Electronic Notes in Theoretical Computer Science 120 (2005) 97–110 

[www.elsevier.com/locate/entcs](http://www.elsevier.com/locate/entcs)

Orbit Complexity and Entropy for Group Endomorphisms

(Extended Abstract)

Robert Kenny [1](#_bookmark0) ,[2](#_bookmark0)

*The University of Western Australia School of Mathematics and Statistics (M019)*

*35 Stirling Highway Crawley 6009, Western Australia*

Abstract

We consider the pointwise inequality (orbit complexity ≤ topological entropy), known in the case of computable maps and computable metric spaces, for endomorphisms of locally compact groups with an arbitrary upper semicomputable distance. Weaker conditions on the effectiveness of the product and metric neighbourhoods are observed which, in R*n*, are transferred to a norm-induced metric and used to prove a version of the inequality on locally compact abelian groups.

*Keywords:* Orbit complexity, topological entropy, effective uniform equivalence

# Introduction

Let (*X, d*) be a metric space and *T* : *X* → *X* a uniformly continuous map. The topological entropy *hd*(*T* ) ∈ [0*,* ∞] is a well-known quantity in dynam- ical systems (dependent on the uniformity induced by *d*) which in a certain sense measures how difficult initial segments of orbits are to specify. To dis- cuss the complicatedness of an individual orbit quantitatively, there are many possible indicators, including Brin-Katok local entropy with respect to an in- variant measure, and dimension-like characteristics of the orbit closure, such

1 The author has been supported by an Australian Postgraduate Award

2 Email: [rkenny@maths.uwa.edu.au](mailto:rkenny@maths.uwa.edu.au)

1571-0661 © 2005 Elsevier B.V. Open access under [CC BY-NC-ND license](http://creativecommons.org/licenses/by-nc-nd/3.0/).

doi:10.1016/j.entcs.2004.06.037

as topological entropy, Hausdorff dimension or box dimension. There is also the orbit complexity *K*sup of Brudno [[3](#_bookmark19)], which perhaps should be called ‘al- gorithmic entropy’, though it is quite coarse by the standards of algorithmic information theory. On compact spaces, complexity and entropy have various relations, including sup*x*∈*X K*sup(*x, T* ) = *hd*(*T* ). Motivated as an extension to (metrizable separable) noncompact phase spaces *X*, an equivalent defini-

tion of orbit complexity *S*¯ was proposed by Galatolo ([[6](#_bookmark22)]), with respect to a

given numbering *ν* :⊆ N → *A* of a dense subset *A* ⊆ *X*. This highlighted the use of assumptions from computable analysis to prove pointwise results about complexity and entropy. At the same time, many of Galatolo’s re- sults apply in less effective situations, and since complexity depends only on the uniform class of the metric, this leads us to hope for information about intuitively noncomputable noncompact systems also. The notable exception

is the inequality *S*¯(*x, T, ν,* (*X, d*)) ≤ *hd*(*T* ), whose general proof requires an

effective means of separating points. In this work we observe how one can proceed in the direction of this inequality when (*X, d, ν*) has an effectively separable semicomputable metric structure and *T* is an endomorphism of a compactly generated locally compact abelian group; such groups are known

to have the form R*a* × Z*b* × *F* where *F* is the maximal compact subgroup. The main item of interest is the trick used to reduce ‘calculation’ of *S*¯ for linear *S* : (R*a, d*) → (R*a, d*) and a translation-invariant metric *d* to the correspond-

ing calculation for a norm - see Section [4](#_bookmark11), where an appropriate class of *ν* is

introduced and linear algebra set in these terms. For definiteness, we should state the final result. Orbit complexity, entropy & relevant inequalities are reviewed in Section [3](#_bookmark3). Basic definitions and notation are given in Section [2](#_bookmark2).

Theorem 1.1 *Let G be a locally compact compactly generated abelian group with invariant metric d, ν* :⊆ N → *G effectively separable and semicomputable such that* + : *G* × *G* → *G is approximable. If there exists a ν-computable sequence dense in the maximal compact subgroup F, then any continuous group homomorphism T* : *G* → *G which is approximable with respect to ν, d has*

*S*¯(*x, T, ν,* (*G, d*)) ≤ *hd*(*T* ) *for all x* ∈ *G.*

Throughout we work in the framework of classical mathematics, including the Axiom of Choice, and use the theory of recursive functions ⊆ N → N in this context. Although Church’s thesis is used freely, *recursively enumerable* (r.e.), *partial recursive* (p.r.), *total recursive* (t.r.) are preferred to “computably enumerable” (c.e.), etc. to avoid confusion with computable analysis concepts.

# Notation

Throughout, *f* :⊆ *X* → *Y* denotes a partial function with domain ∅ ⊆ dom *f* ⊆ *X*; *f* (*x*) is defined (*f* (*x*) ↓) for *x* ∈ dom *f* and undefined (*f* (*x*) ↑) for *x* ∈ *X* \ dom *f* , while if *A* ⊆ *Y* then *f* −1*A* := {*x* ∈ dom *f* |*f* (*x*) ∈ *A*}. *f* : *X* → *Y* always denotes a total function. We often identify N and {0*,* 1}∗ via the lexicographical ordering *λ,* 0*,* 1*,* 00*,.. .*, where *λ* is the empty string, and use as a volume function the binary length lh2(*a*)= [log2(*a* + 1)♩. ⟨·⟩ is used to denote standard tupling functions of various arities.

Given a space *X*, for convenience we abuse the notation of Galatolo by calling a partial function *ν* :⊆ N → *X* an *interpretation* of *X*, i.e. a *numbering* of some (possibly empty) countable subset *A* := *ν*(dom *ν*). Usually, however, *ν* will at least be dense. Many-one reducibility is defined as usual by: *ν* ≤ *λ* iff there exists p.r. *f* :⊆ N → N such that dom *ν* ⊆ *f* −1 dom *λ* and *ν*|dom *ν* = (*λ* ◦ *f* )|dom *ν*. Similarly we denote *ν* ≡ *λ* ⇐⇒ (*ν* ≤ *λ*) ∧ (*λ* ≤ *ν*). For

a metric *d* on *X*, recalling a sequence (*xn*)∞

1

⊆ *X* is (strictly) normed if

*d*(*xm, xn*) *<* 2− min{*m,n*} for all *m, n* ≥ 1, define the derived interpretation

N *ν* = N*dν* by N*dν*(*p*) = lim*n*→∞(*ν* ◦ *φp*)(*n*) and dom N*dν* = {*p* ∈ N|N ⊆

*φ*−1 dom *ν* and ((*ν* ◦ *φp*)(*n*))∞ normed & convergent}, for some fixed (total)

*p* 1

acceptable numbering *φ*0*, φ*1*,...* of the p.r. functions N → N. We call any *x* ∈

*X*c = *X*c*,ν* := N*dν*(dom N*dν*) *computable*. In this way a standard numbering *ν*Rc of the computable reals Rc is obtained from a standard (total) numbering *I*D of the dyadic rationals D := {*m.*2*n*|*m, n* ∈ Z} ⊆ (R*,* |·|). We also denote left- (right-) computable reals by Rlc (Rrc).

