Electronic Notes in Theoretical Computer Science 210 (2008) 15–31 

[www.elsevier.com/locate/entcs](http://www.elsevier.com/locate/entcs)

POVMs and Naimark’s Theorem Without Sums

Bob Coecke[1](#_bookmark0) *,*[2](#_bookmark0)

*Oxford University Computing Laboratory, Wolfson Building, Parks Road,*

*OX1 3QD Oxford, UK.*

# E´ric Oliver Paquette[3](#_bookmark0) *,*[4](#_bookmark0)

*Universit´e de Montr´eal,*

*Laboratoire d’Informatique Th´eorique et Quantique,*

*CP 6128, succursal centre-ville, Montr´eal, Canada H3C 3J7.*

Abstract

We provide a definition of POVM in terms of abstract *tensor structure* only. It is justified in two distinct manners. i. At this abstract level we are still able to prove Naimark’s theorem, hence establishing a bijective correspondence between abstract POVMs and abstract projective measurements (cf. [[12](#_bookmark11)]) on an extended system, and this proof is moreover *purely graphical*. ii. Our definition coincides with the usual one for the particular case of the Hilbert space tensor product. We also provide a very useful *normal form* result for the classical object structure introduced in [[12](#_bookmark11)].

*Keywords:* POVM, Naimark’s theorem, †-compact category, classical object, CPM-construction.

# Introduction

The work presented in this paper contributes to a line of research which aims at recasting the quantum mechanical formalism in purely *category-theoretic terms* [[2](#_bookmark4),[3](#_bookmark6),[12](#_bookmark11),[21](#_bookmark24)], providing it with compositionality, meaningful types, additional degrees of axiomatic freedom, a comprehensive operational foundation, and in particular,

1 B.C. is supported by EPSRC Advanced Research Fellowship EP/D072786/1 *The Structure of Quantum Information and its Ramifications for IT* and EPSRC Grant EP/C500032/1 *High-level methods in quantum computation and quantum information*. He thanks Dan Browne, Peter Selinger and Reinhard Werner for useful feed-back on an earlier version of the presented results.

2 Email: [coecke@comlab.ox.ac.uk](mailto:coecke@comlab.ox.ac.uk)

3 E.O.P. thanks Oxford University Computing Laboratory for its hospitality during his visit in which this work was realised, and for which he enjoyed financial support from Gilles Brassard’s Chaire du recherche du Canada en informatique quantique. He also thanks Andr´e M´ethot for feedback on an earlier version.

4 Email: [eopaquette@isodensity.ca](mailto:eopaquette@isodensity.ca)

1571-0661 © 2008 Published by Elsevier B.V. Open access under [CC BY-NC-ND license.](http://creativecommons.org/licenses/by-nc-nd/3.0/)

doi:10.1016/j.entcs.2008.04.015

high-level mechanisms for reasoning i.e. *logic*. The computational motivation for this line of research, if not immediately obvious to the reader, can be found in earlier papers e.g. [[2](#_bookmark4)]. Particularly informal physicist-friendly introductions to this program are available [[7](#_bookmark10),[8](#_bookmark12),[9](#_bookmark13)]. This program originates in a paper by Samson Abram- sky and one of the authors [[2](#_bookmark4)], and an important contribution was made by Peter Selinger, establishing an abstract definition of mixed state and completely positive map in purely multiplicative terms [[21](#_bookmark24)]. The starting point of this paper is a re- cent category-theoretic definition for projective quantum measurements which does not rely on any additive structure, due to Dusko Pavlovic and one of the authors [[12](#_bookmark11)]. We refer to this manner of defining quantum measurements as *coalgebraically*. We show that the usual notion of POVM (e.g. [[6](#_bookmark8),[13](#_bookmark16),[19](#_bookmark22)]) admits a purely multi- plicative category-theoretic counterpart, in the sense that it is supported both by a *Naimark-type* argument with respect to the coalgebraically defined ‘projective’ quantum measurements, and by the fact that we recover the usual notion of POVM when we consider the category of Hilbert spaces and linear maps.

Recall that a *projective measurement* is characterised by a set of projectors

{P*i* : H → H}*i*, i.e. for all *i* we have P*i* ◦ P*i* = P*i* = P†, such that Σ P*i* = 1H,

*i*

*i*

which implicitly implies that for *i* /= *j* we have P*i* ◦ P*j* = 0. To each *i* we assign an *outcome probability* Tr(P*i* ◦ *ρ*). More generally, a *POVM* is a set of positive operators {*Fi* : H → H}*i*, i.e. *Fi* = *f* † ◦ *fi* for some linear operator *fi*, such that

