*1 *E*, ℝ *with J′(E) a closed set in E*\**. Suppose also that J*

## A slight modification of Theorem 1.26 in [Ref. 3](#_bookmark6) (see also Cor- ollary 4 in this paper) gives the following lemma with a dilatation type condition.

Lemma 3. *Let* *E*, .  *be a Banach space and* *F*, .  *be a normed*

*admits a sequence of almost critical points. Then J has at least one* critical point in E.

Proof. *We consider the complete metric space J′(E) of E*\* *and define* the functional *φ on J′(E) by*

1 2

*space. If A is a closed set in E, f : A → F is continuous, and*

** : *J**E*  ℝ

*k*  0 : **  0,  *f* *x*  *f*  *y*  *k * *x*  *y*

*x*, *y*  *A*,

*J**u*  **  *J**u*   *J**u**E*\*.

2 1

*then f(A) is closed.*

Then *φ* is lower semicontinuous and bounded from below on *J′*(*E*). Let *ε* ∈ (0, 1). From the weak Ekeland variational prin- ciple, there exists *u*\* in *E* such that

# Main results

 *J**u*\**E*\* 

 *J**v**E*\*  **

 *J**u*\*  *J**v**E*\*,

*v*  *E*.

Theorem 1. *Let E be a reflexive Banach space, Ω be a bounded and* weakly closed set of E with J  *C*1 *E*, ℝ*, and let J′ be strongly con-* tinuous on *Ω. Suppose also that J satisfies J* *u* *E*\*  *k J**u* *E*\*

## We deduce that *u*\* is a critical point of *J*. If this is not true

then *J**u*\*  0. Let (*vn*) the sequence of almost critical point of *J*. Then we obtain that

*for all u ∈ Ω, where φ : Ω→Ω is a function such that φ(u) ≠ u for all*

*u ∈ Ω, and 0 < k < 1 is a constant . Then J has at least one critical* point in *Ω.*

 *J**u*\**E*\* 

 *J**vn* *E*\*  **

 *J**u*\*  *J**vn* 

*E*\*,

*n*  N.

Proof. *We first show J′(Ω) is closed. Let g*  *J**. There exist a se- quence gn*  *J**, such that* lim*n* *gn*  *g, and so there exist (un)*

## Because *J′*(*vn*) → 0 as *n* → *∞*, by passing to the limit, we obtain that

*⊂ Ω with* lim*n* *J**un*   *g. Since Ω is bounded and E is reflexive,* there exist *un*   *un*  *such that un* ~ *u*  *. Since J′ is strongly*

 *J**u*\**E*\*  **  *J**u*\**E*\*,

*k k*

*continuous then*

*g*  lim *J**unk*   *J**u*  *J*.

*n* 

## We consider the complete metric space *J′*(Ω), and define the functional *ψ* on *J′*(Ω) by

** : *J*  ℝ

*J**u*  **  *J**u*   *J**u**E*\*.

## Then *ψ* is lower semicontinuous and bounded from below on *J′*(Ω). Let **  1  *k*  0, 1. From the weak Ekeland varia-

1  *k*

tional principle, there exists *u*\* in Ω such that

which is a contradiction. ■

As a consequence of the last theorem, we obtain the fol- lowing corollary.

Remark 1. *The two geometric conditions in the Mountain pass theorem* suffice to get a sequence of almost critical points (see [Ref. 2](#_bookmark5)).

Corollary 1. *Let E be a Banach space, and let J*  *C*1 *E*, ℝ *satisfy* J(0) *= 0. Assume that J′(E) is a closed set in E*\* *and there exist posi-* tive numbers *ρ and α such that*

1. *J*(*u*) ≥ *α if u*  ** ,
2. *there exists e* ∈ *E such that e*  ** *and J*(*e*) < *α*.

 *J**u*\**E*\* 

 *J**v**E*\*  **

 *J**u*\*  *J**v**E*\*,

*v*   such that

*v*  *u*\*.

*Then J admits at least one critical point u. It is characterized by*

## We claim that *u*\* is a critical point of *J*. If this is not true

*J**u*  0,

*J* *u*  infmax*J* ** *t* 

**  *t*0,1

## then

*J**u*\*  0*.* Now

*where*

 *J**u*\*  1  **  *J**v* , *v*  , such that

*v*  *u*\*,

*E*\* 1  ** *E*\*

## and in particular for *v*  ** *u*\*, we have

1  **

  **  *C* 0, 1, *E* ** 0  0, ** 1  *e*.

Corollary 2. *Let E be a Banach space and let the functional*

*J*  *C*1 *E*, ℝ *satisfy*

 *J**u*\**E*\* < 1  **

 *J*** *u*\**E*\*,

*k*  0 : **  0,

 *J**u*  *J**v*  *k u*  *v*

*u*, *v*  *E*.

## (1)

i.e.,

*E*\* *E*

*J*** *u*\**E*\* > *k * *J**u*\**E*\*.