If *ν*, *λ* are interpretations of *X*, *Y* , recall a map *f* : *ν*(dom *ν*) → *λ*(dom *λ*) is usually called (*ν, λ*)-effective if *f* ◦ *ν* ≤ *λ* (with the usual composition for partial maps). We will abuse this notation by calling instead total *T* : *X* → *Y* (*ν, λ*)*-effective* if *T* |*ν*(dom *ν*) has the above property - this should not cause confusion as almost all maps mentioned in this paper will be total. If (*X, d, ν*), (*Y, d*'*, λ*) are understood, we will call a map *f* : *X* → *Y approximable* if some

p.r. *F* :⊆ N × Q+ → N has (*a, η*) ∈ (*ν* ◦ *F* )−1*B*(*fν*(*a*); *η*) for all *a* ∈ dom *ν*, *η* ∈ Q+, and *T* : *X* → *X effectively iterable* if some p.r. *F* :⊆ N2 × Q+ → N has *F* (*a, j, η*) ∈ *ν*−1*B*(*Tjν*(*a*); *η*) for all *a* ∈ dom *ν*, *j* ∈ N *η* ∈ Q+. In this terminology, *f* is approximable iff *f* ◦ *ν* ≤ N *λ* iff it is (*ν,* N *λ*)-effective. If *f* is approximable and eff. uniformly continuous (with respect to the metrics *d*, *d*'), it will be (N *ν,* N *λ*)-effective, hence effectively iterable if (*X, d, ν*)= (*Y, d*'*, λ*). A triple (*X, d, ν*) is *effectively separable* if some r.e. *A* ⊆ dom *ν* has dense image, or *semicomputable* (s.c.) if some p.r. *f* :⊆ N3 → Q+ has (dom *ν*)2×N ⊆

dom *f* and (*f* (*a, b, k*))∞

*k*=0

strictly decreasing with limit *d*(*ν*(*a*)*, ν*(*b*)) whenever

*a, b* ∈ dom *ν*. Equivalently, we could require some p.r. *F* :⊆ N × Q+ × N → N

enumerates, up to dom *ν*, all *ν*-names in ideal balls, i.e. dom *ν* × Q+ × N ⊆

dom *F* and *F* (*a, ϵ,* N) ∩ dom *ν* = *ν*−1*B*(*ν*(*a*); *ϵ*) for all *a* ∈ dom *ν*, *ϵ* ∈ Q+. Here, let *Aa,є* := *F* (*a, ϵ,* N) \ dom *ν*. Such (*X, d, ν*) (when *ν* is dense and total) are also called “computable metric spaces” in the literature; we use this term only when distances are also approximable from below. If (*X, d, ν*) is s.c., so are (*X, d,* N *ν*) and (*X, d, λ*) for any *λ* :⊆ N → *X* with *λ* ≤ *ν*.

# Complexity, entropy & group quotients

Recall the *Kolmogorov complexity* of *n* ∈ N with respect to a p.r. function

*f* :⊆ N → N is the length of a shortest input which produces the output *n*,

*Kƒ* (*n*)= min{lh2(*m*)|*m* ∈ dom *f* ∧ *f* (*m*)= *n*}*,*

with min ∅ = +∞, and that there exists an *additively optimal D*, such that: (∀p.r. *g*)(∃*cD,g* ∈ N)(∀*n*) (*KD*(*n*) ≤ *Kg*(*n*)+ *cD,g*);

for example ∃*c*∀*n*(*KD*(*n*) ≤ lh2(*n*)+ *c*) (see e.g. [[4](#_bookmark20)] or [[10](#_bookmark25)]). Consider now a metric space (*X, d*) equipped with an interpretation *ν* :⊆ N → *X*, and for

p.r. *f* :⊆ N → N, *W* : N∗ → N injective and effective (with respect to a standard numbering, see e.g. [[13](#_bookmark26)] or [[8](#_bookmark24)]) and (*xi*)∞ ⊆ *X*, similarly define

0

*ν є,W,ƒ*

F

((*xi*)*m*−1)= min{*Kƒ* (*W* (*n*0*,... , nm*−1))|*d*(*ν*(*ni*)*, xi*) *< ϵ* for 0 ≤ *i* ≤ *m*−1}

for all *m* ≥ 1, *ϵ >* 0; again if *f* is additively optimal, then for each *W* , *W* ', *f* '

0

there exists *c* ∈ N such that

(∀*m* ≥ 1)(∀*x*0*,... , xm*−1 ∈ *X*)F *ν* ((*xi*)*m*−1) ≤ F *ν* ' ' ((*xi*)*m*−1)+ *c*

*є,W,ƒ* 0 *є,W ,ƒ* 0

(a proof of this and equivalence with the slightly different coding of points in [[6](#_bookmark22)] are omitted). In particular, fixing *f* , *W* (and *K* = *Kƒ* ) will not change

*S*¯ ((*x* )∞*,* (*X, d, ν*)) := lim sup 1 F *ν*

((*x* )*m*−1)*.*

*є i* 0

*m*→∞ *m*

*є,W,ƒ i* 0

Although taking such a growth rate destroys many interesting properties of *K* (moreover, a divisor *g*(*n*) := *n* with log2 *...* log2 = o (*g*) means we can replace *K* with other complexities [[11](#_bookmark27)]), it is well-motivated from the viewpoint of comparison with established entropies in ergodic theory: for *T* : *X* → *X* and *x* ∈ *X*, *S*¯*є*(*x, T,* (*X, d, ν*)) := *S*¯*є*((*T ix*)∞*,* (*X, d, ν*)) is an optimal upper bound

0

on the average information (bits) per iterate needed to specify long initial

segments of *ξ* = (*T ix*)∞ to within *ϵ*; compare with *hd*(*T* ) below. Noting *S*¯*є*

*i*=0

is nonincreasing in *ϵ* and taking *S*¯(*x, T,* (*X, d, ν*)) := lim*є* 0 *S*¯*є*(*x, T,* (*X, d, ν*)), similarly *S*¯(*ξ,* (*X, d, ν*)), we obtain a quantity called *orbit complexity at x* [[6](#_bookmark22)]

when *ν* is dense, dependent only on the uniformity (see Lemma [3.2](#_bookmark6)(vi)).

