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On the Complexity of Convex Hulls of Subsets of the Two-Dimensional Plane

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Abstract

We investigate the computational complexity of computing the convex hull of a two-dimensional set. We study this problem in the polynomial-time complexity theory of real functions based on the oracle Turing machine model. We show that the convex hull of a two-dimensional Jordan domain *S* is not necessarily recursively recognizable even if *S* is polynomial-time recognizable. On the other hand, if the boundary of a Jordan domain *S* is polynomial-time computable, then the convex hull of *S* must be *NP*-recognizable, and it is not necessarily polynomial-time recognizable if *P* =/ *NP* . We also show that the area of the convex hull of a Jordan domain *S* with a polynomial-time computable boundary can be computed in polynomial time relative to an oracle function in #*P* . On the other hand, whether the area itself is a #*P* real number depends on the open question of whether *NP* = *UP* .

*Keywords:* Convex hulls, two-dimensional set, computational complexity, polynomial time, NP.

# Introduction

The convex hull of a set *S* of the two-dimensional plane is the smallest convex set *CH*(*S*) that contains *S*. It is a fundamental concept in mathematics and in computational geometry. For polygons and sets of finite points, there are a number of efficient algorithms to compute their convex hulls (see, for instance, O’Rourke [[14](#_bookmark31)] and de Berg et al. [[7](#_bookmark23)]). In general, however, no efficient algorithms are known to work for all subsets of the two-dimensional plane. In fact, for some set *S*, its convex hull could be very complicated and defies a simple algorithm.

In this paper, we study the complexity of computing the convex hull of a given set *S* ⊆ R2. In particular, we study two problems about the convex hull *CH*(*S*) of a polynomial-time computable set *S* ⊆ R2:

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Membership Problem: For a polynomial-time computable set *S* and a given point z, determine whether z is inside *CH*(*S*).

Area

Problem: For a polynomial-time computable set *S*, compute the two-

dimensional measure of the convex hull of *S*.

There are a number of different formulations of the notion of polynomial-time computable sets in the two-dimensional plane. In this paper, we will use three notions introduced in Chou and Ko [[3](#_bookmark21)]: polynomial-time computable Jordan do- mains (i.e., sets whose boundaries are polynomial-time computable Jordan curves), polynomial-time recognizable sets, and strongly polynomial-time recognizable sets. Our main results can be summarized as follows:

1. There exists a Jordan domain *S* ⊆ R2 which is polynomial-time recognizable such that its convex hull is not even recursively recognizable.
2. If a set *S* ⊆ R2 is a Jordan domain and its boundary is polynomial-time computable, or if *S* is strongly polynomial-time recognizable, then its convex hull *CH*(*S*) is strongly nondeterministic polynomial-time recognizable.
3. If *P* /= *NP* , then there exists a Jordan domain *S* ⊆ R2 whose boundary is polynomial-time computable such that its convex hull *CH*(*S*) is not polynomial- time recognizable.
4. If a set *S* ⊆ R2 is a Jordan domain and its boundary is polynomial-time computable, or if *S* is strongly polynomial-time recognizable, then the area of its convex hull *CH*(*S*) is computable in polynomial-time with the help of an oracle function in #*P* .
5. If *F P*1 /= #*P*1, then there exists a Jordan domain *S* ⊆ R2 whose boundary is polynomial-time computable such that the area of its convex hull *CH*(*S*) is not a polynomial-time computable real number. [4](#_bookmark2)

Our basic computational model for real-valued functions and two-dimensional sets is the oracle Turing machine. For the general theory of computable analysis based on the Turing machine model, see, for instance, Pour-El and Richards [[15](#_bookmark32)] and Weihrauch [[21](#_bookmark37)]. For the theory of computational complexity of real functions based on this computational model, see Ko [[12](#_bookmark29)]. Chou and Ko [[3](#_bookmark21)] extended this theory to the study of computational complexity of two-dimensional sets. Computational complexity of problems related to two-dimensional sets has recently been studied in several directions. Rettinger and Weihrauch [[17](#_bookmark34)], Rettinger [[16](#_bookmark33)], Braverman [[1](#_bookmark19)], and Braverman and Yampolsky [[2](#_bookmark20)] studied the the computational complexity of Julia sets. Chou and Ko [[4](#_bookmark22)] studied the problem of finding paths in a two-dimensional domain. Ko and Yu [[13](#_bookmark30)] studied the problem of computing single-valued analytic branches of logarithm and square-root functions on a two-dimensional domain. All these works used Turing machines and oracle Turing machines as the basic model.

4 *FP*1 and #*P*1 are functions in *FP* and #*P* , respectively, whose inputs are strings from a singleton alphabet.

# Definitions and Notation

* 1. *Discrete complexity classes*

In this paper, we will work on both discrete and continuous objects. The basic objects in discrete complexity theory are binary strings *w* ∈ {0*,* 1}∗. We write *l*(*w*) to denote the length of a string *w* (and reserve the notation |*x*| for the absolute value of a real or complex number *x*).

The fundamental discrete complexity classes we are interested in are the class *P* of sets accepted by deterministic polynomial-time Turing machines (TMs), and the corresponding function class *FP* of functions computable by deterministic polynomial-time TMs. In addition to these classes, we are also interested, in this paper, in the following complexity classes (see, e.g., Du and Ko [[9](#_bookmark24)] for the formal definitions):

*NP* : Sets that are accepted by nondeterministic polynomial-time TMs.

*UP* : Sets that are accepted by nondeterministic polynomial-time TMs that have, on any input, at most one accepting computation.

#*P* : Functions that compute the number of accepting computations of a nonde- terministic polynomial-time TMs.

*P* #*P* : Sets that are accepted by deterministic polynomial-time oracle TMs with the help of an oracle function *f* ∈ #*P* (we also write *P* #*P* [1] for the sets for which the oracle function *f* ∈ #*P* is asked at most once during the computation).