*i*

Σ*i Fi* = 1H, and to each *i* we now assign an *outcome probability* Tr(*Fi* ◦ *ρ*). By

positivity and by cyclicity of the trace we can rewrite this outcome probability as Tr(*fi* ◦ *ρ* ◦ *f* †). While in the case of projective measurements the state of the system undergoes a change *ρ* '→ P*i* ◦ *ρ* ◦ P*i*, for a POVM one typically is only concerned with the probabilities of outcomes, so the *type* of a POVM is

*i*

POVM : *quantum* (*mixed*) *n*-*states* → *classical* (*mixed*) *n*-*states .*

Using the fact that classical *n*-states can be represented by [0*,* 1]-valued diagonal

*n* × *n*-matrices with trace one we can write

POVM :: *ρ* '→ Σ Tr(*fiρf* †)|*i*⟩⟨*i*|

*i*

*i*

where we used standard Dirac notation to represent the canonical projectors {|*i*⟩⟨*i*|}*i*

with respect to the computational base {|*i*⟩}*i*.

# Abstract CPMs and projective measurements

For the basic definitions of †-compact categories and their interpretation as se- mantics for quantum mechanics we refer to the existing literature [[3](#_bookmark6),[12](#_bookmark11),[21](#_bookmark24)] and references therein. The connection between such categories and graphical calculi is in [[1](#_bookmark5),[4](#_bookmark7),[5](#_bookmark9),[14](#_bookmark17),[15](#_bookmark18),[16](#_bookmark19),[17](#_bookmark20),[18](#_bookmark21),[20](#_bookmark23),[21](#_bookmark24)] and references therein. We recall here the CPM- construction due to Selinger [[21](#_bookmark24)] and the coalgebraic characterisation of projective measurements due to Pavlovic and one of the authors [[12](#_bookmark11)]. This coalgebraic char- acterisation of projective measurements comprises the definition of *classical object* which captures the behavioral properties of classical data by making explicit the ability to copy and delete this data.

* 1. *Mixed states and completely positive maps*

A morphism *f* : *A* → *A* is *positive* if there exists an object *B* and a morphism *g* : *A* → *B* such that *f* = *g*† ◦ *g*. Graphically this means that we have the following decomposition:

*A A* =

*f*

*B*

*A A*

*g*†

*g*

A morphism *f* : *A* ⊗ *A*∗ → *B* ⊗ *B*∗ is *completely positive* if there exists an object *C*

and morphisms *g* : *A* ⊗ *C* → *B* and/or *h* : *A* → *B* ⊗ *C* such that *f* is equal to

*A B*

*g*

*g*∗

*C*

*A*∗ *B*∗

and/or

*A B*

*C*

*h*

*h*∗

*A*∗ *B*∗

A *mixed state ρ* : *I* ⊗ *I*∗ → *A* ⊗ *A*∗, which is a special case of a completely positive map, is the *name* of a positive map (for some *h* = *g*†):

*A*



*f*

*A*∗

Name of *f*

positivity

*A*

*A*∗



*g*

*g*†

*A* = *A*

*A*∗ *A*∗



*h*

*h*∗

*ρ*

Mixed state

— note that we rely here on the canonical isomorphism *I* *I* ⊗ *I*∗. Given any

†-compact category, define CPM(C) as the category with the same objects as C, whose morphisms *f* : *A* → *B* are the completely positive morphism *f* : *A* ⊗ *A*∗ → *B* ⊗ *B*∗ in C, and with composition inherited from C. As shown in [[21](#_bookmark24)], if C is †- compact then so is CPM(C), and the morphisms of CPM(FdHilb) are the usual completely positive maps and mixed states.

Remark 2.1 It is worth noting that this *purely multiplicative* definition of com- pletely positive maps (i.e. it relies on tensor structure alone) incarnates the *Kraus representation* [[19](#_bookmark22)], where the usual summation is now implicitly captured by the *internal trace-* and/or *cotrace-*structure on CPM(C) [[10](#_bookmark14)], i.e. the half-circles in the pictures representing completely positive maps.

There also is a canonical ‘almost’ embedding of C into CPM(C) defined as

*P ure* :: C → CPM(C): *f* '→ *f* ⊗ *f*∗ *.*

From now on, we will omit (−)∗ on the objects and (−)∗ on the morphisms in the “symmetric image” which is induced by the CPM-construction.

* 1. *Classical objects*

The type we are after for a quantum measurement is

*A* → *X* ⊗ *A*

expressing that we have as input a quantum state of type *A*, and as output a measurement outcome of type *X* together with the collapsed quantum state still of type *A*. We distinguish between *quantum data A* and *classical data X* by our ability to freely copy and delete the latter. Hence a classical object ⟨*X, δ, ϵ*⟩ is defined to be an object *X* together with a *copying operation δ* : *X* → *X* ⊗ *X* and a *deleting operation ϵ* : *X* → *I*, which satisfy some obvious behavioral constraints that capture the particular nature of these operations. Let *λX* : *X* *I* ⊗ *X* be the natural isomorphism of the monoidal structure and let *ηX* : *I* → *X*∗ ⊗ *X* be the *unit* of the †-compact structure for object *X*.

