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Paraconsistent Arithmetic with a Local Consistency Operator and Global Selfreference

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Abstract

The Provability Logic and Proof-Theory of the system of Paraconsistent Arithmetic PRACI are presented. PRACI is based on the paraconsistent predicate calculus CI corresponding to the C-system Ci introduced by Carnielli et al. [[7](#_bookmark9)]. PRACI can support an infinity of contradictions *B* ∧ ¬*B* without trivializing, but reject identifications between different numbers such as 0 = 1. In PRACI a new propositional connective

* (*.*) is added, so that ◦*A* can be read as “*A* is consistent”. We obtain a system with a *local selfreference*, based

on the *local consistency assertions* ◦*A,* and a *global selfeference*, based statements involving PrPRACI (*.*) *.* The *fundamental relation* PrT(#◦*B*) → ¬ PrT(#*B*) betweeen local and global consistency is investigated. It states that in a paraconsistent setting, the provability of the non-trivialty of Arithmetic could be reduced

to that of some suitable local consistency assertions, so that we can speak of a possible weakened Hilbert’s program.

*Keywords:* Paraconsistent Arithmetic, Provability Logic, Local and Global Selfreference, Weakened Hilbert’s program.

# Introduction

In this paper we explore Provability Logic and Proof-Theory of the system of Para- consistent Arithmetic PRACI, based on the paraconsistent predicate calculus CI, which extends the propositional C-system Ci introduced by Carnielli et al. [[7](#_bookmark9)]. As already pointed out in [1] for the presentation of the theory PCA, we propose systems of Paraconsistent Arithmetic which are essentially new w.r.t. most of para- consistent arithmetical theories existing in the preceding relevant literature. In our thinking paraconsistent reasoning, intuitionistic reasoning and classical reasoning can be compared as different methods to investigate the *same* elementary math- ematical objects: that is, we introduce a paraconsitent reasoning about *standard*

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contradictions *B* ∧ ¬*B* without trivializing, but reject identifications between dif- numbers. Thus, the theories PCA and PRACI can support an infinity of relevant ferent numbers, such as 0 = 1 and so on. An interesting feature of the C-system

based arithmetical theories proposed here, is that a new propositional connective

* (*.*) is added, expressing a notion of consistency referred to a particular formula, so that ◦*A* can be read as “*A* is consistent”. PRACI exhaustively formalizes the properties of local consitency assertions. Thus, by further introducing the global (i.e. referred to a system) canonical provability predicate PrPRACI (*.*) *,* we obtain a system where two kinds of selreference are possible: a *local selfreference*, based on the *local consistency assertions* ◦*A,* and a *global selfeference*, based on the provabil- ity logic statements involving PrPRACI (*.*) *.* In the paper the free cut-elimination and paraconsistency properties of PRACI are presented (Sections 3 and 4), the peculiar features of PRACI-Provability Logic are illustrated (Section 5), the in- tensional meaning of the paraconsistent negation is discussed (Section 5), and the

6). That is, starting from the result *<<* ▶ PrCI(#◦*B*) → ¬ PrCI(#*B*) is PRACI- *fundamental relation* betweeen local and global consistency is investigated (Section provable for each sentence *B* which has not the form ◦*F >>* (Theorem [6.5](#_bookmark2)), a kind of

weakened Hilbert’s program can be suggested. Indeed, some weaker versions of the previous implication, referred to PRACI, i.e. including the predicate PrPRACI(*.*) and not symply PrCI(*.*)*,* are PRACI-provable. Therefore, in a paraconsistent set- ting, the provability of the non-trivialty (which corresponds to classical consistency) of Arithmetic could be reduced to that of some suitable local consistency assertions. Thus, very elementary and constructive extensions of PRACI, with the same in- duction rule and withoud adding mathematical information, could prove the non triviality of PRACI.

# The paraconsistent sequent predicate calculus CI

In Benassi-Gentilini [1] the sequent version BC of the system bC is presented. bC is the basic system of the hierarchy of paraconsistent C-systems, introduced by Carnielli, Marcos and other authors [[7](#_bookmark9),[8](#_bookmark10),[9](#_bookmark11),[10](#_bookmark12),[11](#_bookmark13),[12](#_bookmark14)], whose language is the extension of the classical one through a monadic propositional connective ◦(*.*)*,*which plays an essential role in the introduction of the paraconsistent negation. As we shall see,

both ◦(*.*) and the propositional negation ¬ result as intensional logical operators.

The intended meaning of ◦*B* is “*B* is consistent ” that is “ *<<B* and not *B>>* does

not hold”. Thus, ◦*B* is a kind of formal translation of a metatheoretic statement, as for the provability predicate PrT(*.*) happens. We call the formulas of the form ◦*B local consistency assertions*. The C-system Ci is the bC-extension that explicitly expresses the *local consistency* properties: indeed, bC cannot prove theorems of the form ◦*B,* while Ci has a relevant class of such theorems. ◦*B* is neither bC- nor

Ci-equivalent to ¬(*B* ∧ ¬*B*). We present here the sequent version CI of Ci*.* We recall that (see [[5](#_bookmark3),[23](#_bookmark24),[14](#_bookmark16)]) a sequent *S* is an expression of the form *X* ▶ *Y* where *X* and *Y* are finite (possibly empty) sets of formulas. *X* is called the *antecedent* of *S*,