Several relations between complexity and various entropies are worth men- tioning, even though we shall only deal directly with *hd*(*T* ). Firstly, if (*X, d*)

is compact, for continuous *T* there always exist Borel probability measures *µ*

which are *T* -invariant and ergodic (e.g. [[12](#_bookmark28)]). It is known ([[3](#_bookmark19)])

1. *K*sup(*x, T* )= *hµ*(*T* ) *µ*-a.e. for any ergodic *T* -invariant *µ,*

where *hµ*(*T* ) is the measure-theoretic entropy (e.g. [[12](#_bookmark28)]), and moreover

1. sup *K*sup(*x, T* )= *hd*(*T* )*.*

*x*∈*X*

Thus *K*sup (which is topologically defined) simultaneously carries all the tra- ditional ergodic-theoretic information about invariant measures. If also *X* is a computable metric space, one of the results of [[6](#_bookmark22)] states *S*¯(·*, ν*) = *K*sup, so *S*¯(·*, ν*) shares these properties. More generally (by a direct proof not dis- cussed here) the lower Brin-Katok local entropy *h*−(*T, x*) bounds *S*¯(·*, ν*) below *µ*-almost everywhere; since this coincides[[2](#_bookmark18)] *µ*-a.e. with *hµ*(*T* ), and *S*¯ ≤ *K*sup ([[6](#_bookmark22), Thm 10]), we obtain from the variational principle[[12](#_bookmark28)] sup*µ hµ*(*T* )= *hd*(*T* ) that ([1](#_bookmark4)) and ([2](#_bookmark5)) hold for any dense *ν*. For an endomorphism of a locally com- pact group, Haar measure may not be ergodic or invariant, but it satisfies (∀*x*)*h*−(*T, x*)= *hd*(*T* ) [[1](#_bookmark17)], so to get analogues of ([1](#_bookmark4)) and ([2](#_bookmark5)) it suffices to prove the bound *S*¯ ≤ *hd*(*T* ). We now proceed to stating elementary properties of *S*¯, but first it is useful to have a notion of complexity independent of *T* [[5](#_bookmark21)]:

*µ*

*µ*

Definition 3.1 Given interpretation *ν* :⊆ N → *X*, the *complexity of a point*

*x* ∈ *X with respect to ν* is *C*¯(*x, ν,* (*X, d*)) = lim sup*m*→∞ 1 C*ν* (*x*) where, for

*m m*

each *m* ∈ N, C*ν* (*x*)= min{*K*(*a*)|*a* ∈ *ν*−1*Bd*(*x*; 2−*m*)}.

*m*

Lemma 3.2 *For metric space* (*X, d*)*,* (*xi*)∞ ⊆ *X, x* ∈ *X and interpretations*

0

*ν, λ* :⊆ N → *X:*

1. *If* (*xi*)∞ ⊆ *Y* ⊆ *X and ν*(dom *ν*) ⊆ *Y , then C*¯(*x*0*, ν,* (*X, d*)) = *C*¯(*x*0*, ν,* (*Y,*

0

*dУ* )) *and S*¯*є*((*xi*)∞*, ν,* (*X, d*)) = *S*¯*є*((*xi*)∞*, ν,* (*Y, dУ* )) *for all ϵ >* 0*.*

0 0

1. *If ν* ≤ *λ then C*¯(*x, λ, X*) ≤ *C*¯(*x, ν, X*) *and S*¯*є*((*xi*)∞*, λ,X*) ≤ *S*¯*є*((*xi*)∞*, ν,X*)*.*

0 0

1. *C*¯(*x, ν, X*)= *C*¯(*x,* N *ν, X*) *and S*¯*є*+*y* ((*xi*)∞*, ν,X*) ≤ *S*¯*є*((*xi*)∞*,* N *ν, X*)*.*

0 0

1. *S*¯*є* ((*x*(1)*, x*(2)))∞ *, ν*1 × *ν*2*,* (*X*1 × *X*2*, d*) ≤ Σ *S*¯*є*((*x*(*i*))∞ *, νi,* (*Xi, di*))

Σ

*j*

*j*

*j*=0

*i*

*j*

*j*=0

*and C*¯((*x*1*, x*2)*, ν*1×*ν*2*,* (*X*1×*X*2*, d*)) ≤ *i C*¯(*xi, νi,* (*Xi, di*))*, where d*((*x*1*, x*2)*,*

(*y*1*, y*2)) := max*i di*(*xi, yi*)*.*

1. *If* Ψ : *X* → *Y is uniformly continuous, for all ϵ >* 0 *there exists δ >* 0

*such that S*¯*є*((Ψ*xi*)∞*,* Ψ ◦ *ν, Y* ) ≤ *S*¯*δ* ((*xn*)∞*, ν,X*) *(hence S*¯ *is invariant*

0 0

*under uniformly equivalent metrics). If* Ψ *is α-Ho¨lder continuous (*0 *< α* ≤ 1*), C*¯(Ψ*x,* Ψ ◦ *ν, Y* ) ≤  1 *C*¯(*x, ν, X*)*.*

*α*

1. *If x* ∈ *ν*(dom *ν*) *then C*¯(*x, ν,* (*X, d*)) = 0*, and S*¯(*x, T, ν,* (*X, d*)) = 0 *if*

*T* : *X* → *X is effectively iterable.*

1. *If T* : *X* → *X is effectively iterable and Lipschitz with constant C* ≥ 1*,*

*S*¯(*x, T, ν,* (*X, d*)) ≤ log2 *C.C*¯(*x, ν,* (*X, d*)) *for all x* ∈ *X.*

1. *For any k* ≥ 1 *and* 0 ≤ *r* ≤ *k* − 1*,*

F *ν* ((*x* )*km*+*r*−1)

¯ *i*

*Sє*((*xi*)∞*, ν,X*) = lim sup *є* 0 *,*

0

*m*→∞

C*ν*

*km* + *r*

(*x*)

