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Programming Examples Needing Polymorphic Recursion

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Abstract

Inferring types for polymorphic recursive function definitions (abbreviated to *polymorphic recur- sion*) is a recurring topic on the mailing lists of popular typed programming languages. This is despite the fact that type inference for polymorphic recursion using ∀-types has been proved un-

decidable. This report presents several programming examples involving polymorphic recursion

and determines their typability under various type systems, including the Hindley-Milner system, an intersection-type system, and extensions of these two. The goal of this report is to show that many of these examples are typable using a system of intersection types as an alternative form of polymorphism. By accomplishing this, we hope to lay the foundation for future research into a decidable intersection-type inference algorithm.

We do not provide a comprehensive survey of type systems appropriate for polymorphic recursion, with or without type annotations inserted in the source language. Rather, we focus on examples for which types may be inferred without type annotations, with an emphasis on systems of intersection- types.

*Keywords:* polymorphic recursion, intersection types, finitary polymorphism , examples

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# Introduction

*Background and Motivation*

Type inference in the presence of polymorphic recursion using ∀-types (the familiar “type schemes” of SML) is undecidable

[[10](#_bookmark13),[11](#_bookmark14),[4](#_bookmark7)]. Attempts to work around this limitation include explicit type anno- tations by the user [[8](#_bookmark11)] and user-tunable iteration limits [[17](#_bookmark20)]. However, both of these approaches require the programmer to be actively engaged in the type checking process, thereby defeating the goal of automatic type inference and transparent type checking. There is also an implementation of SML that al- lows the user to switch between the standard type system (which is restricted to monomorphic recursion) and a type system augmented with polymorphic recursion using ∀-types, in an attempt to prove that “hard” examples of poly- morphic recursion do not arise in practice [[1](#_bookmark4)]. Yet, practical examples of programs requiring polymorphic recursion continually appear in discussions on the mailing lists of programming languages such as SML, Haskell, and OCaml.

*Contribution of the Report*

This document attempts to lay the foundation for further research into the typability of implicit polymorphic recursion by discussing several examples which fail to type under the standard type system of SML – also called the Hindley-Milner system. The examples are written (mostly) in SML syntax (one example is presented in Haskell syntax) and are accompanied by the corresponding error found by the SML/NJ type checker. A few of the examples are also shown in Haskell syntax with its corresponding GHC error message for the side purpose of comparing the error reporting of the SML/NJ and GHC compilers.

We also discuss examples which remain untypable using the Hindley-Milner system augmented with polymorphic recursion with ∀-types – also called the Milner-Mycroft system – but are typable using an intersection-type system. These examples support the use of intersection types as an alternative to ∀- types to represent polymorphism.

In addition, we elucidate the need for what we call “infinite-width” inter- section types by examples. However, we do not extend our standard (finite- width) intersection type system in this way, because we do not know a straight- forward extension of the standard system and developing one is beyond the scope of this report. Consequently, we resort to polymorphic recursion with

∀-types for these examples; i.e., we present examples which are not typable

using our intersection-type system, but are with ∀-types. An example is also given which requires both intersection types and ∀-types. Lastly, we present a polymorphic recursive program that is not typable with either intersection types or ∀-types.

*Organization of the Report*

The paper is organized as follows. First we define the types that we deal with, and then we present the rules of several type systems we consider later in the report, starting with the Hindley-Milner system which we here denote HM; this is done in Section [2](#_bookmark2). In the remaining sections, we introduce several simple and natural examples of polymorphic recursion to motivate the augmentation of system HM. We develop type systems that allow polymorphic recursion using only ∀-types (HM∀), only intersection types (S), and both universal and intersection types (S∀). Using these systems we show that we can construct valid typing derivations for most examples. The following chart summarizes the typability of the examples developed in this report with respect to the four type systems we define. [3](#_bookmark1) The last column in the chart, with the heading “Minor Alteration”, indicates whether an example can be “easily” altered to make it typable under system HM.

3 System S is called “S” for lack of a better name.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Example | HM | HM∀ | S | S∀ | Minor Alteration |
| Double |  | C | C | C | C |
| Mycroft |  | C | C | C | C |
| Sum List |  | C | C | C | C |
| Composition |  | C | C | C | C |
| Compiler Pass |  | C | C | C | C |
| Confusing |  | C | C | C |  |
| Matrix Transpose |  |  | C | C | C |
| Vector Addition |  |  | C | C | C |
| Collect |  | C |  | C |  |
| Bar |  | C |  | C |  |
| Construct List |  |  |  | C |  |
| Delay |  |  |  |  |  |

The above table is a little misleading in the following respect. The table indicates that certain examples are typable in our system of intersection types

(S) but not in our system of ∀-types (HM∀). Whereas HM∀ restricts ∀- quantifiers to appear only in the outermost position of type expressions, S imposes no similar restriction on occurrences of ∧ in type expressions. See Section [8](#_bookmark3) for further discussion of this matter.