*Y* the *succedent* of *S*. We will use the symbols *X, Y,* Λ*,* Γ*,...* as meta-expressions

*A*1*, A*2*,... , An* ▶ *B*1*, B*2*,... , Bm* is ∧*iAi* −→ ∨*jBj* and such equivalence holds both for sets of formulas, *A, B, C, D,...* for formulas. The intended meaning of a sequent in a classical and in a paraconsistent setting. Given a rule *S*1*...Sn* , the sequents

*S*

*S*1*, ..., Sn* are the *premises* of the rule, the sequent *S* is the *conclusion* of the rule.

sequent rules. The writing Λ*,* Γ stands for Λ ∪ Γ. The proofs are trees, whose leaves are *axioms*, and whose branches are formed by

and only if it proves each sequent of the form ▶ *A*. A BC- or CI-based system We say that a sequent formulated system or theory *trivializes* (or *is trivial* ) if is trivial if and only if it proves the empty sequent ▶ . A system T is *negation consistent* if T cannot prove any formula of the form *B* ∧ ¬*B.* We recall that a formula *A* is in *prenex form* if *A* is Q1*....*Q*nB* where each Q*j* ∈ {∀*xj,* ∃*xj*}*j*=1*,...,n* and *B* is quantifier-free; in general, any formula *F* is not CI-equivalent to a formula

*D* in prenex form, since the interdefinability of quantifiers does not hold in the

that ∨*,* ∧*,* ¬ link more than →*,* and that → links more than ↔ *.* The sequent system paraconsistent setting presented here. In writing formulas we adopt the convention BC is given by (see also [[1](#_bookmark4),[12](#_bookmark14)]):

BC−*Axioms*: *A* ▶ *A*

BC−*Positive propositional logical rules:*

*B,* Γ ▶ Δ

*A* ∧ *B,* Γ ▶ Δ

∧ −*L*

*B,* Γ ▶ Δ

*B* ∧ *A,* Γ ▶ Δ

∧ −*L*

Γ ▶ Δ*,A* Λ ▶ *X, B*

Γ*,* Λ ▶ Δ*, X,A* ∧ *B*

∧ −*R*

Γ ▶ Δ*,A*

Γ ▶ Δ*,A* ∨ *B*

∨ −*R*

Γ ▶ Δ*,A*

Γ ▶ Δ*,B* ∨ *A*

∨ −*R*

*A,* Γ ▶ Δ *B,* Λ ▶ *X*

*A* ∨ *B,* Γ*,* Λ ▶ Δ*,X*

∨ −*L*

*A,* Γ ▶ Δ*,B*

Γ ▶ Δ*,A* → *B*

−→ −*R*

Γ ▶ Δ*, AB,* Λ ▶ *X*

*A* → *B,* Γ*,* Λ ▶ Δ*,X*

−→ −*L*

BC−*Negation rules:*

*A,* Γ ▶ Δ

¬− *L*1

¬¬*A,* Γ ▶ Δ

*A,* Γ ▶ Δ

* *A,* Γ ▶ Δ*,A*
* *A,* ¬*A,* Γ ▶ Δ¬− *L*3

Γ ▶ Δ*,* ¬*A*

¬− *R*

We call the formula ◦*A* in the rule ¬− *L*3 *constraint formula* of the rule.

BC−*Quantiﬁer rules*:

[*t/x*] *A,* Γ ▶ Δ

∀*xA,* Γ ▶ Δ [*b/x*] *A,* Γ ▶ Δ

∃*xA,* Γ ▶ Δ

∀− *L*

∃− *L*

Γ ▶ Δ*,* [*b/x*] *A* Γ ▶ Δ*,* ∀*xA*

Γ ▶ Δ*,* [*t/x*] *A*

Γ ▶ Δ*,* ∃*xA*

∀− *R*

∃− *R*

where *t* is an arbitrary term and *b* is a free variable which does not occur in

Γ*,* Δ. Moreover, *t* may be not fully quantified while *b* must be uniformly replaced by *x* (see [[23](#_bookmark24)]).

BC−*Structural rules:*

Weakening rules:

Γ ▶ Δ

*W* − *R*

Γ ▶ Δ

*W* − *L*; Cut rule:

Γ ▶ Δ*, AA,* Λ ▶ *X*

*Cut*

Γ ▶ Δ*,A*

*A,* Γ ▶ Δ

Γ*,* Λ ▶ Δ*,X*

The system CI is given by adding to BC the following *proper* CI*-rules*:

Γ ▶ Δ*,* ◦*A*

¬◦*A,* Γ ▶ Δ¬− *L*4

*A* ∧ ¬*A,* Γ ▶ Δ

Γ ▶ Δ*,* ◦*A R* Ci

In the rule *R* Ci the formula *A* ∧ ¬*A* in the premise antecedent is the *R* Ci −*auxiliary formula,* the formula ◦*A* in the conclusion succedent is the *R* Ci −*principal formula.* Note that system CI only has a rule introducing the con-

from BC by replacing the pair ¬− *L*3*,* ¬− *L*1*,* with the rule: Γ ▶ Δ*,A* ¬− *L*2. nective ◦(*.*)*.* Moreover, the classical predicate calculus LK [[19](#_bookmark21),[23](#_bookmark24)] can be obtained

¬*A,* Γ ▶ Δ

Theorem 2.1 *The system* CI *admits of cut-elimination.*

For the proof see [[13](#_bookmark15)].

following sequents are CI-provable: ▶ ◦*B* → ¬(*B* ∧ ¬*B*) , ▶ ¬◦*B* ↔ *B* ∧ ¬*B* . But It must be stressed that CI cannot define the connective ◦(*.*)*.* In particular, the

▶ ¬(*B* ∧ ¬*B*) → ◦*B is not* CI-provable.