*C*¯(*x, ν,* (*X, d*)) = lim sup *kn*+*r ,*

*n*→∞ *kn* + *r*

and *S*¯*є*((*xik*+*j*)∞ *, ν,* (*X, d*)) ≤ *kS*¯*є*((*xi*)∞*, ν,* (*X, d*))*.*

*i*=0

0

1. *If ϵ, η >* 0 *and T is ν-approximable and uniformly continuous, then*

*S*¯ (*x, T, ν,* (*X, d*)) ≤ 1 *S*¯ (*x, Tk, ν,* (*X, d*

))*,*

*є*+*y*

*k є k,F*

*hence S*¯(*x, Tk, ν,* (*X, dk,F* )) = *kS*¯(*x, T, ν,* (*X, d*))*.*

Here *dk,F* (*x, y*) := max0≤*i*≤*k*−1 *d*(*Tix, T iy*) (*k* ≥ 1) are metrics uniformly equivalent to *d* (eff. unif. equivalent if *T* is eff. unif. continuous) which can be used also to give a definition of topological entropy (e.g. [[12](#_bookmark28)]). Namely, for compact *K* ⊆ *X* a set *Y* ⊆ *K* is a (*n, ϵ*)*-spanning subset* if the closed *dn,F* -balls

¯

*Bd*

*i*=0

*n,T*

(*y*; *ϵ*) (= ∩*n*−1*T* −*iB*(*Tiy*; *ϵ*)) (*y* ∈ *Y* ) cover *K*; obviously such *Y* can be

chosen to be of minimal (finite) cardinality *S*' (*K, T, ϵ, n*) := |*Y* |. One then

*d*

defines *hd*(*K, T* ) := lim*є* 0 lim sup*n*→∞ 1 log2 *S*' (*K, T, ϵ, n*) and the *topological*

*n d*

*entropy hd*(*T* ) := sup*K hd*(*K, T* ), depending only on the uniform structure,

so denoted *h*top(*T* ) in the compact case. Here lim sup*n*→∞ 1 log2 *S*' (*K, T, ϵ, n*)

*n d*

may be interpreted as an optimal upper bound for the information necessary

to “distinguish up to *ϵ*” a long initial segment of an arbitrary forward orbit (*T ix*)∞ starting in *K*.

0

Now we review the situation for quotients of metric groups. Namely, let *G* be a metrizable topological group, *H* a closed subgroup, choose a metric *d* on *G* invariant under the right multiplications *R**h*(*h* ∈ *H*), and denote

by *d*˜ the corresponding metric on the left-coset space *G/H*: *d*˜(*xH, yH*) =

inf*h*∈*H d*(*x, yh*). This makes the projection *π* : *G* → *G/H, x* '→ *xH* uniformly continuous, and any uniformly continuous *T* : *G* → *G* with the property

1. *T* (*xH*) ⊆ (*T x*)*H* for all *x* ∈ *G,*

projects naturally to a uniformly continuous factor *S* : *G/H* → *G/H, xH* '→

(*T x*)*H*, i.e. *S* ◦ *π* = *π* ◦ *T* . When also *G* is compact and

1. *τ* (*h*) := (*T x*)−1*T* (*xh*) ∈ *H* is independent of *x* ∈ *G,*

it is known that ([[1](#_bookmark17), Theorem 19])

*h*top(*T* )= *h*top(*S*)+ *h*top(*τ* );

in particular this applies with *τ* = *T* |*H* when *T* is a continuous group homo- morphism and *T* (*H*) ⊆ *H*. In fact, the following statement holds (the proof

is omitted); note for purposes of application that continuous homomorphisms are always uniformly continuous with respect to the left uniformity.

Proposition 3.3 *Assume G is locally compact & metrizable, d is a left- invariant metric, T (uniformly continuous with respect to d) satisﬁes (*[*3*](#_bookmark7)*) and (*[*4*](#_bookmark8)*), and H is compact. Then*

*dH*(*x, y*) := sup *d*(*xh, yh*)

*h*∈*H*

*is a uniformly equivalent metric invariant under Lg*(*g* ∈ *G*) *and Rh*(*h* ∈ *H*)*, π* : (*G, d*) → (*G/H, d*˜*H* ) *and S* : (*G/H, d*˜*H* ) → (*G/H, d*˜*H* ) *are uniformly continuous, and*

*hd*(*T* )= *hd*˜*H* (*S*)+ *h*top(*τ* )

In one direction at least, there is an analogous pointwise result for com- plexity, which we now give:

Theorem 3.4 *Suppose G is a metrizable separable topological group, H a compact subgroup, d a metric on G invariant under Lg (g* ∈ *G) and Rh (h* ∈ *H), and the group operation is ν-approximable for some dense ν* :⊆ N → *G. For any ν-approximable and uniformly continuous map T* : *G* → *G satisfying*

*(*[*3*](#_bookmark7)*) and (*[*4*](#_bookmark8)*), and any x* ∈ *G,*

*S*¯(*x, T, ν,* (*G, d*)) ≤ *S*¯(*xH, S, π* ◦ *ν,* (*G/H, d*˜)) + *h*top(*τ* )*.*

Proof (Sketch) One checks from the conditions on *T* that *τ* is a continuous endomorphism of *H*. By the use of uniform equivalence of *dn,F* , *d*˜and Lemma [3.2](#_bookmark6)(xi),(xii), it is enough to prove

˜

1. *S*¯*θ* (*x, Tn, ν,* (*G, dn,F* )) ≤ *S*¯(*xH, Sn,π* ◦ *ν,* (*G/H, d*˜*n,F* )) + log2 *S*' (*H, τ, ϵ, n*)

*d*|*H*

for large *n* and arbitrary small *θ*, *ϵ*. In line with this, we claim an *ϵ*-approximation (*nj*)*k*−1 of (*Sjn*(*xH*))*k*−1 with respect to *π* ◦ *ν*, *d*˜*n,F* and a suitable “coding”

0

*j*=0

(*γj*)*k*−1 ⊆ {1*,... ,* |*E*|} of the trajectory (*T jnx*)*k*−1 with respect to a fixed (*n, ϵ*)-

0 0

spanning subset *E* ⊆ *H* (with respect to *τ* , *d*|*H*) are enough to determine a

4*ϵ*-approximation of (*T jnx*)*k*−1 with respect to *ν* and *dn,F* . Namely, choosing *γj*

0

based on the error approximating *T jnx* by *ν*(*nj*) and using left-invariance of *d*,

we obtain a 2*ϵ*-approximation to *T jnx* in the form *ν*(*nj*)*.aγ*

*j*

where (*ai*)|*E*| = *E*,

and then use uniform equivalence of *d*, *dn,F* to check these products can be approximated uniformly in *x* = *ν*(*nj*). Since *ν*-approximations to *ai* may be made independently of *k*, in the limit the desired inequality holds.