*FP* #*P* : Functions that are computable by deterministic polynomial-time oracle TMs with the help of an oracle function *f* ∈ #*P*

The classes *NP* , *UP* and #*P* have nice characterizations in terms of class *P* . In the following, we let *A * denote the size of a finite set *A*.

Proposition 2.1 *(a) A set A* ⊆ {0*,* 1}∗ *is in NP if and only if there exist a set*

*B* ∈ *P and a polynomial function p such that, for any w* ∈ {0*,* 1}∗*,*

*w* ∈ *A* ⇐⇒ (∃*u, l*(*u*)= *p*(*l*(*w*))) ⟨*w, u*⟩ ∈ *B.*

* + 1. *A set A* ⊆ {0*,* 1}∗ *is in UP if and only if there exist a set B* ∈ *P and a polynomial function p such that, for any w* ∈ {0*,* 1}∗*,*

*w* ∈ *A* ⇐⇒ (∃*u, l*(*u*)= *p*(*l*(*w*))) ⟨*w, u*⟩ ∈ *B*

⇐⇒ (∃ a unique *u, l*(*u*)= *p*(*l*(*w*))) ⟨*w, u*⟩ ∈ *B.*

* + 1. *A function φ* : {0*,* 1}∗→N *is in* #*P if and only if there exist a set B* ∈ *P and a polynomial function p such that, for any w* ∈ {0*,* 1}∗*,*

*φ*(*w*)= {*u* ∈ {0*,* 1}∗ : *l*(*u*)= *p*(*l*(*w*))*,* ⟨*w, u*⟩ ∈ *B*}*.*

It is known that *P* ⊆ *UP* ⊆ *NP* ⊆ *P* #*P* and *FP* ⊆ #*P* ⊆ *FP* #*P* . Whether any

of these inclusive relations is proper is not known and is a major open question in complexity theory.

For any of the above function classes C, we write C1 to denote the class of functions *φ* : {0}∗→N that are in C. These classes also satisfy the relation *F P*1 ⊆ #*P*1 ⊆ *FP* #*P* , and whether any of the relations is a proper inclusion is also open.

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* 1. *Complexity of real functions and two-dimensional sets*

The basic objects in the Turing machine-based continuous computation are dyadic rationals D = {*m/*2*n* : *m* ∈ Z*,n* ∈ N}. Each dyadic rational *d* has infinitely many binary representations, with arbitrarily many trailing zeros. For each *n* ∈ N, we let D*n* denote the class of dyadic rationals which have a binary representation of at most *n* bits to the right of the binary point; that is, D*n* = {*m/*2*n* : *m* ∈ Z}.

A real number has a few basic representations. The most basic one is the *Cauchy*

*function representation*. We say a function *φ* : N→D *binary converges* to a real number *x*, or is *a Cauchy function representation of x*, if (i) for all *n* ≥ 0, *φ*(*n*) ∈ D*n*, and (ii) for all *n* ≥ 0, |*x* − *φ*(*n*)| ≤ 2−*n*. A real number *x* may have many Cauchy function representations. However, there is a unique function *φx* : N→D that binary converges to *x* and satisfies the condition *x* − 2−*n < φx*(*n*) ≤ *x* for all *n* ≥ 0. We call this function *φx* the *standard Cauchy function* for *x*. We say a real number *x* is *computable* if it has a computable Cauchy function representation. A real number *x* is *polynomial-time computable* (or, simply, *P-computable*) if it has a Cauchy function representation *φ* : {0}∗→D in *FP* . [5](#_bookmark4) We write *P* R to denote the set of all *P* - computable real numbers. Similarly, we write #*P* R (or, *P* #*P* ) to denote the set of real numbers which have a Cauchy function representation *φ* : {0}∗→D such that the function *φ*'(0*n*) = *φ*(0*n*) · 2*n* is in #*P* (or, respectively, in *FP* #*P* ). We note that the relation between *P* R and #*P* R depends on that between *F P*1 and #*P*1: *F P*1 = #*P*1 if and only if *P* R = #*P* R (see Theorem 5.32 of Ko [[12](#_bookmark29)]).

R

To compute a real-valued function *f* : R→R, we use oracle TMs as the com- putational model. We say an oracle TM *M computes* a function *f* : R→R if, for a given oracle *φ* that binary converges to a real number *x* and for a given input *n >* 0, *Mφ*(*n*) halts and outputs a dyadic rational *e* such that |*e* − *f* (*x*)| ≤ 2−*n*. We say a function *f* : R→R is *polynomial-time computable* (or, simply, *P-computable*) if there exists a polynomial-time oracle TM that computes *f* .

We write z, *Z* or ⟨*x, y*⟩, where *x, y* ∈ R, to denote a point in the two-dimensional plane R2. For any two points z1 = ⟨*x*1*, y*1⟩ and z2 = ⟨*x*2*, y*2⟩ in R2, we write |z1 −z2| to denote the distance (*x*1 − *y*1)2 + (*x*2 − *y*2)2 between them. For any point x ∈ R2 and a closed set *A* ⊆ R2, we write dist(x*, A*)= dist(*A,* x)= min{|x−y| : y ∈ *A*}.

√

The notions of computable and polynomial-time computable real functions can

be extended naturally to functions *f* : R→R2 and functions *f* : R2→R2. In par- ticular, we say a Jordan curve (simple, closed curve) Γ in R2 is *polynomial-time computable* if there exists a *P* -computable function *f* : [0*,* 1]→R2 such that the range of *f* is Γ, *f* is one-to-one on [0*,* 1) and *f* (0) = *f* (1). For any set *S* ⊆ R2, let

5 Note that the input integers *n* to *φ* are written in the form of the unary representation 0*n*.

*∂S* denote the boundary of *S*, i.e., the set of all points z ∈ R2 such that any neigh- borhood *N* (z; *ϵ*) of z contains points in *S* and points not in *S*. We say a bounded open set *S* ⊆ R2 is a *Jordan domain* if its boundary *∂S* is a Jordan curve, and say it is *P-computable* if *∂S* is a *P* -computable Jordan curve.