Theorem 2.2 [[12](#_bookmark11)] *Classical objects can be equivalently deﬁned as* :

1. *special* †*-compact Frobenius algebras* ⟨*X, δ, ϵ*⟩ *which realise*

*ηX* = *δ* ◦ *ϵ*† *,*

*where specialness means* 1*X* = *δ*† ◦ *δ and the* †*-Frobenius identity*

*δ* ◦ *δ*† = (1*X* ⊗ *δ*) ◦ (*δ*† ⊗ 1*X* )

*depicts as*

*δ*†

*δ*

=

*δ*†

*δ*

1. *special X-self-adjoint internal commutative comonoids* ⟨*X, δ, ϵ*⟩*, where X-self- adjointness stands for*

*δ* = (1*X* ⊗ *δ*†) ◦ (*ηX* ⊗ 1*X*) ◦ *λX* and *ϵ* = *η*† ◦ (1*X* ⊗ *ϵ*†) *.*

*X*

*which are graphically represented as*

= *δ*† =



*ϵ*



*ϵ*†

*δ*

*In particular do we have self-duality of X i.e. ηX realises X*∗ := *X, and also δ*

*and ϵ prove to be self-dual i.e. δ*∗ = *δ and ϵ*∗ = *ϵ.*

* 1. *Coalgebraically deﬁned projective measurements*

Classical objects, being internal commutative comonoids, canonically induce com- mutative comonads, so we can consider the Eilenberg-Moore coalgebras with respect to these. This results in the following characterization of quantum spectra as the *X-self-adjoint coalgebras* for those comonads. Given a classical object ⟨*X, δ, ϵ*⟩, a *projector-valued spectrum* is a morphism P : *A* → *X* ⊗ *A* which is *X*-complete

i.e. (*ϵ* ⊗ 1*A*) ◦P = *λA*, and which also satisfies

*A*  P *X* ⊗ *A A*  P *A* ⊗ *X*

P

J

1X ⊗P

J

and

J

1X ⊗P†

J

*X* ⊗ *A δ*⊗1A *X* ⊗ *X* ⊗ *X I* ⊗ *A η*X ⊗1A *X* ⊗ *X* ⊗ *A*

to which we respectively refer as *X-idempotence* and *X-self-adjointness* and are respectively depicted as

*X*

*δ*

P

*X* =

*A A A*

*X* *X*

*X*



P

P

P

=

*A A A*

*X*

*A A*

P†

Remark 2.3 It is most definitely worth noting that *X*-idempotence exactly incar- nates *von Neumann’s projection postulate*, in a strikingly resource-sensitive fashion: repeating a quantum measurement has the same effect as merely copying the data obtained in the first measurement.

As shown in [[12](#_bookmark11)], in FdHilb these projector-valued spectra are in bijective corre- spondence with the usual projector spectra defined in terms of self-adjoint linear operators. In particular, the classical object

C⊕*n ,* | *i*⟩ '→ | *ii*⟩ *,* | *i*⟩ '→ 1

yields the projector spectra of all *n*-outcome measurements on a Hilbert space of dimension *k* ≥ *n*, where *X*-idempotence assures projectors to be idempotent (P2 = P*i*) and mutually orthogonal (P*i* ◦ P*j*/=*i* = 0), *X*-self-adjointness assures them to be

*i*

self-adjoint (P† = P*i*), and *X*-completeness assures Σ*i*=*n* P*i* = 1H i.e. probabilities

*i*

arising from the Born-rule add up to 1.

*i*=1

Given this representation theorem, and the fact that such a projector-valued spectrum already admits the correct type of a quantum measurement, one might think that projector-valued spectra are in fact quantum measurements. Unfortu- nately this is not the case: a projector-valued spectrum preserves the relative phases encoded in the initial state. In other words, the off-diagonal elements of the density matrix of the initial state expressed in the measurement basis do not vanish. But this can be easily fixed. In [[12](#_bookmark11)] it was shown that these redundant phases can be eliminated by first embedding C into CPM(C) and then post-composing the image P⊗P∗ of a projector-valued spectrum P under *P ure* with 1*A* ⊗ Decohere ⊗ 1*A* where

Decohere := (1*X* ⊗ *η*†

*X*

⊗ 1*X* ) ◦ (*δX* ⊗ *δX* ): *X* ⊗ *X* → *X* ⊗ *X*

or, graphically,

*X*



*X*

*δ*

*X*

*δ*

*X*

Note that Decohere is indeed a morphism in CPM(C). One also verifies that equivalently one can set Decohere = *δ* ◦ *δ*†. Conclusively, a projective measurement is a composite

M := (1*A* ⊗ Decohere ⊗ 1*A*) ◦ (P ⊗ P∗)

where *X* carries a classical object structure and P is a corresponding projector- valued spectrum, and is of type *A* → *X* ⊗ *A* in CPM(C).