1

# Weak semicomputability

Given (*X, d*1*, ν*) and *f* : *X* → *X* which is *ν*-approximable with respect to *d*1, if another metric *d*2 is such that id : (*X, d*1) → (*X, d*2) is effectively uniformly continuous, it is easy to check *f* is *ν*-approximable with respect to *d*2 (or note N*d*1 *ν* ≤ N*d*2 *ν* directly). In this context the following observation is interesting:

Proposition 4.1 *For any (real or complex) normed space* (*X, *·)*, for any translation-invariant metric d inducing the topology of X, the map* Id : (*X, d*) → (*X, *·  ) *is effectively uniformly continuous.*

Proof. Denote *Uk* := *Bd*(0; 2−*k*) (*k* ≥ 1) and note these have the property (2*Uk*+1 ⊆) *Uk*+1 + *Uk*+1 ⊆ *Uk*. Certainly there exists *N* such that *UN* ⊆ *B*·(0; 1) (by topological equivalence of *d* and ·), so given Q+ e *ϵ <* 1, pick *k* = [log2 1 ♩ + *N* + 1. We then have *Uk*+1 ⊆ 2−1*Uk* ⊆ *...* ⊆ 2−(*k*−*N*+1)*UN* ⊆ *B*·(0; *ϵ*), using the scaling property of ·.

*є*

Since some algorithms need to be able to recognise nearby points, we would like to obtain a form of semicomputability invariant under such changes of metric. The following is the weakest of several obvious definitions, and rather ill-formed; note that *U* := ∪*n*∈dom *ν* ∪*k*∈N *U* (*n*) may be a (dense if *ν* is) proper open subset of *X*. However in our algebraic setting it is strong enough to get somewhere, as the properties immediately following the definition show.

*k*

Definition 4.2 In a metric space (*X, d*), an interpretation *ν* :⊆ N → *X* is *weakly semicomputable* if there exist open neighbourhoods *U* (*n*) ⊆ *Bd*(*ν*(*n*); 2−*k*) of *ν*(*n*) (*n* ∈ dom *ν, k* ∈ N) and some p.r. *F* :⊆ N3 → N with dom *ν* × N2 ⊆ dom *F* such that

*k*

(∀*n* ∈ dom *ν*)(∀*k* ∈ N)(∃*An,k* ⊆ N \ dom *ν*){*F* (*n, k, l*)|*l* ∈ N} = *An,k*∪˙ *ν*−1*U* (*n*)

*k*

Proposition 4.3 *(i) For an effectively separable weakly semicomputable in- terpretation ν of a topological group X with right-invariant metric d, if the product is ν-approximable then so is the inverse, and identity e* ∈ *Xc.*

1. *Given ν* :⊆ N → *X, if homeomorphism* Ψ : (*X, d*) → (Ψ(*X*)*, d*') *is effectively uniformly continuous (in the forward direction) and* (*X, d, ν*) *is weakly semicomputable, then so is* (Ψ(*X*)*, d*'*,* Ψ ◦ *ν*)*.*
2. *If* (*X, *·) *is a real normed space, ν* :⊆ N → *X is effectively separable and weakly semicomputable, and* +: *X* × *X* → *X is ν-approximable, then* 1 Id : *X* → *X is* (*µ, µ*)*-effective, where µ* = N·*ν.*

2

1. *In metric space* (*X, d*)*, if ν, λ* :⊆ N → *X have λ* ≤ *ν and ν is weakly semicomputable, then so is λ.*

Properties (ii) & (iv) are obvious from the definitions, or at least have rather canonical methods of proof. So we give proofs only for (i) & (iii).

Proof of (i) Assume without loss of generality |*X*| *>* 1, pick *a* ∈ *ν*−1(*X* \{*e*}) arbitrarily, let *C* ⊆ dom *ν* be an r.e. set with dense image, and let p.r. *P* :⊆ N2 × Q+ → N witness approximability of the group operation. To see *e* ∈ *X*c, consider an algorithm which, on input *i* ∈ N, dovetails calculating and

searching for *P* (*n, a,* 2−*k*) in an enumeration of *A...*∪˙ *ν*−1*U* (*a*) over all *k* ≥ *i* + 1,

*i*+1

*n* ∈ *C*, halting and outputting *n* if found. By density of *ν*(*C*), there exists some

*n* ∈ *C* with *ν*(*n*) ∈ *U* (*a*) *ν*(*a*)−1, and then (∃*k* ≥ *i* + 1)(*ν* ◦ *P* )(*n, a,* 2−*k*) ∈ *U* (*a*) ,

*i*+1 *i*+1

so the algorithm must halt. Conversely, any output *n* has *n* ∈ *C* ⊆ dom *ν* ⇒

*P* (*n, a,* 2−*k*) ∈ *ν*−1*U* (*a*) , so *d*(*ν*(*n*)*, e*)= *d*(*ν*(*n*)*ν*(*a*)*, ν*(*a*)) *<* 2−*k* + 2−*i*−1 ≤ 2−*i*, and the output correctly provides an approximation to *e* within 2−*i*.

*i*+1

So let t.r. *f* : N → N have (∀*n*)(*f* (*n*) ∈ *ν*−1*B*(*e*; 2−*n*)), and consider an algorithm which on input *a, i* ∈ N dovetails computing and searching for

*c* := *P* (*b, a,* 2−*k*) in an enumeration of *A...*∪˙ *ν*−1*U* (*ƒ*(*i*+2))

*i*+2

over all *k* ≥ *i* + 2,

*b* ∈ *C*, halting and outputting *b* if a match is found. Certainly each output *b* has *b* ∈ dom *ν*, so if *a* ∈ dom *ν* then *d*(*ν*(*b*)*, ν*(*a*)−1) = *d*(*ν*(*b*)*ν*(*a*)*, e*) *< d*(*ν*(*b*)*ν*(*a*)*, ν*(*c*)) + 2−*i*−2 + 2−*i*−2 *<* 2−*i*. Conversely, some *b* ∈ *C* must have

*ν*(*b*) ∈ *U* (*ƒ*(*i*+2))*ν*(*a*)−1, and then (∃*k* ≥ *i* + 2)(*ν* ◦ *P* )(*b, a,* 2−*k*) ∈ *U* (*ƒ*(*i*+2)), so

*i*+2 *i*+2

the algorithm must halt, showing ·−1 : *X* → *X* is approximable.