1. Paraconsistent Arithmetic with a *local consistency*

*operator*

The system of CI-*based Paraconsistent Recursive Arithmetic* PRACI is so defined. The language of PRACI is that of Primitive Recursive Arithmetic PRA, plus the monadic propositional connective ◦(*.*) of the C-systems. For the proof theory of classical Arithmetic we refer to [[4](#_bookmark5),[6](#_bookmark8),[17](#_bookmark19),[19](#_bookmark21),[23](#_bookmark24)].

We choose a version of PRA with the only predicate = (*., .*), the individual constant 0, symbols for numerals, and a function letter for each primitive recursive function. We assume as identified the writings = (*t, s*) and *t* = *s*. All the primitive recursive predicates *R* different from = (*., .*) are expressed by their characteristic function *XR,* and *XR*(*t*1*, ..., tn*)=1 means that *R*(*t*1*, ..., tn*) holds, *XR*(*t*1*, ..., tn*)= 0 means that *R*(*t*1*, ..., tn*) does not hold. Moreover, we establish that each proper axiom set we shall present in this work is closed under term substitution.

PRACI is given by the system CI plus the set AxPRACI of proper axioms and the rule Ind.

AxPRACI is the following axiom set:

* 1. Arithmetical axioms defining primitive recursive function (in the following sequents all the explicitly indicated variables *xi*, *yj*, are free):

1j)Definitions of the basic recursive functions (*zero function, successor function, projection function*):

▶ *Zk* (*x*1*, ..., xk*)=0 for each *k* ≥ 1 (*Zk zero* function);

*S* (*x*)=0 ▶ (*S successor* function);

*S*(*x*)= *S*(*y*) ▶ *x* = *y*

▶ *Pk* (*x*1*, ..., xk*)= *xi* for each *k* ≥ 1 ,*i* ≤ *k* (*Pk projection* function);

*i* *i*

1jj) Composition schema:

▶ *f* (*x*1*, ..., xm*)= *h* (*g*1 (*x*1*, ..., xm*) *, ..., gm* (*x*1*, ..., xm*))

where *g*1*, ...., gm* are *n*-ary function letters and *h* is a *m*-ary function letter.

1jjj) Recursion schema :

▶ *f* (*x*1*, ..., xn,* 0) = *g* (*x*1*, ..., xn*)

▶ *f* (*x*1*, ..., xn,S* (*y*)) = *h* (*x*1*, ..., xn, y,f* (*x*1*, ..., xn, y*))

where *g* is a *n*-ary function letter and *h* is a *n* + 1-ary function letter.

* 1. Equality axioms:

▶ *x* = *x*

*x*1 = *y*1*, ..., xn* = *yn* ▶ *f* (*x*1*, ..., xn*)= *f* (*y*1*, ..., yn*)

*x*1 = *y*1*, x*2 = *y*2*, x*1 = *x*2 ▶ *y*1 = *y*2

*Convention*:

for each natural number *m* we write *m* for the term *S...S*(0)*,* with *m* occurences of *S* on the left. The terms of the form *m* are called *numerals*. We will briefly write 1*,* 2*, ...,*for numerals, each time no confusion arises.

The induction rule Ind of PRACI has the following form:

*F* (*x*) *,X* ▶ *Y, F* (*S* (*x*))

*F* (0) *,X* ▶ *Y, F* (*t*)

Ind

where *F* (*x*) is an atomic formula; the free variable *x*, called the eigenvariable of the

rule, does not occur in *X, Y, t* ; *t* is an arbitrary term which we say *introduced by* Ind; *F* (0) *,F* (*t*) are the *principal formulas* of Ind; *F* (*x*), *F* (*S* (*x*)) are the *auxiliary formulas* of Ind.

in the schemata of AxPRACI Moreover, the axiom *S* (*x*) = 0 ▶ gives rise to Note that the usual definitions by recursion of sum and product are included a set of *bottom particles* (see [[9](#_bookmark11)]) for PRACI, so that any formula of the form

*S*(*t*) = 0 trivializes PRACI. We note that the deduction apparatus of classical Primitive Recursive Arithmetic PRA is given by the classical predicate calculus LK plus AxPRACI and the rule Ind. *Classical full Arithmetic* PA is the ex- tension of PRA by induction rules admitting arbitrary induction formulas, and analogously is defined the CI-*based Full Paraconsistent Arithmetic* PACI*.* Note that the PRACI-language strictly includes the PRA-language due to the occur- rence of the connective ◦(*.*). The system of *Paraconsistent Recursive Arithmetic* PCA introduced in [[1](#_bookmark4)] is the subsystem of PRACI obtained by restricting CI to BC. We recall that a term *t* of the PRACI-language is called a *ground term* if no

for the class of formulas ∃*xA*(*x*) with *A* ∈ Δ0*,* Π1 for the class of formulas ¬*A* with variables occur in it. We write Δ0 for the class of primitive recursive relations, Σ1 *A* ∈ Σ1 (see also [[6](#_bookmark8),[22](#_bookmark25)]). Note however that, e.g., the class Π1 is not the same as

in the classical case since the translation into a prenex form fails in PRACI. We

will write N for the standard model of PRA and N for the natural number set. As usual, at the metatheoretic level of the discourse, we assume the consistency of classical Arithmetic.