We will slightly relax this measurement notion by dropping *X*-completeness,

something which is quite standard in quantum information literature where rather than Σ*i Fi* = 1H one regularly only requires Σ*i Fi* ≤ 1H for POVMs. The same relaxation applies to our definition of projector-valued spectra.

# Normalisation

We now provide a normal form result for expressions involving the structural wit- nesses of †-compactness and classical objects.

A *classical network* is a morphism obtained by composing, tensoring and taking adjoints of *δ*, *ϵ* (and hence also of *η* and identities) and the natural isomorphisms of the symmetric monoidal structure.

Depicting *δ* and *ϵ* as





a classical network is connected if its pictorial representation forms a topologically connected whole of dots and lines, which means that there is a *path* from any input, output, or dot to any other input, output, or dot.

Set *δ*0 := *ϵ*† and *δ*1 := 1*X* and, for *n* ≥ 2,

*δn* := (*δ* ⊗ 1*X*⊗n−2 ) ◦ (*δ* ⊗ 1*X*⊗n−3 ) ◦ *...* ◦ (*δ* ⊗ 1*X* ) ◦ *δ.*

For *n >* 2, *δn* is depicted as

.

where there are *n* output wires.

Classical networks of the form

*δn* ◦ *δ*†

*m*

: *X*⊗*m* → *X*⊗*n.*

are completely determined by their number of inputs and outputs. For instance, the pair (0*,* 1) defines *ϵ*†, the pair (1*,* 2) defines *δ*, the pair (2*,* 2) defines *δ* ◦ *δ*† etc.

*m*

We can depict the classical network *δn*

* *δ*† as



.

.

where the number of wires going in is *m* and the number of wires going out is *n*, except for *δ*1 ◦ *δ*† = 1*X* which we depict by a wire without a dot.

1

We introduce *rewriting rules* which will realise the normalisation process:

Fusion rule: We direct Frobenius identity:

(1 ⊗ *δ*†) ◦ (*δ* ⊗ 1) ~= *δ* ◦ *δ*† = (*δ*† ⊗ 1) ◦ (1 ⊗ *δ*)

~= =



1st Annihilation rule: We direct (co)monoid (co)unit laws:

(*ϵ* ⊗ 1) ◦ *δ* ~=

1 = (1 ⊗ *ϵ*) ◦ *δ δ*† ◦ (*ϵ*† ⊗ 1) ~=

1 = *δ*† ◦ (1 ⊗ *ϵ*†)

~= = ~= =



2nd Annihilation rule: We direct specialness:

*δ*† ◦ *δ* ~= 1

~=



Note that each of these rules reduces the number of dots in classical networks.

Lemma 3.1 [normalisation] *Each connected classical network admits a normal*

*m*

*form δn*

* *δ*†

*which only depends on its number of inputs and outputs, and is realised*

*using the above described rewriting rules.*

Proof: We sketch the ‘proof by rewriting’ and illustrate each rewriting step on a

generic example, namely the connected classical network



*Step 1:* Replace all occurrences of *η* (*η*†) by *δ* ◦ *ϵ*† (*ϵ* ◦ *δ*†). This substitution does not affect connectedness. Let the resulting number of dots be *N* .

~=



*Step 2:* Use bifunctoriality to move all *ϵ*’s and *ϵ*†’s out of the ‘main body of the expression’ in order to obtain a composition of the form

*Eє* ◦ *Eδ,δ*† ◦ *Eє*†

where *Eє* is a tensor product of identities and *ϵ*’s, *Eδ,δ*† a classical network without

*ϵ*’s nor *ϵ*†’s, and *Eє*† a tensor products of identities and *ϵ*†’s.

~=



*Eє*†

*Eδ,δ*†

*Eє*

*Step 3:* Since the components *Eє*† and *Eє* are completely disconnected, the com- ponent *Eδ,δ*† has to be connected. Induction on *Eδ,δ*† using the fusion rule to ‘move *δ*†’s before *δ*’s’, using the 1st annihilation rule to cancel out components of the form *δ*† ◦ *δ*, and using (co)associativity and (co)commutativity of *δ* and *δ*† results in an expression of the form *δk* ◦ *δ*† with *k, l >* 0. Indeed, confluence is witnessed by the

*l*

fact that:

* + both rules reduce the total number of dots with at least one,
    - as long as the number of dots is at least two we will always be able to apply one of the rules at least one more time due to connectedness,
    - we start with a finite number *N* of dots so rewriting terminates,
    - a classical network with either one or no dots can always be rewritten in the normal form by (co)associativity and (co)commutativity.