Proof of (iii) (Sketch) By an obvious algorithm one can effectively find *b*

such that 2*ν*(*b*) is 2−*k*+1-close to *ν*(*a*), and the scaling property of ·  guaran- tees an approximation to 1 *ν*(*a*). On the other hand, 1 Id is an open mapping,

2 2

so we will eventually find *b* ∈ *ν*−1( 1 *U* (*a*)), and the algorithm (demonstrating

2 *k*

1Id is approximable) must halt. Since 1 Id is a bounded linear operator, the

2 2

*µ*-effectivity follows.

Remark 4.4 The converse of Proposition [4.1](#_bookmark12) is not true, since it is easy to construct an invariant metric *d* on R*d* for which Id : (R*d, *·) → (R*d, d*) is not eff. unif. continuous for any norm  ·; let *Uk* = *B*·(0; 2−*mk* ) where strictly increasing (*mk*)∞ ⊆ N is not bounded by any t.r. function, note these still have (∀*k*)(*Uk*+1 + *Uk*+1 ⊆ *Uk*) and ∩*kUk* = {0}, and apply a standard construction [[9](#_bookmark29), pg 68] to get an invariant metric *d* with *Uk*+3 ⊆ *Bd*(0; 2−*k*) ⊆ *Uk* for all *k*.

1

With this notion established, we elaborate on the linear algebra situation under the assumptions of (iii) above. Since vector addition + and negation

* : *X* → *X* are trivially eff. uniformly continuous (for any fixed bi-invariant metric), for scalar multiplication *k* : R × *X* → *X,* (*α, x*) '→ *αx* we find *k*D := *k*|D×*X* is (*I*D × *ν,* N·*ν*)-effective. The proof of the next lemma is omitted.

Lemma 4.5 *For d-dimensional real normed space* (*X, *·  )*, weakly s.c. ν such that* +*, k*D *are approximable, and linearly independent v*1*,..., vm* ∈ *Xc:*

* 1. *If V* = spanR{*v*1*,... , vm*}*,* ∅ /= *J* ⊆ {1*,... , m*} *and VJ* = spanR{*vi*|*i* ∈

*J*}*, the linear map P V* : *V* → *X,* Σ*m αivi* '→ Σ *αivi is ν-approximable*

*VJ*

*i*=1

*i*∈*J*

*X*

= *P*

*as a partial function* ⊆ *X* → *X. For m* = *d, PVJ is a* (*µ, µ*)*-*

*VJ*

*effective total linear map for µ* := N  ·*ν. For J* = {*j*}*, the functional*

*V* :⊆ *X* → R*,* Σ*m αivi* '→ *αj is* (*ν, I*D)*-approximable with respect to* |·|*,*

*p*

*i*=1

*vj*

*and a* (*µ, ν*Rc )*-effective total function if m* = *d.*

* 1. *If u* = Σ*m αivi* ∈ *Xc for some* (*αi*)*m* ⊆ R *then* (*αi*)*m* ⊆ Rc*. If m* = *d,*

*i*=1

1

1

*then Xc* = spanRc {*v*1*,... , vd*}*, and any linear S* : *X* → *X with S*(*Xc*) ⊆

*Xc has* [*S*](*v*1 *,...,vd* ) ∈ *Md*×*d*(Rc)*.*

Actually, it may be more helpful to consider the latter statement in a slightly different form. Denote by D the class of invariant metrics inducing the topology of *X*, and by A the class of interpretations *ν* :⊆ N → *X* such that some *d* ∈D makes (*X, d, ν*) eff. separable & weakly s.c. with + approximable. Without loss of generality we assume *d* is induced by a fixed norm ·.

Corollary 4.6 *For any basis v*1*,... , vd,*

*λ* :⊆ N → *X,* ⟨*q*1*,... , qd*⟩ '→ Σ *ν*Rc (*qi*)*.vi*

*d*

*i*=1

*is weakly s.c., eff. separable and makes vector addition effective. Conversely, for any ν* ∈ A*, basis v*1*,... , vd* ∈ *Xc,ν and λ as above we have* N·*ν* ≡ *λ.*

So we might say we have identified the “general form” of interpretations *ν, ν*' ∈ A of the topological group (*X,* +), up to equivalence N *ν* ≡ N *ν*'. In fact, assuming *ν* eff. separable with + approximable, a basis *v*1*,... , vd* ∈ *X*c*,ν* , and results & notation from [[8](#_bookmark24)] including the normed limit operator *N* :⊆ *X*N → *X*, we find *ν* is weakly s.c. iff the Cauchy representation *ρ* induced by *ν*, ·  lies in the minimal class of representations making effective the structure S = (*X, v*1*,... , vd,* +*,* −*,* 1 Id*,N* ) (which is not r-effectively categorical). For current purposes, though, we use only interpretations, for which we list a few more properties: *ν* ∈A⇒N · *ν* ∈ A, while a general *d* ∈D witnessing *ν* ∈A may be replaced with a norm · such that (*X, *·  *, ν*) is semicomputable (this is essentially part of the proof of Corollary [4.6](#_bookmark14), which was omitted). On the other hand, it is easy to see examples of weakly s.c. but non-s.c. interpretations:



2

Example 4.7 In (R*, *· ) = (C*,* |·|), let *v*1 = 1, *v*2 = *e*i*θ* for some *θ* ∈ (0*, π* )

2 2

with *θ* ∈ Rlc \ Rc, and *λ* :⊆ N → C*,* ⟨*q*1*, q*2⟩ '→ *ν*Rc (*q*1)*.v*1 + *ν*Rc (*q*2)*.v*2.

Since cos |(0*, π* ) is strictly decreasing and computable, cos *θ* ∈ Rrc \ Rlc, hence

2

|*v*1 − *v*2|2 = 2(1 − cos *θ*) ∈ Rlc \ Rrc, so (R2*, *· *, λ*) is not semicomputable, but is weakly s.c., eff. separable and makes addition effective.