including AxPRACI, where W ∈ {LK*,* BC*,* CI}, it is important to establish In order to study arithmetical theories T of the form W+ AxT + Ind, AxT whether the proofs in T admits of the elimination of some class of cut inferences. In

[[1](#_bookmark4)] we have already considered the case with W ≡ BC. According to the canonical exposition of Buss [[5](#_bookmark3)], p 43, we employ the notion of *free cut* :

Definition 3.1 Let P be a proof in W+ AxT + Ind*,* W ∈ {LK*,* BC*,* CI} *.* We say that a formula occurrence *B* in P is *anchored* if *B* is the direct descendant either of

a formula occurring in an initial sequent belonging to AxT or of a principal formula of an induction rule. A cut inference in P is called a *free cut* if either : i) the cut formula is not atomic and both the cut formula occurrences in the premises are not anchored, or ii) at least one cut formula occurrence in the premises is not anchored and is introduced in P by weakenings or logical axioms only. A cut inference which is not free is said to be *anchored*.

In Buss [[5](#_bookmark3)], pp. 43-47, the following result is proven:

Theorem 3.2 *(Classical free-cut elimination theorem) Let* T ≡ LK+ AxT + Ind*,*

*be a theory of arithmetic with* AxT *closed under term substitution. Then* T *admits*

*of free-cut elimination.*

The following is a classical consequence of the theorem mentioned above:

Proposition 3.3 *Cut-elimination of non atomic cuts holds in* PRA*.*

The extension of the free-cut elimination theorem to the system PRACI holds:

Lemma 3.4 *Let* U ≡ CI + AxV *with* AxV *closed under term substitution. Then*

U *admits of free-cut elimination.*

Theorem 3.5 *(Free-cut elimination theorem) Let* T ≡ CI+AxT+Ind *be a theory of arithmetic with* AxT *closed under term substitution. Then* T *admits of the*

*elimination of free cuts.*

For the sake of brevity we cannot present the proofs here.

Corollary 3.6 *Cut-elimination of non atomic cuts holds in* PRACI*.*

Definition 3.7 Let T ≡ U + AxT*,* U ∈ {PRA*,* PCA*,* PRACI}, AxT possibly empty axiom set. Then a proof P in T is called *normal* if free cuts do not occur

in P.

Lemma 3.8 *Let X* ▶ *Y be a sequent such that X and Y are not both empty and include at most atomic formulas. Then X* ▶ *Y is* PRACI*-provable if and only if it is* PRA*-provable.*

Proof. If *X* ▶ *Y* is PRACI-provable, then, it admits of a normal PRACI-proof the other hand, if *X* ▶ *Y* is PRA-provable it admits of a normal PRA-proof P in Q in which only atomic formulas occur. Therefore, Q is a PRA-proof too. On which only atomic formulas occur, and P is also a PRACI-proof.

Let’s conclude the section by checking the non triviality and the negation con- sistency of PRACI:

Definition 3.9 We call *reduced translation* of a PRACI-formula *B* into PRA the

form ◦*F* with ¬(*F* ∧ ¬*F* )*.* We call reduced PRA-translation of a PRACI-tree *Q* formula *B*∗ obtained from *B* by replacing each occurrence of a *B*-subformula of the

the tree *Q*∗ in the conservative extension PRACI + ¬− *L*1 obtained from *Q* by replacing each formula with its reduced translation.

Proposition 3.10 PRACI *is non trivial and negation consistent.*

reduced PRA-translation, a proof tree Q\* of the empty sequent in PRA + ¬− *L*1: Proof. From a PRACI-proof tree Q of the empty sequent we would obtain, by the indeed each *R* Ci instance is translated into a ¬− *R* rule. But this is absurd under

our assumptions.

We remark that, through the reduced translation, we also obtain a kind of interpetation of PRACI into the standard model N. However *this is not a proper semantics for* PRACI, which, for example, must falsify an infinity of formulas

¬(*B* ∧ ¬*B*) . PRACI-semantics could be defined as an extension of the semantics

for Ci presented in [[7](#_bookmark9)], but a different approach is proposed in [[13](#_bookmark15)].

# Paraconsistency properties of PRACI

No contradiction in the *classical* language is either proved or rejected by BC or CI. However, in the arithmetical systems PCA and PRACI we have a more complex situation since, as already discussed in [[1](#_bookmark4)], PCA rejects the identifications between

different numbers. Since this is guaranteed by the axiom *S* (*x*)= 0 ▶*,* PRACI too rejects the identification between different numbers*.* We call *numerical absurdity*

any atomic formula *s* = *t* where *s* and *t* are ground terms which are respectively

is trivialized by each sequent ▶ *s* = *t*, where *s* = *t* is any numerical absurdity, and PRACI-provably equal to *m* and *n*, with *m, n* different numerals. Thus PRACI furthermore by a lot of classical contradictions involving numerical absurdities:

Proposition 4.1 PRACI *is trivialized by sequents of the form* ▶ (*m* = *n*)∧¬(*m* = *of the form* ▶ (*A* ∧ ¬*A*)*, where only the classical connectives at most occur in A. n*)*, where m* = *n is a numerical absurdity and, moreover, by an inﬁnity of sequents* Theorem 4.2 *Let* ▶ ¬(*A* ∧ ¬*A*) *be a* PRACI*-provable sequent such that in A*

*classical connectives at most occur, through a normal proof Q. Then* ¬(*A* ∧ ¬*A*)

*must be a descendant in Q of a sequent s* = *t* ▶ *such that s* = *t has at least one instance which is a numerical absurdity, and atomic formulas having numerical*

*absurdities as instances necessarily occur in A.*

The proofs are similar to that of the analogous theses for PCA (see [[1](#_bookmark4)]). Of course, PRACI maintains many relevant paraconsistency properties already owned by PCA:

Theorem 4.3 *Sequents of the form* ▶ (*m* = *m*) ∧ ¬(*m* = *m*)*, m any numeral, do not trivialize* PRACI*.*

Proposition 4.4 *Sequents of the form* ▶ ¬(*t* = *t* → *t* = *t*)*,t any term, do not trivialize* PRACI*.*

Theorem 4.5 *Sequents of the form* ▶ Pr*V*(#*A*∧¬*A*)∧¬ Pr*V*(#*A*∧¬*A*) *, where* V *is any non trivial recursively axiomatized system extending* PRACI*, do not trivialize*

PRACI*.*

Proposition 4.6 *There is a denumerable inﬁnity of formulas A such that* PRACI

*does not prove any sequent of the form* ▶ ¬(*A* ∧ ¬*A*)*.*

For the sake of brevity we do not present the proofs of the theses above. The

reader can see the similar proofs of the corresponding results for PCA in [[1](#_bookmark4)]. How- ever, as expected, the paraconsistency properties of PRACI cannot be the same as for PCA*,* for example:

Proposition 4.7 *Each sequent of the forms* ▶ ◦*F* ∧ ¬◦*F or* ◦◦*F* ▶ *trivializes*

PRACI *and does not trivialize* PCA*.*

Proof. Consider the following PRACI-proof:

◦*F* ▶ ◦*F*

◦*F,* ¬◦*F* ▶

* *F* ∧ ¬◦*F* ▶ . As to the second part

of the thesis, it follows from the properties of normal proofs in PCA*.*

# Global Selfreference and Provability Logic of PRACI

In [[1](#_bookmark4)] we have already illustrated the formalization of metatheory inside PCA. As to the basic notions and properties the extension to PRACI is straightfor- ward.The writing #*E* stands for the g¨odel-number of any expression *E* of the PCA-language. Therefore, we can define a binary primitive recursive predicate *Prov*PRACI(*., .*) such that *Prov*PRACI(*m, n*) holds iff *m* is the g¨odel-number of a PRACI- proof of the sentence with g¨odel-number *n*. We recall that in our language each primitive recursive predicate *R* is expressed by the characteristic function *XR*, so that we employ the fuction *X*Pr ov−PRACI, and *Prov*PRACI(*m, n*) corresponds to *X*Pr ov−PRACI(*m, n*) = 1 in the PRACI-language. However, we will briefly write *Prov*PRACI(*m, n*) for *X*Pr ov−PRACI(*m, n*) = 1, and so on. The

formula ∃*yX*Pr ov−PRACI(*y,* #*B*) = 1 means “the sentence *B* is PRACI-provable”

and we also write it as PrPRACI (#*B*). [2](#_bookmark1) In general if *K* is a recursive relation be-

tween terms *t*1*, ..., tn* we formally express it by *XK* (*t*1*, ..., tn*)= 1*, XK* characteristic function, and its recursive complementary relation by *XK* (*r*1*, ..., rn*)= 0: *note that we do not establish in* PRACI *any link between XK* (*r*1*, ..., rn*)=0 *and the formula*

¬(*XK* (*r*1*, ..., rn*) = 1), since, as happens for PCA, the *logical* PRACI-negation is

boolean. PrPRACI (·) is the non recursive Σ1-provability predicate of PRACI. We not the boolean negation. Conversely, in the classical PRA the logical negation is recall that a slightly different canonical predicate PrPRACI[*.*] is definable, such that

if *B* is any open formula with free variables *x, y, ...*, then PrPRACI[#*B*] has the same *x, y, ...*as free variables and means “the formula *B* is PCA-provable”; PrPRACI[#*B*] coincides with PrPRACI (#*B*) if *B* is closed. We briefly write PrPRACI (#*B*) even if *B* is open, with the convention that PrPRACI (#*B*) has the same free variables

2 As to the classical Provability Logic see [[2](#_bookmark6),[3](#_bookmark7),[21](#_bookmark23)]. For a possible conjectural approach see [[18](#_bookmark20)]. For the links with modal logic see [[14](#_bookmark16),[15](#_bookmark17),[16](#_bookmark18),[20](#_bookmark22),[22](#_bookmark25)].

PrPRACI (#*X* ▶ *Y* ) means “sequent *X* ▶ *Y* is PRACI-provable” and it is evident, as *B*. We extend in a straightforward way the godelization to sequents, and so the same meaning as PrPRACI (# ▶ *B*). For each recursively axiomatized theory T since we are working with sequent formulated systems, that PrPRACI (#*B*) has of the form PRACI + AxT, AxT proper axiom set, we can analogously define a

provability predicate PrT (*.*) for T.