~=



*Eδ,δ*†



*F*

*Step 4:* In *Eє* ◦ (*δk* ◦ *δ*†) ◦ *E* † , by connectedness, all *ϵ*’s (*ϵ*†’s) can be cancelled out

*l є*

by the 2nd annihilation rule.

~=



*Eє*† *F*

*Eє*

Hence we obtain the desired normal form.

It is easy to see that this lemma induces a rewriting scheme for the ‘classical com- ponent of more general expressions’, i.e. the part only involving classical object structure, simply by normalising all (maximal) classical networks it comprises while considering the ‘boundary’ of the classical component as its inputs and outputs. We will make this more precise in future writings.

# Abstract POVMs

In the same vein as the notions of *X*-self-adjointness, *X*-idempotence, and also *X*- unitarity introduced in [[12](#_bookmark11)], we now define the appropriate generalisations of scalars, their inverses, isometries, and positivity of morphisms. This means that we will introduce new classes of morphisms whose types include *X*, which we interpret as a *X*-indexed family of morphisms. Most generally, an *X-morphism* is any morphism of type *f* : *X* ⊗ *A* → *B* where *X* is a classical object. A more general high-level treatment will be in [[11](#_bookmark15)].

Definition 4.1 An *X-isometry* is a morphism V : *X* ⊗ *A* → *B* for which

V*δ* := (1*X* ⊗ V) ◦ (*δ* ⊗ 1*A*): *X* ⊗ *A* → *X* ⊗ *B*

is an isometry i.e. it satisfies

V† ◦ V*δ* = 1*X*⊗*A .*

*δ*

Definition 4.2 A morphism *f* : *A* → *A* ⊗ *X* is *X-positive* if there exists an *X*- morphism *g* : *B* → *A* ⊗ *X* such that

*A*



*f*

*δ*

*X*

*A*

*X* = *A*

*g*

*B*

*g*

*A*

*X* *X*

In the second picture, the fact that the trapezoid on the left points with its sharp corner to the left, as compared to trapezoid on the right of which the sharp corner points to the right, indicates that it is “daggered” as compared to the one on the right. This graphical convention will be reused in what follows.

Recall that a *polar decomposition* of a linear operator *M* is a factorisation of

*M* = *V* ◦ *H* where *V* an isometry and *H* is positive.

Definition 4.3 We say that an *X*-morphism *f* : *A* → *B* ⊗ *X* is *X-polar decom- posable* if there exists an *X*-positive morphism *g* : *A* → *X* ⊗ *A* and an *X*-isometry V : *X* ⊗ *A* → *B* such that *f* = V*δ* ◦ *g* i.e. *f* can be depicted as

*X*

*δ*

*g*

V

*A A B*

Definition 4.4 An *X-scalar* is a morphism *f* : *I* → *X*. An *X*-scalar *t* : *I* → *X* is an *X*-*inverse* of *s* : *I* → *X* iff, setting *λI* : *I* *I* ⊗ *I*, we have

*δ*† ◦ (*s* ⊗ *t*) ◦ *λI* = *ϵ*† *.*

In FdHilb *X*-scalars are *n*-tuples of complex numbers. An *X*-scalar’s *X*-inverse in FdHilb is the *n*-tuple consisting of the component-wise inverses to the given *n*-tuple. In our context, *X*-scalars will arise when tracing out *A* in a morphism

*f* : *A* → *A* ⊗ *X*, yielding the *X*-scalar Tr*A*

*I,X*

(*f* ) : *I* → *X*. Graphically an *X*-scalar

is represented as



*s*

*X*

From now on, we will work within CPM(C). Classical objects will however always be defined in C, and then embedded in CPM(C) via *P ure*.

Definition 4.5 [POVM] Let ⟨*X, δ, ϵ*⟩ be a classical object. A *POVM* on a system of type *A* which produces outcomes in *X* is a morphism

*A*

*A*

*f*

*δ*

*δ*

*f*

*A*

*A*

*X*

satisfying

*X*

*A A*

=

*f*

*A*

*X*

*δ*

*δ*

*f*

*X*

*A*

*A A*

and for which *f* ∈ C(*A, X* ⊗ *A*) is *X*-polar-decomposable.

Hence, within CPM(C) the type of such a POVM is indeed *A* → *X*. In FdHilb the requirement on *X*-polar-decomposability is of course trivially satisfied since any linear map admits a polar decomposition.