2

Now we extend the above considerations to C*d*. Namely, extend *ν* :⊆ N → R*d* to *ν*ˆ :⊆ N → C*d* by dom *ν*ˆ = {⟨*a, b*⟩|*a, b* ∈ dom *ν*} and *ν*ˆ(⟨*a, b*⟩) = *ν*(*a*)+ i*ν*(*b*). Considering R*d* ⊆ C*d* and taking a metric *d* on R*d* to *d*ˆ(*x, y*) :=

max{*d*(঩*x,* ঩*y*)*, d*( *x,* *y*)}, if 0 ∈ *ν*(dom *ν*) we get

*C*¯(*x, ν,* (R*d, d*)) = *C*¯(*x, ν*ˆ*,* (C*d, d*ˆ))*, S*¯*є*((*xj*)∞*, ν,* (R*d, d*)) = *S*¯*є*((*xj*)∞*, ν*ˆ*,* (C*d, d*ˆ))

0 0

for any *x* ∈ R*d*, (*xj*)∞ ⊆ R*d*, *ϵ >* 0.

0

One also checks the following assumptions of approximability (with respect to *ν*, *d*) on real operations +, (unary) −, *k*D are sufficient for corresponding complex operations to be approximable with respect to *ν*ˆ, *d*ˆ: + for +; − for

i*.*Id; − and *k*D for complex scalar multiplication *k*ˆD : (D+iD)×C*d* → C*d*; and −

for conjugation conj : C*d* → C*d*. If linear *T* : R*d* → R*d* is approximable, clearly the complexification *T*ˆ : C*d* → C*d, x*1 + i*x*2 '→ (*T x*1)+ i(*T x*2) is approximable, and if *d* ∈ D witnesses *ν* ∈ A, *T* will be (*µ, µ*)-effective as before, so *T*ˆ is *ν*ˆ- approximable with respect to *d*ˆ' for any norm-induced metric *d*' on R*d*, and also (*ai,j*)*i,j* = [*T* ](*v*1 *,...,vd*) ∈ *Md*×*d*(Rc) for any basis *v*1*,... , vd* ∈ *X*c*,ν* . Considering the generalised eigenspace *Sλ* for eigenvalue *λ* (for *λ* ∈ C recall *Sλ* = R*d* ∩ (*S*ˆ*λ* ⊕ *S*ˆ¯ ), where *S*ˆ*a* := ∪*k*∈N ker(*T*ˆ − *a.*Id)*k*), the next two results (well-known in some form) show computable bases for *S*ˆ*λ* exist for all eigenvalues *λ*.

*λ*

Proposition 4.8 *If A* ∈ *Md*×*d*(Rc) *has an eigenvalue λ* ∈ Rc(i)*, the gener- alised eigenspace S*ˆ*λ of T*ˆ : C*d* → C*d,x* '→ *Ax has a basis in* Rc(i)*d, and the (real) generalised eigenspace Sλ of T* : R*d* → R*d,x* '→ *Ax has a basis in* R*d.*

c

Proposition 4.9 Rc(i) *is algebraically closed, i.e. every f* (*X*) ∈ Rc(i)[*X*]

*splits over* Rc(i)*.*

Finally, we need the following simple estimate.

Lemma 4.10 *For any norm *· *on X and dense ν* :⊆ N → *X such that* +*,*

*k*D *are approximable, we have C*¯(*y, ν,* (*X, *·  )) ≤ *d for all y* ∈ *X.*

This can be proven using (e.g.) binary expansions of coefficients with respect to an arbitrary basis in *ν*(dom *ν*). A similar bound can be obtained for a left-invariant Riemannian metric on a Lie group, with assumptions similar to Theorem [1.1](#_bookmark1), though the proof is longer and more technical than that of the above lemma. When *X* is a computable metric space, in general ([[7](#_bookmark23)]) we have

(∀*x*)(*C*¯(*x, ν,* (*X, d*)) ≤ dimb(*X, d*)) where dimb is the upper box dimension,

but in the semicomputable case this is not clear. For general *ν*, this is not true

at all; for any *C >* 0, separable metric space *X* and nowhere dense subset *A*

one can construct dense *I* : N → *X* such that (∀*x* ∈ *A*)(*C*¯(*x, I,* (*X, d*)) ≥ *C*). From the current bound, we note we can now prove the theorem of this section:

Theorem 4.11 *Considering* (R*d,* +)*, if ν* ∈A *is witnessed by d* ∈D *and lin-* *ear map T* : R*d* → R*d with distinct eigenvalues λ*1*,..., λr* ∈ C *is approximable with respect to ν, d, then*

1. *S*¯(*x, T, ν,* (R*d, d*)) ≤ Σ log2 *λi* dim *Sλ ,*

*j*

*ij*

*j*∈Γ

*where* (*λij* )*j*∈Γ *are the eigenvalues with* *λij* ≥ 0 *and* *λij* ≥ 1*.*

By the well-known formula for topological entropy of a linear map[[1](#_bookmark17)], the upper bound here is just *hd*(*T* ). We also need one more technical lemma.

Lemma 4.12 *In a metrizable topological group G with right-invariant metric d and effectively separable weakly semicomputable interpretation ν* :⊆ N → *G, if the product* · : *G* × *G* → *G is approximable and the connected component G*0 *of the identity is open then each restriction of ν to a connected component is effectively separable.*

Now we can prove Theorem [1.1](#_bookmark1), although we will denote the objects in that statement by *G*', *d*', *ν*', *F* ', *T* '; since by [[9](#_bookmark29), Thm 9.8] *F* ' is compact and there is a homeomorphism & group isomorphism Ψ : *G*' → R*a* × Z*b* × *F* ' =: *G* for some *a, d* ∈ N, we can use *d*(*x, y*) := *d*'(Ψ−1*x,* Ψ−1*y*), *T* := Ψ ◦ *T* ' ◦ Ψ−1 and *ν* := Ψ ◦ *ν*' (these plainly give the same entropy and complexity).