As to the PRACI-representation of the non triviality of any paraconsistent rec. axiom. system T extending PRACI, we remark that the situation is different with respect to the PRA-representation of the consistentcy of any classical rec. axiom.

formulas stating the consistency of U, that are all PRA-equivalent to ¬ PrU (# ▶), system U extending PRA. In the classical case we have an infinity of different non PRACI-provable sequents or formulas, then both ¬ PrT (#*S*) and ¬ PrT (#*F* ) which we also write *Con*(U)*.* In the paraconsistent case, if *S* and *F* are two different even if *S* and *F* are PRACI-equivalent. Moreover, observe that ¬ PrT (#*F* ) is a express the non-triviality of T, *but they are not* PRACI-*equivalent*, and this holds form ∀*xA*(*x*)*.* We call *global selfrefence sentences of* PRACI those including also Π1-formula in our paraconsistent sense, i.e. it cannot be equivalently written in the subformulas of the form PrT (#*S*) *,* with T any PRACI-extension.

The deep difference between paraconsitent and classical selfreference is an- nounced by the following *non-transparency* results:

*the sequent r* = *t, B*(*r*) ▶ *B*(*t*) *is in general not* PRACI*-provable, so that* PRACI Proposition 5.1 *Let B*(*z*) *be any formula in which the free variable z occurs; then is not transparent.*

PRACI-unprovability of sequents of the form *r* = *t,* ¬(*f* (*r*) = *k*) ▶ ¬(*f* (*t*) = *k*)*,* The proof is similar to that for PCA (see [[1](#_bookmark4)]). An elementary example is the with *r, t, k* suitable terms, *f* any unary function letter.

Also the local consistency connective ◦(*.*) has a non transparent behaviour, that

is:

Proposition 5.2 *Sequents of the form x* = *y,* ◦*A*(*x*) ▶ ◦*A*(*y*)*, such that x, y are different free variables and A*(*z*) *is any atomic formula with z free variable, are in*

*general not* PRACI*-provable.*

Proof. Suppose ad absurdum that *x* = *y,* ◦(*Z*(*x*)= 0) ▶ ◦(*Z*(*y*)= 0) is the root of a normal PRACI-proof Q, where *Z* is the zero-function. By free-cut elimination

property, by recalling that all non-logical axioms of PRACI are sequents of atomic formulas, and that the PRACI-induction acts on atomic formulas only, each ances-

and never is the constraint formula of a ¬− *L*3 rule . If we delete such wakenings, tor in Q of ◦(*Z*(*x*) = 0) in the antecedent must be introduced by weakenings only, get a normal proof Q’ of *x* = *y* ▶ ◦(*Z*(*y*)= 0)*.* We assume that in Q and Q’ all the and also delete weakenings introducing atomic formulas, in the most general case we in the most general case, a *R* Ci −premise of the form *x* = *y, Z*(*y*)=0 ∧ ¬*Z*(*y*)=0 atomic cuts occur above any propositional rule, so that the premise of the Q’-root is,

▶ . Then, by the free-cut elimination, we have that *x* = *y, Z*(*y*) = 0 ▶ must be

provable. Since ▶ *Z*(*y*) = 0 is a PRACI-axiom, we get *x* = *y* ▶ as a theorem, which is absurd.

The definitive arguments to conclude that ¬ and ◦(*.*) are in PRACI intensional connectives will be given in the sequel through provability logic statements. First,

we mention what standard provabilty logic statements are preserved in PRACI:

Proposition 5.3 *If* PRACI *proves* ▶ *B, then* PRACI *proves* ▶ PrPRACI (#*B*)*.* PrPRACI (#*A* → *B*) ▶ PrPRACI (#*A*) → PrPRACI (#*B*); *(D2) Moreover, the following sequents are* PRACI *-provable: (D1)* PrPRACI (#*A* → *B*) ∧ PrPRACI (#*A*) ▶ PrPRACI (#*B*); *(D3)* ▶ PrPRACI (#*A* ∧ *B*) ↔ PrPRACI (#*A*) ∧ PrPRACI (#*B*); *(D4)*PrPRACI (#*B*) ▶

PrPRACI (# PrPRACI (#*B*))*.*

Furthermore, it must be remarked that the proof-strength of PRACI is relevant, for example:

Theorem 5.4 PRACI *proves the consistency (non triviality) of each consistent (non trivial) ﬁnitely axiomatized subsystem of* PRA *(of* PRACI*).*

Proof. Let V be any finitely axiomatized subsystem as mentioned in the thesis. By fundamental results of classical proof-theory (see [[19](#_bookmark21)]) we have that PRA proves

▶ ¬ PrV (# ▶) and then PrV (# ▶) ▶ *.* By the predicate calculus LK, the atomic sequent *Prov*V(*a,* # ▶) ▶ , *a* free variable, is PRA-provable. Then, by Lemma 3.8, it is PRACI-provable too.