For any set *S* ⊆ R2, let *χS* denote the characteristic function of *S*; i.e., *χS*(x)= 1 if x ∈ *S*, and *χS*(x) = 0 otherwise. Intuitively, *S* is computable (or, polynomial- time computable) if the function *χS* is computable (or, respectively, polynomial- time computable). Since *χS* is discontinuous at the points on *∂S*, the definition based on this concept is too strict. That is, suppose that we define a set *S* to be computable if there is an oracle TM computing *χS*; then, only two trivial sets, R2 and ∅, are polynomial-time computable. Chou and Ko [[3](#_bookmark21)] considered two ways to relax the computability requirements of this concept, and defined the notions of polynomial-time approximable and polynomial-time recognizable sets.

A set *S* ⊆ R2 is called *polynomial-time recognizable* (or, simply, *P-recognizable*) if there exists a polynomial-time oracle TM *M* that, when given two oracles *φ*1*, φ*2 and an input *n >* 0 (written in its unary representation 0*n*), computes *χS*(z) whenever

⟨*φ*1*, φ*2⟩ represents a point z in R2 having a distance greater than 2−*n* from the boundary *∂S*; i.e, the error set

*EM* (*n*)= {z ∈ R2 | (∃⟨*φ*1*, φ*2⟩ representing z) *Mφ*1 *,φ*2 (*n*) /= *χS*(z)} (1) of *M* on input *n* is a subset of {z ∈ R2 | dist(z*, ∂S*) ≤ 2−*n*}.

A set *S* ⊆ R2 is called *strongly recursively recognizable*, (or, *Strongly P-*

*recognizable*) if it is recursively recognizable (or, respectively, P-recognizable) by an oracle TM *M* such that the error set *EM* (*n*) is also contained in R2 − *S* (i.e., errors only occur when the oracles representing a point outside *S*, and has distance

≤ 2−*n* from the boundary).

A set *S* ⊆ R2 is called *polynomial-time approximable* (or, simply, *P- approximable*) if there exists a polynomial-time oracle TM *M* that, when given two oracles *φ*1*, φ*2 representing a point z ∈ R, and an input 0*n*, computes *χS*(z) with possible errors such that the Lebesgue measure of the error set *EM* (*n*), defined in ([1](#_bookmark5)) above, is bounded by 2−*n*.

For any set *S* ⊆ R2, we let *CH*(*S*) be the convex hull of *S*; that is,

3 3

*CH*(*S*)= {z ∈ R2 | (∃z1*,* z2*,* z3 ∈ *S*) (∃*r*1*, r*2*, r*3 ∈ [0*,* 1]) Σ *ri* = 1*,* z = Σ *ri*z*i*}*.*

# Convex hull of a *P* -recognizable set

*i*=1

*i*=1

*P* -recognizability is the most general concept of polynomial-time computability for two-dimensional sets, but some of the important properties of a set are not re- tained in this formulation. For instance, Chou and Ko [[6](#_bookmark25)] pointed out that the distance function *δS*(z)= dist(z*, ∂S*) is not necessarily computable even if *S* is *P* - recognizable. It is not hard to see that this is also true for the notion of convex hulls. As a simple example, suppose *S* consists of four corners of a square [0*, x*] × [0*, x*]

where *x* is a noncomputable real number. Then, *S* is *P* -recognizable since all its points are on the boundary *∂S* and so a trivial oracle TM *M* that always outputs 0 computes *χS* correctly for all points away from the boundary. On the other hand, we note that *CH*(*S*) is exactly the square *R* = [0*, x*] × [0*, x*]. It is not hard to see that *R* is recursively recognizable if and only if *x* is a computable real number.

In the following, we show that, even if *S* is a Jordan domain and is *P* - recognizable, its convex hull *CH*(*S*) is not necessarily recursively recognizable.

Theorem 3.1 *There exists a P-recognizable Jordan domain S of which the convex hull CH*(*S*) *is not recursively recognizable.*

Proof. Let *K* ⊆ N be an r.e., nonrecursive set of integers. Then, there exists a TM *MK* that enumerates the integers in *K*. That is, *MK* prints, on input 0, integers on its output tape one by one such that (i) it prints only integers in *K*, and (ii) every integer in *K* is eventually printed. For *n* ∈ *K*, let *t*(*n*) be the number of moves *MK* takes to print integer *n* on input 0. Without loss of generality, we assume that *t*(*n*) ≥ 2*n* + 1.

Let *O* denote the origin ⟨0*,* 0⟩ of R2 and *C* denote the unit circle, i.e., the circle with center *O* and radius 1. For any *n >* 0, let *an* = 1*/*4 − 2−(*n*+1), *Zn* = ⟨ cos(2*πan*)*,* sin(2*πan*)⟩, and *Cn* be the chord of *C* connecting the points *Zn* and *Zn*+1.