*Related Work*

For other examples of polymorphic recursive programs, specifically nested re- cursive data types similar to the Collect example, see Chris Okasaki’s book [[16](#_bookmark19)]. Simon Peyton-Jones and Mark Shields have written a paper describing the ap- proach taken by GHC when inferring arbitrary high rank types via explicit user-defined type annotations [[9](#_bookmark12)].

*Future Work*

In the future we plan to explore the possibility of a decidable (and hopefully feasible) type inference and checking algorithm for a system of intersection types under which most, if not all, of the examples in this report can be typed. We would also like to investigate whether introducing expansion variables, a

technology developed in conjunction with System I, into our intersection type system will yield any benefits [[14](#_bookmark17)].

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# Types and Type Systems

The syntax of types is specified by the following grammar:

*τ* ∈ Type ::= *α* | *τ* → *τ* | *τ* × *τ* | *τ* list | *τ* ∧ *τ* | int | bool | *... σ* ∈ Scheme ::= *τ* | ∀*α.σ*

Note that we use *τ* as a metavariable ranging over the set Type which com- prises simple types combined with intersection types, and *σ* as a metavariable ranging over the set Scheme which comprises all members of Type each pre- ceded by zero or more ∀ quantifiers. In particular, Type is a proper subset of Scheme.

We list the four different type systems considered in the rest of the report. The basic type system, HM, is analogous to the type system of SML, Haskell, and OCaml, which allows let-polymorphism and only monomorphic recursion. System HM∀ is an extension of system HM that allows polymorphic recursion with ∀-types, and ∀-types in general as long as the ∀ quantifiers are outside all type constructors. System S allows intersection types; S provides polymorphic recursion via intersection types. The last system that we develop is called S∀. System S∀ allows intersection types and ∀-types together; S∀ also requires that ∀ quantifiers are kept outside all type constructors.

We now outline the conventions for reading the following tables. We as- sume there exists a function, type, from term constants to types, such that the type(*c*) is the type of constant *c*. We use ∆ as a context in our typing

judgement. ∆ is a sequence of bindings between term variables and types. However, we also allow ∆ to act as a function from term variables to types, such that ∆(*x*) is the type bound to variable *x*. Lastly, we use the function

FTV from contexts to sets of type variables, such that FTV(∆) is the set of

free type variables that occur in context ∆. First we define system HM.

*System* HM *Typing Rules:* (all types are ∧-free)

type(*c*) = *σ*

∆ ▶ *c* : *σ*

(∀-Const)

∆(*x*) = *σ*

(*σ* closed)

∆ ▶ *x* : *σ* (∀-Var)

∆ ▶ fn *x* =*> M* : *τ* → *τ* ' (Abs)

∆*,x* : *τ* ▶ *M* : *τ* '

∆ ▶ *M* : *τ* → *τ* ' ∆ ▶ *N* : *τ*

∆ ▶ *MN* : *τ* '

(App)

∆ ▶ *M* : *σ* ∆*, x* : *σ* ▶ *N* : *τ* (∀-Let)

∆ ▶ let *x* = *M* in *N* end : *τ*

∆*, x*1 : *τ*1*,... , xn* : *τn* ▶ *N* : *τ*

∆*, x*1 : *τ*1*,... , xn* : *τn* ▶ *Mp* : *τp*

∆ ▶ let val rec *x*1 = *M*1 and *...*

and *xn* = *Mn* in *N* end : *τ*

(Rec)

(1 ≤ *p* ≤ *n*)

∆ ▶ *M* : *σ* (Gen)

∆ ▶ *M* : ∀*α.σ*

(*α* /∈ FTV(∆))

∆ ▶ *M* : ∀*α.σ* (Inst)

∆ ▶ *M* : *σ*[*α* := *τ* ]

∆ ▶ *M*1 : *τ*1 ∆ ▶ *M*2 : *τ*2 (×)

∆ ▶ (*M*1*, M*2) : *τ*1 × *τ*2

∆ ▶ *M* : *τ*1 × *τ*2 (Fst)

∆ ▶ fst(*M* ) : *τ*1

∆ ▶ snd(*M* ) : *τ*2

∆ ▶ *M* : *τ*1 × *τ*2 (Snd)

∆ ▶ *M*1 : bool ∆ ▶ *M*2 : *τ* ∆ ▶ *M*3 : *τ* (If)

∆ ▶ if *M*1 then *M*2 else *M*3 : *τ*

To define system HM∀ we simply augment HM with the rule (∀-Rec).

*System* HM∀ *Typing Rules:* (all types are ∧-free)

All the typing rules of system HM are typing rules of system HM∀ in

addition to the following.