*Then* PRACI *proves* ▶ ◦ PrV (# ▶) *.* Corollary 5.5 *Let* V *be any ﬁnitely axiomatized subsystem of* PRA *or* PRACI*.*

Notwithstanding such meaningful proof power, the main tool of classical Prov- ability Logic fails in PRACI. We recall that Go¨del’s Diagonal Lemma for PRA (see e.g. [[21](#_bookmark23),[22](#_bookmark25)]) states that:

*the free variable u occurs. Then there is a formula B such that A*(#*B*) ↔ *B is* Lemma 5.6 *(Classical Diagonal Lemma) Let A*(*u*) *be a* PRA*-formula in which* PRA*-provable. Moreover, A*(#*B*) *and B have exactly the same free variables.*

We have that:

Theorem 5.7 *Classical Diagonal Lemma does not hold for* PRACI*.*

Theorem 5.8 *(Paraconsistent diagonal lemma for* PRACI*) Let A*(*u*) *be a*

PRACI*-formula in which the free variable u occurs. Then there are a formula*

*B and a ﬁnite set* {◦*Ds*}

*s*=1*,...,d*

*of local consistency assertions such that:*

1. *the sequent* {(∀*xj*)◦*Ds*}

*s*=1*,...,d*

▶ *A*(#*B*) ↔ *B is* PRACI*-provable;*

1. *A*(#*B*) *and B have exactly the same free-variable set V and the free variables*

*occurring in* {(∀*xj*)◦*Ds*}

*s*=1*,...,d*

*belong to V* ;

1. *each formula Ds can be obtained by term renaming from a proper subformula*

*Gs of A*(*u*) *such that Gs has not the form* ◦*F and* ¬*Gs too is an A*(*u*)*-subformula.*

The classical form of diagonal lemma can be re-obtained in PRACI for a sig- nificant particular case, due to the PRACI- provability of some classes of local consistency assertions:

*curs, such that each negated A*(*u*)−*subformula has the form* ¬◦*G. Then the thesis* Proposition 5.9 *Let A*(*u*) *be a* PRACI*-formula in which the free variable u oc- of the Diagonal Lemma holds in* PRACI *for A in the classical way.*

The lack of diagonal lemma in its standard form shows that the proofs of G¨odel’s theorem cannot be the same as in the classical case. On the other hand, the an- tecedent of Paraconsistent Diagonal Lemma shows the relevant role of local consis- tency assertions in providing new formulations of important principles of classical selfreference.

We know that in the classical case *A* ▶ PrPRA (#*A*) is PRA-provable for each *A* which is a Σ1−formula (see [[2](#_bookmark6),[22](#_bookmark25)]). In the paracosistent case such property in general does not hold:

Proposition 5.10 *The sequent B* ▶ PrPRACI (#*B*) *is in general not* PRACI*- quantiﬁer free non negated formula such that* ▶ ¬*A is not* PRACI*-provable, then provable for any arbitrary quantiﬁer free formula B; in particular, if A is any*

¬*A* ▶ PrPRACI (#¬*A*) *is not* PRACI*-provable.*

Proposition 5.11 *If* ◦*B is not a* PRACI−*theorem, then* ◦*B* ▶ PrPRACI (#◦*B*)

*is not* PRACI*-provable.*

Proof. Assume ad absurdum that a normal PRACI-proof Q of ◦*B* ▶

PrPRACI (#◦*B*) exists. By free-cut elimination, ◦*B* in the antecedent can be

straint formula of a ¬ − *L*3 rule. If we delete such weakenings we have that introduced in Q by weakenings only, and no ancestor of it can be the con-

▶ PrPRACI (#◦*B*) is provable. Then, since PRACI is not trivial, a normal proof

P of ▶ PrPRACI (#◦*B*) exists, such that the root succedent is introduced in P also by a ∃-*R* inference having the premise of the form: ▶ *X*Pr ov−PRACI(*t*1*,* #◦*B*) = 1*, ...., X*Pr ov−PRACI(*tm,* #◦*B*) = 1*,* with *t*1*, ..., tm*, closed terms, *m* ≥ 1. But, by hypotheses, each formula in such premise is a false ground recursive relation, and

the normal proof of such premise is a PRA-proofs too. This is absurd, since PRA

does not prove any disjunction of false ground recursive relations.

Therefore, the PRACI-logical negation ¬ anf the local consistency connective

for example, ¬1= 1 ▶ PrPRACI(#¬1 = 1) and ◦(1 = 1) ▶ PrPRACI(#◦(1 = 1)) are ◦(*.*) are intensional, and, moreover, cannot be expressed through Σ1-formulas. Thus, relation the property is preserved , that is *X*=(*x, y*) = 0 ▶ PrPRACI(#*X*=(*x, y*) = not PRACI-provable. Observe however that for the boolean negation of a recursive 0)*, x, y*, free variables, is PRACI-provable.

We wish to emphasize the fact that the sentences expressing the non triviality of a PRACI-based system T are in general not PRACI-equivalent. In particular:

Proposition 5.12 *Let B any non* T*-provable sentence of a paraconsistent system*

T ≡ PRACI + AxT *whose non-triviality is not* PRA*-provable. Then neither*

¬ PrT(# ▶) ▶ ¬ PrT(#*B*) *nor* ¬ PrT(#*B*) ▶ ¬ PrT(# ▶) *are* PRACI*-provable.* Proof. First, we establish that, if *S* is any non T-provable sequent, then PrT(#*S*) ▶ CI, *Prov*T(*a,* #*S*) ▶ in PRACI and then, by Lemma 3.8, *Prov*T(*a,* #*S*) ▶ should is not PRACI-provable. Indeed, if so it is we would have, by predicate calculus be PRA-provable, against the hypotheses. Thus, since ▶ ¬ PrT(#*S*) cannot be

PRACI-provable, the thesis follows from free-cut elimination.

lent roles w.r.t. classical PRA. For example, ¬ PrPRACI(# ▶) ↔ ¬ PrPRA(# ▶) is Remark that the non-triviality statements mentioned above do not play equiva- PRA-provable, but ¬ PrPRACI(#¬(1 = 1 ∧¬1= 1)) → ¬ PrPRA(#¬(1 = 1 ∧¬1=

1)) is false in the standard model N.