We now define a function *f* : [0*,* 1]→R2 whose image is a Jordan curve Γ. On [1*/*4*,* 1], the image of *f* is the circle *C* on the second, third, and fourth quadrants; i.e., *f* (*t*) = ⟨ cos(2*tπ*)*,* sin(2*tπ*)⟩, for *t* ∈ [1*/*4*,* 1]. Next, for each *n >* 0, if *n* /∈ *K*, then *f* is linear on [*an, an*+1], with *f* (*an*) = *Zn* and *f* (*an*+1) = *Zn*+1; i.e., *f* maps [*an, an*+1] linearly to the chord *Cn*. If *n >* 0 and *n* ∈ *K*, then *f* maps [*an, an*+1] to the chord *Cn* with a bump in the middle, where the bump has width 2−*t*(*n*) and height

*hn* = 1 − cos(2−(*n*+2)*π*). To be more precise, let *X*'

*n*

be the middle point of the chord

−−→

*Cn*, and *Xn* the intersection point of the circle *C* and the halfline *OX*. Define *Pn*

and *Qn* to be the two points on *Cn* with distance 2−*t*(*n*)−1 from *X*'

*n*

(with *Pn* closer

to *Zn* and *Qn* closer to *Zn*+1). [6](#_bookmark7) The function *f* is piecewise linear on [*an, an*+1] with *f* (*an*) = *Zn*, *f* ((*an* + *an*+1)*/*2 − 2−*t*(*n*)−*n*−3) = *Pn*, *f* ((*an* + *an*+1)*/*2 = *Xn*, *f* ((*an* + *an*+1)*/*2+ 2−*t*(*n*)−*n*−3) = *Qn*, and *f* (*an*+1) = *Zn*+1. (Figure [1](#_bookmark8) shows the curve Γ between *Zn* and *Zn*+1.) This completes the definition of function *f* . Note that *f* is a continuous function but is not computable.

Let *S* be the interior of the Jordan curve Γ. We claim that *S* is *P* -recognizable. First, it is easy to see that the set *S*0 that is enclosed by the curve *f* [1*/*4*,* 1] plus all chords *Cn*, for *n >* 0, is *P* -recognizable. Next let *Bk* be the area enclosed by the chord *Cn* and the circle *C* from *Zn* to *Zn*+1, and let *Sk* = *S* ∩ *Bk*. If *k* /∈ *K*, then *Sk* = ∅; and if *k* ∈ *K*, then *Sk* is a small bump of width 2−*t*(*n*) and height *hn*. Now, consider the following algorithm for the membership problem of *S*:

Oracles: ⟨*φ*1*, φ*2⟩ representing a point z ∈ R2.

Input: *n >* 0.

6 Note that *t*(*n*) ≥ 2*n* + 1 implies that 2−*t*(*n*)−1 ≤ 2−2*n*−2, and the distance between *Zn* and *X*' is

sin(2−*n*−2*π*) *>* 2−*n*−2. Therefore, *Pn* and *Qn* are between *Zn* and *Zn*+1.

*n*

*Z n* +1



to *O*

*Q*

*n*

**. .**

*Yn*

**.**

*Pn*

**.** *Xn*

to *O*

2  *an*

*Zn*

to *O*

Fig. 1. The curve *∂S* between *Zn* and *Zn*+1

1. Ask the oracles to get a dyadic point w ∈ D2

*n*+1

with |w − z| *<* 2−(*n*+1).

1. If w ∈ *S*0, then answer yes;
2. Else if w /∈ *Bk* for any *k* ≤ *n*, then answer no;
3. Else if w ∈ *Bk* for some *k* ≤ *n*, then simulate TM *MK* for *n* moves, and answer yes if and only if *MK* prints *k* within *n* moves and w ∈ *Sk*.

To see that the above algorithm solves the membership problem of *S* correctly, assume that z is a point in R2 with dist(z*,* Γ) *>* 2−*n*. Then, if z ∈ *S*0 or if z lies outside *C*, then the answer given by the algorithm is correct. Next, suppose z ∈ *Bk* for some *k >* 0. If *k* /∈ *K*, or if *k* ∈ *K* and *t*(*k*) ≤ *n*, then again the answer is correct. Finally, suppose z ∈ *Bk* with *k* ∈ *K* and *t*(*k*) *> n*. Then, *Sk* is a small bump of width 2−*t*(*k*) *<* 2−*n*, and so all points in *Sk* have distance at most 2−(*n*+1) from the boundary Γ. Thus, the answer no is correct for z if it has distance *>* 2−*n* from Γ.

Next, we verify that this algorithm works in polynomial time. It is apparent that steps (1)—(3) and the first half of step (4) can be done in time polynomial in

*n*. For the second half of step (4), we note that if *t*(*k*) *> n*, then we can simulate *Mk* for *n* steps and answer no. Otherwise, if *t*(*k*) ≤ *n*, then we can calculate *t*(*k*) in *O*(*n*) moves, and compute points *Xn, Pn, Qn* correctly within error 2−(*n*+1) in time polynomial in *n*. From these points, we can then determine whether w ∈ *Sk* if w has distance *>* 2−(*n*+1) from the line segments *PnXn*, *XnQn*. This completes the proof that *S* is *P* -recognizable.

Now, let us consider the convex hull *CH*(*S*) of set *S*. For each *n >* 0, let *Tn* = *CH*(*S*) ∩ *Bn*. Note that the curve Γ lies completely within *C* and it includes all points *Zn*. Therefore, *Tn* depends only on the curve Γ between *Zn* and *Zn*+1.

That is, for *n* /∈ *K*, *Tn* = ∅; and for *n* ∈ *K*, *Tn* is equal to Δ*ZnXnZn*+1, the triangle with the vertices *Zn*, *Xn* and *Zn*+1. Now, suppose that *CH*(*S*) is recursively recognizable. Then, we can determine whether *n* ∈ *K* as follows:

*n*

Let *Yn* be the middle point in *X*' *Xn*, and determine whether *Yn* is inside *CH*(*S*) with error ≤ 2−2*n*−6. Answer *n* ∈ *K* if and only if *Yn* ∈ *CH*(*S*).