∆*, x*1 : *σ*1*,... , xn* : *σn* ▶ *N* : *τ*

∆*, x*1 : *σ*1*,... , xn* : *σn* ▶ *Mp* : *σp*

∆ ▶ let val rec *x*1 = *M*1 and *...*

and *xn* = *Mn* in *N* end : *τ*

(∀-Rec)

(1 ≤ *p* ≤ *n*)

Note that we allow both rules (∀-Rec) and (Rec) to co-exist within system HM∀. This is acceptable because (Rec) is simply a special case of (∀-Rec). System S uses only intersection types.

*System* S *Typing Rules:* (all types are ∀-free)

type(*c*) = *τ*

∆ ▶ *c* : *τ* (∧-Const) (*τ* closed)

∆(*x*) = *τ*

∆ ▶ *x* : *τ* (∧-Var)

∆*,x* : *τ* ▶ *M* : *τ* '

∆ ▶ fn *x* =*> M* : *τ* → *τ* ' (Abs)

∆ ▶ *M* : *τ* → *τ* ' ∆ ▶ *N* : *τ*

∆ ▶ *MN* : *τ* ' (App)

∆ ▶ *M* : *τ* ' ∆*,x* : *τ* ' ▶ *N* : *τ*

∆ ▶ let *x* = *M* in *N* end : *τ* (∧-Let)

∆*, x*1 : *τ*1*,... , xn* : *τn* ▶ *N* : *τ*

∆*, x*1 : *τ*1*,... , xn* : *τn* ▶ *Mp* : *τp*

∆ ▶ let val rec *x*1 = *M*1 and *...*

and *xn* = *Mn* in *N* end : *τ*

(∧-Rec)

(1 ≤ *p* ≤ *n*)

∆ ▶ *M*1 : *τ*1 ∆ ▶ *M*2 : *τ*2 (×)

∆ ▶ (*M*1*, M*2) : *τ*1 × *τ*2

∆ ▶ *M* : *τ*1 × *τ*2

∆ ▶ fst(*M* ) : *τ*1

(Fst) ∆ ▶ *M* : *τ*1 × *τ*2

∆ ▶ snd(*M* ) : *τ*2

(Snd)

∆ ▶ *M* : *τi i* ∈ *I*

∆ ▶ *M* : ∧*i*∈*I τi*

(∧)

(size(*I*) ≥ 2), (size(*I*) is finite)

∆ ▶ *M* : *τ τ* ≤ *τ* '

∆ ▶ *M* : *τ* ' (Sub)

*τ* ≤ *τ*

(S-Refl) *τ*1 ≤ *τ*2 *τ*2 ≤ *τ*3

*τ*1 ≤ *τ*3

(S-Trans)

*τ*1 ≤ *τ* ' *τ* ' ≤ *τ*2

*τ* ' ≤ *τ*1 *τ* ' ≤ *τ*2

1 2 (S-Fun)

*τ* ' → *τ* ' ≤ *τ*1 → *τ*2

1 2 (S-Pair)

*τ* ' × *τ* ' ≤ *τ*1 × *τ*2

1 2 1 2

*τi* ≤ *τ* ' *i* ∈ *I I* ⊆ *J*

*i*

'

(S-∧)

∧*i*∈*J τi* ≤ ∧*i*∈*I τi*

This system has been proved sound. The proof can be found in appendix B. Lastly, we define system S∀.

*System* S∀ *Typing Rules:*

All the typing rules of system S are typing rules of system S∀ in

addition to the following.

type(*c*) = *σ*

∆ ▶ *c* : *σ*

(∀-Const)

∆(*x*) = *σ*

(*σ* closed)

∆ ▶ *x* : *σ* (∀-Var)

∆ ▶ *M* : *σ* ∆*, x* : *σ* ▶ *N* : *τ* (∀-Let)

∆ ▶ let *x* = *M* in *N* end : *τ*

∆*, x*1 : *σ*1*,... , xn* : *σn* ▶ *N* : *τ*

∆*, x*1 : *σ*1*,... , xn* : *σn* ▶ *Mp* : *σp*

∆ ▶ let val rec *x*1 = *M*1 and *...*

and *xn* = *Mn* in *N* end : *τ*

(∀-Rec)

(1 ≤ *p* ≤ *n*)

∆ ▶ *M* : *σ* (Gen)

∆ ▶ *M* : ∀*α.σ*

(*α* /∈ FTV(∆))

∆ ▶ *M* : ∀*α.σ* (Inst)

∆ ▶ *M* : *σ*[*α* := *τ* ]

Note that (∧-Const), (∧-Var), (∧-Let), and (∧-Rec) in system S are special cases of (∀-Const), (∀-Var), (∀-Let), and (∀-Rec) in system S∀.