Theorem 4.6 *In the category* FdHilb *the abstract POVMs of Deﬁnition* [*4.5*](#_bookmark2) *ex-*

*actly coincide with the assignments ρ* '→ Σ*i* Tr(*giρg* )|*i*⟩⟨*i*| *corresponding to POVMs*

†

*i*

*deﬁned in the usual manner* (cf. Section 1)*.*

Proof. Consider a POVM as in Definition [4.5](#_bookmark2). In FdHilb a classical object is of the form C⊕*n* and induces canonical base vectors | *i*⟩ : C → C⊕*n*. Set

*f*ˆ*i* := (⟨*i* |⊗ 1*A*) ◦ *f* : *A* → *A* and *fi* := (| *i*⟩⟨*i* |⊗ 1*A*) ◦ *f* : *A* → *X* ⊗ *A.*

In particular do we have *f* = Σ*i*=*n fi*. Hence, we can rewrite the POVM as

*i*=1

tr*A* Decohere ◦ Σ *fi* ⊗ Σ *fj*∗ ◦ − = tr*A* Decohere ◦ Σ(*fi* ⊗ *fj*∗) ◦ −

*i j i,j*

= tr*A* Σ(*fi* ⊗ *fi*∗) ◦ − *.*

*i*

Passing from CPM(C) to standard Dirac notation, i.e. from | *i*⟩⊗ | *i*⟩∗ to | *i*⟩⟨*i* | and from (*f* ⊗ *f*∗) ◦− to *f* (−)*f* †, also using *fi* = (| *i*⟩⊗ 1*A*) ◦ *f*ˆ*i*, we obtain

Σ Tr(*f*ˆ*i*(−)*f*ˆ†)|*i*⟩⟨*i*|*.*

*i*

*i*

Using the polar decomposition of *f*ˆ*i* and cyclicity of the trace we get

Σ Tr(*f*ˆ*i*(−)*f*ˆ†)|*i*⟩⟨*i*| = Σ Tr(*Uigi*(−)*g*†*U* †)|*i*⟩⟨*i*|

*i i i*

*i* *i*

= Σ Tr(*gi*(−)*g*†)|*i*⟩⟨*i*|

*i*

*i*

which is the intended result. Finally, the abstract normalisation condition tells

us that indeed, *f* † ◦ *f* = 1*A* and so *g*† ◦ *g* = 1*A*. The converse direction constitutes analogous straightforward translation in the graphical language.

Theorem 4.7 [Abstract Naimark theorem] *Given an abstract POVM, there exists an abstract projective measurement on an extended system which realises this POVM. Conversely, each abstract projective measurement on an extended system yields an abstract POVM.*

Proof: We need to show that there exists a projective measurement *h* : *C* ⊗ *A* → *C* ⊗ *A* ⊗ *X* in C together with an auxiliary input *ρ* : *I* → *C* in CPM(C) such that they produce the same probability as a given POVM defined via *f* : *A* → *A* ⊗ *X*, as in Definition [4.5](#_bookmark2), provided we trace out the extended space after the measurement. Graphically this boils down to

*A*

auxiliary input

*X*

*A*

*f*

*δ*

*δ*

*f*

*A*

*ρ*

=

*X*

projective

measurement

*C C*

*A h A δ*

*δ*

Trace

*X*

*X*

*A*

*A h A*

*C C*

i.e. an equality between two morphisms of type *A* → *X* in CPM(C). Firstly we exploit *X*-polar-decomposability of *f* . Factoring out *f* graphically yields

*A*

*A*

*A*

*f*

*δ*

*δ*

*f*

*A*



*k* U

*δ δ*

*δ*

*k*  U

*δ*

*A*

*X* *X*

=

*X* *X*

*A A A*

which by graphical manipulation and coassociativity of *δ* rearranges as

*A A*



*k*

U U

*δ*

*δ*  *δ*

*δ*

*k*

*A*

*k*

*δ*

*δ*

*k*

*A*

*X*

*X*

=

*X* *X*

*A A*

*A*

The pale square on the right-hand-side vanishes U being an *X*-isometry. Set

*C C*



*g*

*g*

*t*

*δ*†

*C C A A*

*h*

*A A* :=

*X* *X*

where *k* = *g*† ◦ *g* by *X*-positivity as in Definition [4.2](#_bookmark1) (we will consider *g* to be fixed

for the reminder of the proof), where *t* := Tr*A*(*f* ) −1 is an *X*-scalar, [5](#_bookmark3) and where the *δ*† with three input wires is (1*X* ⊗ *δ*†) ◦ *δ*† — which is meaningful by associativity of the comultiplication. Let

*C C*



*A*

*g*

*δ*†*X t*

*g*

*A*

*ρ*

:=

*C C*

We now check *X*-idempotence of *h*. We have



*C*

*A*

*C*

*g*

*g*

*A*

*g*

*g*

*X δ*†

*X*

*t*

*δ*†

*t*

*C*

*C C A*



*h*

*h*

*A A* ≡ *X*

*X*

*X* *X*

Via *X*-positivity of *f* , the pale square on the previous picture becomes *δ* ◦ *s* where *s* := Tr*A*(*f* ) is an *X*-scalar which is inverse to the *X*-scalar *t*. Factoring out the *X*-scalars, using normalisation and cancelling relative inverse *X*-scalars, we obtain the following equality between the pale squares below