Note that *hn* = 1 − cos(2−(*n*+2)*π*) ≥ 2−2*n*−4, and the length of *Cn* is 2 sin(2−(*n*+2)*π*) ≥ 2−*n*−2. [7](#_bookmark10) Now, it is not hard to see that the distance between *Yn* and the boundary of *CH*(*S*) is greater than *hn/*4, no matter whether *n* ∈ *K* (or, equivalently, whether *Yn* ∈ *CH*(*S*)). Thus, the above algorithm determines whether *n* ∈ *K* correctly. This is a contradiction to the assumption that *K* is not a recursive set. HH

# Convex hull of a *P* -computable Jordan domain

In this section, we consider the complexity of convex hulls of *P* -computable Jordan domains. In order to characterize the complexity of convex hulls, we need to extend the notion of *P* -recognizable sets to *NP* -recognizable sets.

Definition 4.1 (a) A set *T* ⊆ R2 is *NP-recognizable* if there exists a polynomial- time nondeterministic oracle TM *M* such that, on oracles ⟨*φ*1*, φ*2⟩ representing a point z ∈ R2, and on input *n >* 0,

1. For z ∈ *T* with dist(z*, ∂T* ) *>* 2−*n*, *Mφ*1*,φ*2 (*n*) contains at least one accepting path, and
2. For z /∈ *T* with dist(z*, ∂T* ) *>* 2−*n*, *Mφ*1*,φ*2 (*n*) has no accepting paths.
3. A set *T* ⊆ R2 is *strongly NP-recognizable* if it is *NP* -recognizable and the nondeterministic oracle TM *M* also satisfies the following stronger condition

(i') For all z ∈ *T* , *Mφ*1*,φ*2 (*n*) contains at least one accepting path.

Theorem 4.2 *Assume that S* ⊆ [0*,* 1]2 *is a Jordan domain whose boundary ∂S is*

*P-computable. Then, its convex hull CH*(*S*) *is strongly NP-recognizable.*

Proof. Let *Scl* denote the closure of *S*; i.e., *Scl* = *S* ∪ *∂S*. We note that, as *S* is a Jordan domain, *CH*(*Scl*) = *CH*(*S*) ∪ *CH*(*S*)*cl*. Since the notion of *P* - and *NP* -recognizable sets allows the machine to have errors near the boundary of the set, *CH*(*S*) and *CH*(*Scl*) have the same complexity as far as we are only concerned with these complexity notions. So, in the following, we will work directly with the convex hull *CH*(*Scl*) of the closed set *Scl*.

We note that a point z belongs to *CH*(*Scl*) if and only if there exist three points on the boundary *∂S* such that z lies in the triangle *D* formed by these three points. The following algorithm for the membership problem of *CH*(*S*) is based on this idea.

7 By the Taylor expansion of the functions cos and sin, we know that for small *t*, 1 − cos *t* ≥ *t*2*/*2 − *t*4*/*24 ≥

*t*2*/*4, and 2 sin *t* ≥ 2(*t* − *t*3*/*6) ≥ *t*.

Assume that the function *f* : [0*,* 1]→R2 represents the boundary *∂S*, and that *f*

is computable in time *p*(*n*) for some polynomial *p*.

Oracles: ⟨*φ*1*, φ*2⟩ representing a point z ∈ R2.

Input: *n >* 0.

* 1. Ask oracles ⟨*φ*1*, φ*2⟩ to get a dyadic point w ∈ D*n*+32 such that |w−z|≤

2−(*n*+2).

* 1. Nondeterministically guess three dyadic numbers *d*1*, d*2*, d*3 ∈ D*p*(*n*+3).
  2. Compute three dyadic points x1*,* x2*,* x3 ∈ D*n*+42 such that |x*i* −*f* (*di*)| ≤

2−(*n*+3) for *i* = 1*,* 2*,* 3.

* 1. Let *D* be the triangle whose three vertices are x1*,* x2 and x3. Accept z

if w is inside *D* or has distance ≤ 2−(*n*+1) from the boundary *∂D* of *D*.

It is clear that the above algorithm works in polynomial time. To see that the above algorithm strongly recognizes *CH*(*Scl*), we first assume that z ∈ *CH*(*Scl*). Then, there must be three numbers *t*1*, t*2*, t*3 ∈ [0*,* 1] such that z lies in the triangle *D*0 formed by three vertices *f* (*t*1)*,f* (*t*2) and *f* (*t*3).

Suppose that, for each *i* = 1*,* 2*,* 3, we have a dyadic number *di* ∈ D*p*(*n*+4) and

dyadic point x*i* ∈ D*n*+42 such that |*di* − *ti*| ≤ 2−*p*(*n*+3), and |x*i* − *f* (*di*)| ≤ 2−(*n*+3). Then, |x*i* − *f* (*ti*)| ≤ 2−(*n*+2). Let *D* be the triangle with x1*,* x2*,* x3 as the three vertices. Then, the Hausdorff distance between *D*0 and *D* is ≤ 2−(*n*+2). Therefore, z either lies inside *D* or has distance ≤ 2−(*n*+2) from *∂Q*. It follows that w either lies inside *D* or has distance ≤ 2−(*n*+1) from *∂Q*. Therefore, the computation path of the algorithm that guesses the numbers *d*1*, d*2*, d*3 will accept z.

Conversely, assume that the above algorithm accepts z, with the guesses *d*1*, d*2*, d*3 ∈ D*p*(*n*+3). Then, the algorithm found a triangle *D* such that w is ei- ther inside *D* or has distance ≤ 2−(*n*+1) from *∂D*. Let *D*1 be the triangle with the three vertices *f* (*d*1)*,f* (*d*2) and *f* (*d*3). Then, the Hausdorff distance between *D* and *D*1 is ≤ 2−(*n*+3). It follows that w is either inside *D*1 or within distance 2−(*n*+1) + 2−(*n*+3) from *∂D*1. Since |w − z| ≤ 2−(*n*+2), and since *D*1 ⊆ *CH*(*Scl*), the point z is either inside *CH*(*Scl*) or within distance 2−*n* from the boundary of *CH*(*Scl*). This shows that the acceptance of the algorithm is correct. HH