# Relations between local and global consistency in PRACI-Provability Logic

Proposition 6.1 *The schema* ◦*A* → *A makes* PRACI *trivial.*

Proof. Consider any formula *B* which has not the form ◦*F* an such that *B* ▶ is

PRACI-provable. Then, by ∧− *R* we get *B* ∧ ¬*B* ▶ from which, by *R* Ci*,* we have

▶ ◦*B.* Thus, from the schema-instance ◦*B* → *B* we get ▶ *B,* that with *B* ▶ produces the empty sequent.

Proposition 6.2 *Let B any sentence such that B* ▶ PrPRACI(#*B*) *is* PRACI*- provable. Then the foollwing sequents are* PRACI*-provable: i)* ▶ PrPRACI(#*B*) ∨

* *B and ii)* ▶ ¬◦*B* → PrPRACI(#*B*)*.*

and ¬− *L*4*.*  Proof. Both theses follow from the application of the proper PRACI rules *R* Ci

Note that the propositions above do not hold for PCA.

*form* ◦*F. Then, if* CI *proves* ▶ ◦*B, the formula B is not* CI*-provable.* Lemma 6.3 *Assume that the formula B is not a negated formula and has not the*

Proof. By the cut elimination for CI-proofs, the proof Q of ▶◦*B* must have a *R* Ci as end-rule, with premise *B* ∧ ¬*B* ▶ *.* By cut elimination, this implies that either *B* ▶ or ¬*B* ▶ is CI-provable. The latter case would be absurd, by the hypotheses on *B* and by cut elimination. Then, *B* ▶ is CI-provable. Therefore, if ▶ *B* would be CI-provable we obtain the empty sequent, against the cut-elimination for CI.

If *cut* − *el*(CI) is the formula representing in the PRA-language the cut- elimiantion property of CI, we have:

Lemma 6.4 *The following sequents are both* PRA*- and* PRACI*-provable: i)* ▶

*cut* − *el*(CI)*; ii)* ▶ ¬ PrCI(# ▶)*.*

Note that for CI the non triviality is a straightforward consequence of cut-

elimination. Conversely, it is obvious that the stated free-cut elimination for PRA

and PRACI does not imply their consistency or non-triviality.

Theorem 6.5 *(Fundamental relation between local and global consistency) The proof of Lemma 6.3 can be translated into* PRA *and* PRACI *so that :*

▶ PrCI(#○*B*) → ¬ PrCI(#*B*)

*is* PRACI*-provable for each sentence B which has not the form* ○*F.*

Proof. The starting point is the PRA-provability of ▶ PrCI(#○*B*) → ¬ PrCI(#*B*), due to the standard representation power of PRA (see [[21](#_bookmark23),[22](#_bookmark25)]). Then, it can be

shown that the PRA-proof must also be a PRACI-proof.

The *fundamental relation* connects the provability of a local consistency asser- tion with the global non triviality of the system. An important fact is that the fundamental relation is true for PRACI too:

*does not prove the empty sequent. Then, if* PRACI *proves* ▶ ○*B, the formula B is* Lemma 6.6 *Assume that the formula B has not the form* ○*F, and that* PRACI *not* PRACI *-provable.*

Q of ▶ ○*B* all atomic cuts occur above each logical rule. Then, recalling that ○*B* Proof. Without any loss of generality we can suppose that in the normal proof cannot be introduced in Q by Induction rules, we proceed as in the proof of Lemma

6.3, by employing free-cut elimination.

The main goal is to select a minimal constructive PRACI-extension that proves the fundamental relation for PRACI.

Let *free*−*cut*−*el*(PRACI) be the formula representing the free-cut elimination property for PRACI. Then we have the following preliminary fact:

Proposition 6.7 *Assume that the formula B has not the form* ○*F, and that* PRACI *does not prove the empty sequent. Then the proof of Lemma 6.6 can be translated into* PRA *and* PRACI *so that :*

*free* − *cut* − *el*(PRACI) ∧ ○ PrPRACI(# ▶) ∧¬ PrPRACI(# ▶

) ∧ PrPRACI(#○*B*) ▶ ¬ PrPRACI(#*B*)

*is* PRACI*-provable.*

A *weakened Hilbert’s program* could be so declared: to find a very weak and constructive PRACI-extension W such that the *fundamental relation for* PRACI,

i.e. the sequent PrPRACI(#○*B*) ▶ ¬ PrPRACI(#*B*), is W-provable. Thus, the problem of proving the non triviality of PRACI (i.e a *global* selfreference statement)

could be reduced to the provability of suitable *local* consistency assertions−that is,

PRACI *almost* would establish its own non triviality. In essence, the conjecture

is the following: the CI-based paraconsistent arithmetical systems have, w.r.t non- triviality, more constructive and efficient proof capabilities than that owned by classical atithmetical systems w.r.t. consistency.

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