## This contradicts the hypothesis

 *J*** *u**E*\*  *k * *J**u**E*\* . ■

*Suppose also that J admits a sequence of almost critical points.*

*Then J has at least one critical point in E.*

Proof. *This is a direct consequence of Theorem 2 using Lemma 3.* ■

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Corollary 3. *Let E be a Banach space, and let J*  *C*1 *E*, ℝ *with J′(E)* a closed set in E\**. Suppose that J is bounded from below. Then J has* at least one critical point.

Proof. *The minimization principle ensures the existence of almost* critical points. The conclusion follows from Theorem 2. ■

## Ω is weakly closed, *u* ∈ Ω. Let *C* be the constant of the con- tinuous embedding of *H*1(0, 1) in *L*2(0, 1). We have

 *J**un*   *J**u**E*٨ = sup *J**un* *h*  *J**u**h*

*h H*1 1

1

0          

= sup *f t*, *un t f t*, *u t h t dt*

*h H*1 1

Corollary 4. *Let E be a reflexive Banach space, let Ω a bounded* and weakly closed subset of E with J  *C*1 *E*, R*, and let J′ be* strongly continuous on *Ω. Suppose also that J admits a sequence of* almost critical point in *Ω. Then J has at least one critical point in*

*Ω.*

1

 sup 1  *f* *t*, *un* *t*   *f* *t*, *u* *t* *dt* 2 2

 

*h H*1 1 0

1 *h*2 *t* *dt* 2

0

1

= sup  1  *f* *t*, *u* *t*   *f* *t*, *u* *t* *dt* 2 2 *h*

0

1

*n* 2

Proof. *We show J′(Ω) is closed. Indeed let g*  *J**. There exists*

*h H*1 1

1

1

*L* 0,1

*g*   *J* *such that* lim

*g*  *g, and there exists (u ) ⊂ Ω*

 *C* sup   *f* *t*, *u* *t*   *f* *t*, *u* *t* *dt* 2 2 *h*

*n n*  *n n*

*h H*1 1 0

*n H*1

*with* lim*n* *J**un*   *g. Since Ω is bounded and E is reflexive, there* 1

*exists* *u*

  *u*  *such that u*

~ *u*  *. Since J′ is strongly con-*

 *C*

1  *f* *t*, *u* *t*   *f* *t*, *u* *t* *dt* 2 2.

*nk n nk*

0 *n* 

*tinuous then*

*g*  lim

*n*

*J**unk*

  *J**u* *J*

## Let *K* be the constant of the continuous embedding of *H*1(0, 1) in *C*[0, 1], and note that

Following the same steps in the proof of Theorem 2, we obtain the result. ■

*f* *t*, *un* *t*   *f* *t*, *u* *t*   2

sup

*t*,*y*0,1*K* ,*K* 

*f* *t*, *y* ,

# Application

## We consider the functional *J* defined on *E*  *H*1 0, 1 by

*J* *u*  1 *u**t*  *f* *t*, ** *d* *dt*,

0 0

lim *f* *t*, *un* *t*   *f* *t*, *u* *t* .

*n* 

## From the Lebesgue dominated convergence theorem, we obtain

1

 

lim 1  *f* *t*, *un* *t*   *f* *t*, *u* *t* *dt* 2 2 = 0,

*n*  0

## where *f*  *C* 0, 1  ℝ, ℝ is a continuous function. Suppose that there exist a function ** : R  R and *k* ∈ ]0, 1[ such that

## and so

1 1 lim

*J**un*   *J**u**E*٨  0.

 *f*1  0 *f* *t*, ** *u* *t* *h* *t* *dt*  *k* 0 *f* *t*, *u* *t* *h* *t* *dt* , for all *u*, *h*  *H*1 0, 1.

*n*

## Finally we show *J′* satisfies

 *J*** *u**E*\*  *k * *J**u**E*\*

for all

One may take as examples of *f* and *ϕ*,

*u* ∈ Ω, where *φ* is the Nemytskii’s operator associated with *ϕ*.

Now from (*f*1) we have

*f* *t*, *u*  *q**t* *u*  *k*2 ,

** *s*  *ks*  *k*2  *k*,

*t* 0, 1, *k*  0, 1

*J*** *u* ٨  sup 1 *f* *t*, ** *u* *t* *h* *t* *dt*

*E* 

*h H*1 1 0

## where *q* is a positive function defined on [0, 1].

 *k* sup  *f* *t*, *u* *t* *h* *t* *dt*

*h H*1 1 0

1

Theorem 3. *Suppose that f satisfies (f1). Then J has at least one criti-* cal point.

Proof. *Note that J is well defined and J*  *C*1 *H*1 0, 1, R *with*

 *k * *J**u**E*٨.

## From Theorem 1, *J* has at least one critical point in Ω.

R E F E R E N C E S

*J**u*.*h*  1 *f* *t*, *u* *t* *h* *t* *dt*,

0

for all

*u*, *h*  *H*1 0, 1.

## We now show *J′* is strongly continuous on

  *B* 0, **   *H*1 0, 1.

## Let (*un*) a sequence with (*un*) ⊂ *Ω* and *un* ~ *u* (Ω is bounded in *H*1(0, 1)) and note it converges uniformly to *u* on [0, 1]. Since

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