Corollary 4.3 *Assume that S* ⊆ [0*,* 1]2 *is strongly P-recognizable. Then, its convex hull CH*(*S*) *is strongly NP-recognizable.*

Proof. Assume that an oracle TM *M* strongly *P* -recognizes *S* in time *p*(*n*). We modify the algorithm of Theorem [4.2](#_bookmark9) by replacing steps (2) and (3) with

(2') Guess three points x1*,* x2*,* x3 ∈ D2 , and verify that *M* x*i* (*n* + 3) = 1 for

*n*+3

*i* = 1*,* 2*,* 3,

where *M* x*i* (*n*) denotes the computation of *M* on input *n* with the standard Cauchy functions of x*i* as the oracles. Then, this new nondeterministic oracle TM strongly accepts *CH*(*Sc*). HH

Next, we show that the result of strong *NP* -recognizability of the convex hulls

cannot be improved to *P* -recognizability, unless *P* = *NP* .

Lemma 4.4 *For any set A* ∈ *NP, there exist a P-computable Jordan domain S,a P-computable (discrete) function g* : {0*,* 1}∗→D*, and a polynomial function q, such that, for any w* ∈ {0*,* 1}∗*,*

1. *The distance between g*(*w*) *and the boundary of CH*(*S*) *is at least* 2−*q*(*l*(*w*))*, and*
2. *w* ∈ *A if and only if g*(*w*) ∈ *CH*(*S*)*.*

Proof. Let *A* ∈ *NP* . From Proposition [2.1](#_bookmark3)(a), there exist a polynomial function *p*

and a set *B* ∈ *P* such that, for all *w* ∈ {0*,* 1}∗,

*w* ∈ *A* ⇐⇒ (∃*u, l*(*u*)= *p*(*l*(*w*)))⟨*w, u*⟩ ∈ *B.*

For any *w* ∈ {0*,* 1}∗, we let *iw* be the integer between 0 and 2*l*(*w*) − 1 whose *l*(*w*)- bit binary representation (with possible leading zeroes) is equal to *w*. Also let *w*' denote the successor of *w* in the lexicographic ordering. Now, suppose *l*(*w*)= *n >* 0, we define a dyadic rational number in [0*,* 1*/*4]: *xw* = (1 − 2−(*n*−1) + *iw* · 2−2*n*)*/*4, and an interval: *Iw* = [*xw, xw*' ]. Note that *Iw* has length 2−2*l*(*w*)−2.

Next, for each *u* ∈ {0*,* 1}*p*(*n*), we define two dyadic rationals and two subintervals of *Iw* as follows:

*xw,u* = *xw* + 2−2*n*−4 + *iu* · 2−*p*(*n*)−2*n*−4*,*

'

*x*

*w,u*

= *xw* + 2−2*n*−3 + *iu* · 2−*p*(*n*)−2*n*−4 = *xw,u* + 2−2*n*−4*,*

*Iw,u* = [*xw,u, xw,u* + 2−*p*(*n*)−2*n*−4]*,*

'

*I*

*w,u*

'

*w,u*

= [*x*

'

*w,u*

*, x*

+ 2−*p*(*n*)−2*n*−4]*.*

Now, we describe the boundary *∂S* of the desired Jordan domain *S*. Let *O* be the origin, and *C* the unit circle with center *O* and radius 1. For each *w* ∈ {0*,* 1}∗ of length *n*, let *Zw* = ⟨ cos(2*πxw*)*,* sin(2*πxw*)⟩, and *Cw* the chord connecting *Zw* and *Zw*' . Then, length of *Cw* is equal to 2 sin(2−2*n*−2*π*). We denote it by *leng*(*Cw*). Let *Xw* be the middle point on the arc of *C* between *Zw* and *Zw*' , and *hn* be the distance between *Xw* and the chord *Cw*; that is, *hn* = 1 − cos(2−2*n*−2*π*). Let *Bw* denote the area between the chord *Cw* and the arc of *C* from *Zw* through *Xw* to *Zw*' .

We now divide each chord *Cw* into four line segments of equal length, and further divide each of the two middle segments into 2*p*(*n*) subsegments, each corresponding

to a string *u* ∈ {0*,* 1}*p*(*n*). That is, let *Vw*, *V* ' , and *V* ''

be the points on *Cw* of

*w w*

distance (1*/*4)*leng*(*Cw* ), (1*/*2)*leng*(*Cw* ), and (3*/*4)*leng*(*Cw* ) from *Zw*, respectively.

Also let *Pw,u* be the point on *Cw* of distance (*iu* · 2−*p*(*n*)−2 · *leng*(*Cw*)) from *Vw*, and

'

*P*

*w*

*w,u*

the point on *Cw* of distance (*iu* · 2−*p*(*n*)−2 · *leng*(*Cw*)) from *V* ' . Finally, let *Qw,u*

be the point in *Bw* that is of equal distance from *Pw,u* and *Pw,u*' and has distance

*hn/*2 from the chord *Cw*, and *Q*'

*w,u*

the point in *Bw* that is of equal distance from

'

*P*

*w,u*

*w,u*

and *P* '

' and has distance *hn/*2 from the chord *Cw* (see Figure [2](#_bookmark12)).

to *O*



*Z w*

*Vw* **.**

**.***Qw,u*

*P*

*w,u*

**.**

*Yw***.**

**.***Q*

**.**

*w,u*

*Xw*

*Vw* **. .**

*Vw*

**.**

to *O*

*Z w*

2  *xw*

*Pw,u*

to *O*

Fig. 2. The curve *∂S* around *Cw*

Now, we are ready to define the function *f* : [0*,* 1]→R2 that computes the bound- ary *∂S* of the desired Jordan domain *S*. First, *f* maps [1*/*4*,* 1] to the unit circle *C* on the second, third, and fourth quadrants; i.e., *f* (*t*) = ⟨ cos(2*tπ*)*,* sin(2*tπ*)⟩, for *t* ∈ [1*/*4*,* 1]. Next, on each interval *Iw* = [*xw, xw*' ], *f* maps [*xw, xw* + 2−2*n*−4] lin- early to the line segment *ZwVw*, and maps [*xw* +3 · 2−2*n*−4*, xw*' ] linearly to the line segment *V* ''*Zw*' . For each *u* ∈ {0*,* 1}*p*(*n*), if ⟨*w, u*⟩ /∈ *B*, then *f* maps *Iw,u* linearly to

*w*

*w,u*

the line segment *Pw,uPw,u*', and maps *I*'

*w,u*

linearly to the line segment *P* '

*w,u*' .