*C*



*C*

*A*

*g*

*g*

*s δ*

*t*

*δ*†

*X*

*t*

*δ*†

*A C C*



*g*

*g*

*X*

*t*

*δ*†

*δ*

= *A A*

*X* *X*

*X* *X*

so we indeed obtain *X*-idempotence for *h*. It should be obvious that *h* is also *X*-self- adjoint by construction, so *h* defines a (not necessarily *X*-complete) projector-valued spectrum, and hence defines a projective measurement by adjoining the Decohere- morphism. Next we show that this projective measurement indeed realises the given POVM when feeding-in the mixed state *ρ*, as defined above, to its *C*-input, and when tracing-out the *A*-output. In the following, we will ignore the Decohere-morphism

5 It was observed by Pavlovic and one of the authors that every †-compact category C admits a universal *localization L*C together with a †-compact functor C → *L*C, which is initial for all †-compact categories with †-compact functors from C, and where a †-compact category is *local* iff all of its positive scalars are either divisors of zero, or invertible, where *zero* is multiplicatively defined in the obvious manner. These considerations extend to *X*-scalars. This result will appear in a forthcoming paper.

since, as we will see later, it will cancel as it is idempotent. Now, in

*C C*



*A*

*g*

*g*

*A*

*g*

*δ*†

*A*

*X*

*δ*†*X t*

*t*

*g*

*t*

†

*A*

*δ*

*A*

*g*

*g*

*X*

*A*

*C C*

the pale square is *δ* ◦ *s* by *X*-positivity of *f* . Hence we then obtain

*C*

*X*



*A*

*g*

*A*

*g*

*δ*†*X t*

*s*

*t*

*δ*†

*δ*

*g*

*t*

†

*A*

*δ*

*A*

*g*

*X*

*C*

Via an obvious graph isomorphism we get



*g*

*C*

*g*

*A*

*X*

*δ*†

*X*

*δ*†

*s*

*s*

*t*

*δ*

*t*

*δ*† *X*

*g*

*g*

*X*

*C*

*A*

*A*

*A*

Again, by *X*-positivity of *f* , we obtain



*k*

*A*

*δ*

*X*

*δ*†

*X*

*δ*†

*s*

*s*

*t*

*δ*

*t*

*δ*

*δ*†

*X*

*k*

*X*

*A*

*A*

*A*

The pale square in the previous picture reduces to the Decohere-morphism if first,

we factor out the *X*-scalars, we apply normalisation and cancel out the relative inverse *X*-scalars. Re-adjoining the Decohere-morphism which we omitted, which now cancels out by Decohere’s idempotence, we finally obtain

*A*

*k*

*δ*

*δ*

*k*

*A*

*A*

*X*

*X*

*A*

Conversely, we need to show that each projective measurement on an extended system yields a POVM. A projector-valued spectrum is *X*-positive since its *X*- idempotence and *X*-self-adjointness yield

*A*

*X*

*A A*

*X* =



P

*δ*

*X*

*A*

*X* = *A A*

P P

P P

*X* *X*

Next, observe that for an *X*-complete projector-valued spectrum we always have

P† ◦P = 1*A* since

*A*

*A*

*A*

P P

*X*

*A*

*A*

*A*

=

P

P

*δ*

*ϵ*

and by *X*-self-adjointness of P and *δ* we get

*A* = *A*

*A*

*A*

P

P

*A*

= *A*

*δ*† *ϵ*

P

*ϵ*

*X*

where the first equality uses *X*-idempotence of P and *δ*† ◦ *δ* = 1*A*. The second equality is obtained from the definition of *X*-completeness. Now, when considering a projective measurement on an extended system, using this fact together with *δ*† ◦ *δ* = 1*X* we obtain



*A*

*A*

P

*C*

*δ*

*C*

*δ*

P

*C*

*A*

*A*

*A*

= *C*

*A*

thence satisfying the normalisation condition up to a *C*-dependent scalar. The POVM which we obtain is

*A*

*A*

P

*C*

*δ*

*δ*

P

*C*

*A*

*A*

*X*

*C*

*X*

what completes the proof.

Remark 4.8 Manipulation of classical data in the above proof is extremely sim- plified by the normalisation lemma. A more refined version of this result together with its consequences will be given and discussed in a forthcoming paper [[11](#_bookmark15)].

Remark 4.9 While POVMs are not concerned with the state after the measure- ment, our analysis does produce an obvious candidate for non-destructive gener- alised measurements, sometimes referred to as PMVMs in the literature [[13](#_bookmark16)]. We postpone a discussion to forthcoming writings.