Proof of Theorem [1.1](#_bookmark1) (Sketch) By Theorem [3.4](#_bookmark10) and Proposition [3.3](#_bookmark9) ap- plied to *H* := {(0*,* 0)}× *F* ', it is sufficient to show *S*¯(*πz, S, π* ◦ *ν,* (*G/H, d*˜)) ≤ *hd*˜(*S*) for all *z* ∈ *G*. Noting the connected component *V* ∼= R*a* of the identity in *G/H* has *S*(*V* ) ⊆ *V* (from continuity and *S*(0) = 0) and *L* := *S*|*V* linear,

and that *hdV* (*L*) ≤ *hd*˜(*S*) (for *dV* := *d*˜|*V* ×*V* ), we plan to take a decomposition

*G/H* = *V* ⊕ *W* for which all *w* ∈ *W* have *S*¯(*w, S, π* ◦ *ν,* (*G/H, d*˜)) = 0, and try

to bound *S*¯(*v, L, π* ◦ *ν,* (*G/H, d*˜)) (*v* ∈ *V* ) using Theorem [4.11](#_bookmark15) (note here we are using invariance of the metric *d*˜ to ensure Lemma [3.2](#_bookmark6)(vi) applies to +). For this purpose it is convenient to use the obvious homeomorphism & group isomorphism Φ : *G/H* → R*a* × Z*b*, pick arb. *uj* ∈ *πν*(dom *ν*) corresponding

to R*a* × {(*δi,j*)*b* }, and consider *W* := spanZ{*u*1*,..., ub*}. We also use an

*i*=1

obvious extension *Y* ∼= R*a* × R*b* of *G/H* to fix a norm-induced metric *d*ˆ; then id : (*G/H, d*˜) → (*G/H, d*ˆ) is eff. unif. continuous (from Proposition [4.1](#_bookmark12) and openness of *V* ) and so is *S* : (*G/H, d*ˆ) → (*G/H, d*ˆ) (checked directly).

From these, approximability of *S* (with respect to *d*˜), *uj* ∈ *πν*(dom *ν*) and

Lemma [3.2](#_bookmark6)(vii) one gets *S*¯(*uj, S,π* ◦ *ν,* (*G/H, d*˜)) = 0. On the other hand,

since *V* is an open subgroup one can check the restriction of *π* ◦ *ν* to *V* has +, *L* approximable with respect to *dV* , and by Proposition [4.3](#_bookmark13)(iv) and Lemma [4.12](#_bookmark16), a sufficient condition to apply Theorem [4.11](#_bookmark15) is that (*G/H, d*˜*, π* ◦ *ν*) be

(effectively separable and) weakly semicomputable. This is the reason for requiring some t.r. *f* : N → N have (*ν* ◦ *f* )(N) dense in *H*. Assuming also

p.r. *h*, *P* witnessing the (N *ν,* N *ν*)-, (N *ν* ×N *ν,* N *ν*)-effectivity of − : *G* → *G*

and + : *G* × *G* → *G*, and noting *ν* and *θ* := N*dν* are semicomputable with respect to *d*, we consider the following algorithm:

Algorithm 1 *On input n* ∈ N*, η* ∈ Q+*, dovetail calculation of z* := *P* (⟨*n,* (*h*◦

*f* )(*i*)⟩) *and enumerations of Az,y*∪˙ *θ*−1*Bd*(*θ*(*z*); *η*) *over all i* ∈ N*.*

For any *a* ∈ *θ*−1*π*−1*Bd*˜((*π* ◦ *θ*)(*n*); *η*), there exist *g* ∈ *H* and *ξ* ∈ (0*, η*) such that *θ*(*a*)+ *g* ∈ *Bd*(*θ*(*n*); *ξ*), and then ∃*i* ∈ N such that *d*(*g,* (*θ* ◦ *f* )(*i*)) *< η* − *ξ*, so

*d* (*θ*(*a*)*, θ*(*n*) − (*θ* ◦ *f* )(*i*)) ≤ *d* (*θ*(*a*)*, θ*(*n*) − *g*)+ *d* (*θ*(*n*) − *g, θ*(*n*) − (*θ* ◦ *f* )(*i*))

= *d* (*θ*(*a*)+ *g, θ*(*n*)) + *d* (*g,* (*θ* ◦ *f* )(*i*)) *< η,*

and *a* must appear in the output. Conversely, any output *a* ∈ N either has *a* ∈ *A...* ⊆ N \ dom *θ* or *a* ∈ *θ*−1*Bd*(*θ*(*n*) − (*θ* ◦ *f* )(*i*); *η*) for some *i* ∈ N, in which case

*d*˜((*π* ◦ *θ*)(*a*)*,* (*π* ◦ *θ*)(*n*)) ≤ *d*(*θ*(*a*)*, θ*(*n*) − (*θ* ◦ *f* )(*i*)) *< η,*

and the output is correct. This shows *π* ◦ *θ* is semicomputable with respect to

*d*˜, hence *π* ◦ *ν* ≤ *π* ◦ *θ* is also, and the proof is finished.

# References

1. Bowen, R., *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. 153 (1971), pp. 401–414.
2. Brin, M. and A. Katok, *On local entropy*, in: *Geometric dynamics (Rio de Janeiro, 1981)*, Lecture Notes in Math. 1007, Springer, Berlin, 1983 pp. 30–38.
3. Brudno, A. A., *Entropy and the complexity of the trajectories of a dynamic system*, Trudy Moskov. Mat. Obshch. 44 (1982), pp. 124–149.
4. G´acs, P., “Lecture Notes on Descriptional Complexity and Randomness,” 89 pp.

URL <http://www.cs.bu.edu/~gacs/papers/ait-notes.ps.gz>

1. Galatolo, S., *Pointwise information entropy for metric spaces*, Nonlinearity 12 (1999),

pp. 1289–1298.

1. Galatolo, S., *Orbit complexity by computable structures*, Nonlinearity 13 (2000), pp. 1531–1546.
2. Galatolo, S., *Complexity, initial condition sensitivity, dimension and weak chaos in dynamical systems*, Nonlinearity 16 (2003), pp. 1219–1238.
3. Hertling, P., *A real number structure that is effectively categorical*, MLQ Math. Log. Q. 45

(1999), pp. 147–182.

1. Hewitt, E. and K. A. Ross, “Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations,” Die Grundlehren der mathematischen Wissenschaften, Bd. 115, Academic Press Inc., Publishers, New York, 1963, viii+519 pp.
2. Li, M. and P. Vit´anyi, “An introduction to Kolmogorov complexity and its applications,” Springer-Verlag, New York, 1993, xx+546 pp.
3. Uspensky, V. A. and A. Shen, *Relations between varieties of Kolmogorov complexities*, Math. Systems Theory 29 (1996), pp. 271–292.
4. Walters, P., “An introduction to ergodic theory,” Springer-Verlag, New York, 1982, ix+250 pp.
5. Weihrauch, K., “Computability,” Springer-Verlag, Berlin, 1987, x+517 pp.