If ⟨*w, u*⟩ ∈ *B*, then *f* maps *Iw,u* piecewise linearly to two line segments: *Pw,uQw,u*

*P*

'

and *Qw,uPw,u*' , and maps *I*'

*w,u*

*w,u*

piecewise linearly to two line segments *P* '

'

*w,u*

*Q*

and

'

*Q*

*w,u*

'

*w,u*

*P*

' . This completes the definition of *f* . Finally, we let *g*(*w*) be the point *Yw*

between *O* and *Xw* that has distance 3 · *hn/*4 from *Xw*. Figure [2](#_bookmark12) shows the curve

*∂S* in the area *Bw*.

It is not hard to see that the function *f* and *g* are polynomial-time computable.

We omit the details of the proof.

We now check that the domain *S* satisfies the required conditions. As we argued in the proof of Theorem [3.1](#_bookmark6), our design of the curve *∂S* guarantees that the part of the convex hull *CH*(*S*) within *Bw* depends only on the curve *∂S* between *Zw* and *Zw*' . More precisely, if *w* /∈ *A*, then *CH*(*S*) ∩ *Bw* = ∅, and *Yw* /∈ *CH*(*S*). On the other hand, if *w* ∈ *A*, then *S* ∩ *Bw* contains at least two bumps which lie to the two sides of *Yw*, and so *Yw* ∈ *CH*(*S*). Furthermore, we claim that, no matter whether *Yw* ∈ *CH*(*S*), the distance between *Yw* and the boundary of *CH*(*S*) is greater than 2−*p*(*n*)−4*n*−5.

For the case of *Yw* /∈ *CH*(*S*), we know that the chord *Cw* is part of the boundary

of *CH*(*S*), and dist(*Yw, Cw*) = *hn/*4. For the case of *Yw* ∈ *CH*(*S*), let us assume

that *∂S* passes through two points *Qw,u* and *Q*' . Then, the line segment *Qw,uQ*'

*w,u*

*w,u*

forms part of the boundary of the convex hull *CH*(*S*), and *Yw* has distance *hn/*4

from this boundary. In addition, we know that both *Qw,u* and *Q*'

*w,u*

have distance at

least (2−*p*(*n*)−3 · *leng*(*Cw*)) away from the line *OXw*. It implies that *Yw* has distance at least (2−*p*(*n*)−3 · *leng*(*Cw*)) from other parts of the boundary of *CH*(*S*). That is, no matter whether *Yw* ∈ *CH*(*S*), dist(*Yw, ∂S*) ≥ min{*hn/*4*,* 2−*p*(*n*)−3 · *leng*(*Cw*)}.

Note that *hn* = 1 — cos(2−2*n*−2*π*) ≥ 2−4*n*−3, and *leng*(*Cw*) = 2 sin(2−2*n*−2*π*) ≥ 2−2*n*−2. Therefore, dist(*Yw, ∂S*) ≥ 2−*p*(*n*)−4*n*−5. This completes the proof of the claim. The proof of the lemma is also complete by setting *q*(*n*) = *p*(*n*)+ 4*n* + 5. HH

Theorem 4.5 *Assume that P* /= *NP. Then, there exists a Jordan domain S* ⊆ R2 *whose boundary ∂S is P-computable but whose convex hull CH*(*S*) *is not P- recognizable.*

Proof. Assume that the convex hull *CH*(*S*) of the set *S* constructed in Lemma [4.4](#_bookmark11) is *P* -recognizable. Then, we can determine whether *w* ∈ *A* by asking whether *g*(*w*) is in *CH*(*S*), with error bound *<* 2−*q*(*n*). HH

Corollary 4.6 *Assume that P* /= *NP. Then, there exists a Jordan domain S* ⊆ R2

*which is strongly P-recognizable but whose convex hull CH*(*S*) *is not P-recognizable.*

# Areas of Convex Hulls

In this section, we consider the complexity of computing the area of the convex hull *CH*(*S*) of a *P*-computable Jordan domain *S*. We first recall the results about the complexity of computing the area of a set *T* in the two-dimensional plane.

Proposition 5.1 *(a) If T* ⊆ [0*,* 1]2 *is P-approximable, then area of T is a real number in* #*P* R*.*

1. *If T* ⊆ [0*,* 1]2 *is a P-recognizable Jordan domain with a rectiﬁable boundary, then area of T is in* #*P* R*.*
2. *If F P*1 /= #*P*1*, then there exists a convex set T* ⊆ [0*,* 1]2 *that is P- approximable but its area is not in P* R*.*

*Remarks*. (1) Friedman [[10](#_bookmark26)] proved that the integral ∫ 1 *f* of a *P* -computable function *f* : [0*,* 1]→R is a real number in #*P*R. Parts (a) and (b) of Proposition [5.1](#_bookmark13) are due to Chou and Ko [[3](#_bookmark21)], in which the result of [[10](#_bookmark26)] was extended to the measure of two-dimensional *P* -approximable and *P* -recognizable sets.