Remark 4.10 Notice the delicate role which *X*-completeness and normalisation of the POVMs plays in all this, on which, due to lack of space, we cannot get into. We postpone this discussion to an extended version of the present paper, which is also forthcoming.

# References

1. S. Abramsky (2005) *Abstract scalars, loops, free traced and strongly compact closed categories*. In: Proceedings of CALCO 2005, pp. 1–31, Springer Lecture Notes in Computer Science 3629.
2. S. Abramsky and B. Coecke (2004) *A categorical semantics of quantum protocols*. In: Proceedings of the 19th IEEE Conference on Logic in Computer Science, pp. 415–425, IEEE Computer Science Press. E-print [arXiv:quant-ph/0402130](http://arxiv.org/abs/quant-ph/0402130).
3. S. Abramsky and B. Coecke (2005) *Abstract physical traces*. Theory and Applications of Categories 14,

pp. 111–124. Available from [www.tac.mta.ca/tac/volumes/14/6/14-06abs.html](http://www.tac.mta.ca/tac/volumes/14/6/14-06abs.html) .

1. J. Baez (2004) *Quantum quandaries: a category-theoretic perspective*. In: Structural Foundations of Quantum Gravity, Oxford University Press. E-print [arXiv:quant-ph/0404040](http://arxiv.org/abs/quant-ph/0404040).
2. J. Baez and J. Dolan (1995) *Higher-dimensional algebra and topological quantum field theory*. Journal of Mathematical Physics 36, pp. 6073–6105. E-print [arXiv:q-alg/9503002](http://arxiv.org/abs/q-alg/9503002).
3. P. Busch, P. J. Lahti and P. Mittelstaedt (1991) *The Quantum Theory of Measurement*. Springer Lecture Notes in Physics 2.
4. B. Coecke (2005) *Kindergarten quantum mechanics — lecture notes*. In: Quantum Theory: Reconsiderations of the Foundations III, pp. 81–98, AIP Press. E-print [arXiv:quant-ph/0510032](http://arxiv.org/abs/quant-ph/0510032).
5. B. Coecke (2005) *Quantum information-flow, concretely, and axiomatically.* In: Proceedings of Quantum Informatics 2004, pp. 15–29, Proceedings of SPIE Vol. 5833. E-print available from [arXiv:quant-ph/0506132](http://arxiv.org/abs/quant-ph/0506132).
6. B. Coecke (2006) *Introducing categories to the practicing physicist*. In: What is Category Theory? Advanced Studies in Mathematics and Logic 30, pp. 45–74, Polimetrica Publishing. Available from Bob Coecke’s [homepage](http://web.comlab.ox.ac.uk/oucl/work/bob.coecke/Cats.pdf).
7. B. Coecke (2008) *Axiomatic description of mixed states from Selinger’s CPM-construction*. In: Proceedings of the 4th International Workshop on Quantum Programming Languages (QPL 2006), Electronic Notes in Theoretical Computer Science.
8. B. Coecke, E´. O. Paquette and D. Pavlovic. In preparation.
9. B. Coecke and D. Pavlovic (2007) *Quantum measurements without sums*. In: Mathematics of Quantum Computing and Technology. Chapman & Hall, pp. 559–596. E-print available from [arXiv:quant-ph/0608035](http://arxiv.org/abs/quant-ph/0608035).
10. E. B. Davies (1976) *Quantum Theory of Open Systems*. Academic Press.
11. P. Freyd and D. Yetter (1989) *Braided compact closed categories with applications to low-dimensional* *topology*. Advances in Mathematics 77, pp. 156–182.
12. A. Joyal and R. Street (1991) *The geometry of tensor calculus I*. Advances in Mathematics 88, pp. 55– 112.
13. A. Joyal, R. Street and D. Verity (1996) *Traced monoidal categories*. Proceedings of the Cambridge Philosophical Society 119, pp. 447–468.
14. G. M. Kelly (1972) *Many-variable functorial calculus I*. In: Coherence in Categories, pp.66–105, Springer Lecture Notes in Mathematics 281.
15. G. M. Kelly and M. L. Laplaza (1980) *Coherence for compact closed categories*. Journal of Pure and Applied Algebra, 19, pp. 193–213.
16. K. Kraus (1983) *States, Effects, and Operations*. Springer-Verlag.
17. R. Penrose (1971) *Applications of negative dimensional tensors*. In: Combinatorial Mathematics and its Applications, pp. 221–244, Academic Press.
18. P. Selinger (2007) *Dagger compact closed categories and completely positive maps*. In: Proceedings of the 3rd International Workshop on Quantum Programming Languages (QPL 2005), Electronic Notes in Theoretical Computer Science 170, 139–163. Available from Peter Selinger’s [homepage](http://www.mathstat.dal.ca/~selinger/papers.html#dagger).