0

∫

(2) Friedman [[10](#_bookmark26)] also showed that, if *FP* /= #*P* , then the integral 1 *f* of some *P* -computable function *f* : [0*,* 1]→R is not in *P* R. Du and Ko [[8](#_bookmark27)] and Chou and Ko [[3](#_bookmark21)] extended this result to two-dimensional, *P* -approximable, convex sets.

0

We note that a convex Jordan domain *T* must have a rectifiable boundary. There- fore, if the convex hull *CH*(*S*) of a Jordan domain is *P* -recognizable, then its area is a real number in #*P* R. This observation can be easily extended to *NP* -recognizable

convex hulls. We first need to extend the notion of #*P* -computable real numbers to #*NP* -computable real numbers.

Definition 5.2 We define the class #*NP* (or, #· *NP* ) [8](#_bookmark14) to be the class of functions *φ* : {0*,* 1}∗→N with the following property: There exist a set *B* ∈ *NP* and a polynomial function *p* such that, for any *w* ∈ {0*,* 1}∗,

*φ*(*w*)= {*u* ∈ {0*,* 1}∗ : *l*(*u*)= *p*(*l*(*w*))*,* ⟨*w, u*⟩ ∈ *B*}*.*

We let #*NP* R denote the class of real numbers *x* which have a Cauchy function representation *φ* : {0}∗→D such that the function *φ*'(0*n*) = *φ*(*n*) · 2*n* is a function in #*NP* .

Theorem 5.3 *Assume that S is a P-computable Jordan domain. Then, the area of CH*(*S*) *is a real number in* #*NP* R*.*

Proof. Without loss of generality, assume that *S* ⊆ [0*,* 1]2. Also assume that the boundary of *CH*(*S*) has length bounded by *a*. Assume that *M* is a nondetermin- istic polynomial-time oracle TM that strongly *NP* -recognizes *CH*(*S*), as given in Theorem [4.2](#_bookmark9). For any *n >* 0, let

*B* = {⟨0*n, d*1*, d*2⟩| *d*1*, d*2*,* ∈ D*n,Md*1 *,d*2 (*n*) accepts}*,*

where *Md*1 *,d*2 denotes the computation of the machine *M* using the standard Cauchy functions for *d*1 and *d*2 as the oracles. It is clear that *B* ∈ *NP* . Furthermore, the function

*φ*(0*n*)= {⟨*d*1*, d*2⟩| *d*1*, d*2 ∈ D*n,* ⟨0*n, d*1*, d*2⟩∈ *B*}

is a function in #*NP* such that the function *ψ*(0*n*)= *φ*(0*n*) · 2−2*n* converges to the area of *CH*(*S*) with error |*ψ*(0*n*) — *area*(*CH*(*S*))| ≤ *a* · 2−2*n*+2. HH

Next, we study whether *CH*(*S*) is actually a real number in #*P* R. For this question, we need to review more results about the relations between counting complexity classes in discrete complexity theory.

In his celebrated paper about counting complexity classes, Toda [[18](#_bookmark35)] showed that *PPPH* ⊆ *P* #*P* [1]; that is, if a set is computable in probabilistic polynomial time relative to a set in the polynomial-time hierarchy, then it is computable in polynomial-time with a single query to an oracle function in #*P* . [9](#_bookmark15) Toda and Watanabe [[19](#_bookmark36)] further extended this result to the function classes and showed that #*PPH* ⊆ *FP* #*P* [1]. Since #*NP* is a subclass of #*PPH*, the following result is immediate.

8 In the original paper of Valiant [[20](#_bookmark38)], the notation #*NP* was defined to mean the class #*P NP* . Hemas- paandra and Vollmer [[11](#_bookmark28)] pointed out that, in view of the characterization of #*P* of Proposition [2.1](#_bookmark3)(c), it appears to be more approapriate to define #*NP* to mean the class we defined here, and proposed, in a general framework, the notation # · *NP* for this class. Here, we use #*NP* for its simplicity.

9 Here, *PP* denotes the class of sets accepted by polynomial-time probabilistic TMs with accepting proba- bility greater than 1*/*2, and *PH* denotes the polynomial-time hierarchy, of which *NP* is the first level. For more details, see Du and Ko [[9](#_bookmark24)].

Proposition 5.4 #*NP* ⊆ *FP* #*P* [1]*.*

Combining Propositions [5.1](#_bookmark13) and [5.4](#_bookmark16), we obtain the following results about the area of *CH*(*S*).

Corollary 5.5 *Assume that S* ⊆ R2 *is a P-computable Jordan domain. Then, the area of CH*(*S*) *is a real number in P* #*P .*

R

Corollary 5.6 *The following are equivalent:*

1. *For any P-computable Jordan domain S* ⊆ R2*, the area of CH*(*S*) *is in P* R*.*
2. *F P*1 = #*P*1*.*

Corollary [5.5](#_bookmark17) leaves it open whether the area of *CH*(*S*) is actually in #*P* R. This question is clearly related to the question of whether the discrete classes #*P* and #*NP* are equal. The following nice characterization of this question is due to Hemaspaandra and Volmer [[11](#_bookmark28)].

Proposition 5.7 *NP* = *UP if and only if* #*P* = #*NP.*

Corollary 5.8 *If UP* = *NP, then area of the convex hull CH*(*S*) *of a P-computable Jordan domain S is in* #*P* R*.*

Whether the converse of the above holds remains open. We note that Proposition

[5.7](#_bookmark18) implies that if *UP* /= *NP* then there exists some function *ψ* in #*NP* that is not in #*P* . However, this function *ψ* constructed in the proof in Hemaspaandra and Vollmer [[11](#_bookmark28)] is a simple, characteristic function of a set *A* ∈ *NP* — *UP* . It seems difficult to construct a P-computable Jordan curve *S* of which the area of *CH*(*S*) is related to such a function *ψ*. It would be interesting to find out whether a stronger condition of separating some discrete classes implies that the area of *CH*(*S*) is not in #*P* R.

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