# Typable in HM∀ and S

* 1. *Double*
     1. *Double - Coupled*

The following is a simple example that exposes the untypability of polymorphic recursion in SML.

let val rec double = fn f => fn y => f (f y) and foo = fn v => double (fn x => x + 1) v and goo = fn w => double Math.sqrt w

in (foo 3, goo 16.0) end

*SML Type Checker Reports:*

Error: operator and operand don’t agree [literal] operator domain: real -> real

operand: int -> int in expression:

double (fn x => x + 1)

The definitions of double, foo, and goo are mutually recursive. Therefore the calls to double within the definition of foo and goo are recursive calls. Hence, the Hindley-Milner typing derivation breaks down with the realization that each of these recursive calls is on an argument of a different type.

This example is not typable under system HM. However, we can use either HM∀ or S to type it. Using HM∀ we can write a typing derivation for this example, where the final types assigned are:

double : ∀*α.*(*α* → *α*) → *α* → *α*

foo : int → int

goo : real → real*.*

Using S we can also type this example. If double is given the following intersection type:

((int → int) → int → int) ∧ ((real → real) → real → real)

then the call to double within the body of foo would be able to utilize the first component of the intersection type and the call to double within the body of goo would be able to use the second component. We hold off on a typing derivation in S until the next, more complicated, example.

* + 1. *An Aside: SML/NJ vs. GHC*

As an aside we translate a couple of the examples in this report into Haskell syntax and compare the SML/NJ error messages with the GHC error mes- sages (which uses Algorithm M in contrast to Algorithm W of SML/NJ - for more discussion see [[3](#_bookmark5)]). We choose to translate only those examples which will yield an interesting and different error message. Most of the following examples, when translated, offer error messages that are very similar to the SML/NJ error messages, but differ occasionally in the program location which the compiler targets as problematic. This example, when translated, is no dif- ferent.

intFunc :: Int -> Int

intFunc x = x + 1

doubleFunc :: Double -> Double doubleFunc x = sqrt x

myPair = let (double, foo, goo) =

(\f -> \y -> f (f y),

\v -> double intFunc v,

\w -> double doubleFunc w) in (foo 3, goo 16.0)

*GHC Type Checker Reports:*

Couldn’t match ‘Double’ against ‘Int’ Expected type: Int -> Int Inferred type: Double -> Double

In the first argument of ‘double’, namely ‘doubleFunc’ In a lambda abstraction: \ w -> double doubleFunc w

Both the SML/NJ and the GHC compiler detect the same error but SML/NJ assigns the type:

double : (real → real) → real → real*,*

while GHC assigns the type:

double : (int → int) → int → int*.*

Although this difference is not enormous, it does show an operational disparity between the two compilers.

* + 1. *Double - Uncoupled*

The problem exhibited in the double example above can be alleviated by a technique that we call “uncoupling”. Namely, we make use of the Hindley- Milner let-polymorphism by removing double from the mutual recursive defi- nition and defining it in an outer let.

let val double = fn f => fn y => f (f y)

in let val rec foo = fn v => double (fn x => x + 1) v and goo = fn w => double Math.sqrt w

in (foo 3, goo 16.0) end end

*SML Type Checker Reports:*

No Errors

* 1. *Mycroft*
     1. *Mycroft - Coupled*

The following is the canonical example of polymorphic recursion as discovered by Alan Mycroft [[15](#_bookmark18)].

let val rec myMap = fn f => fn l =>

if (null l) then l

else cons (f(hd l)) (myMap f (tl l)) and sqList = fn l => myMap (fn (x:int) => x \* x) l and compList = fn l => myMap not l

in (sqList [2,4], compList [true,false]) end

*SML Type Checker Reports:*

Error: operator and operand don’t agree [tycon mismatch] operator domain: bool -> bool

operand: int -> int in expression:

myMap (fn x : int => x \* x)

As before, we have three mutually recursive function definitions and two recur- sive calls with arguments of different types. This example, though untypable in system HM, can be typed in a system of polymorphic recursion with ∀- types or intersection types. To witness this either system must be able to handle lists. For the purposes of brevity we will consider hd, tl, cons, and nil to all be primitive constants within our language. With these constants we will be able to handle expressions with list types. Also, we note that the expressions [1,2] and [true, false] are simply syntactic sugar for

cons 1 (cons 2 nil) and cons true (cons false nil) respectively. We are now able to assign the following types under HM∀:

myMap : ∀*α.*∀*β.*(*α* → *β*) → *α* list → *β* list

sqList : int list → int list

compList : bool list → bool list*.*

With S we can assign the following rank-1 types:

myMap : ((int → int) → int list → int list) ∧

((bool → bool) → bool list → bool list)

sqList : int list → int list

compList : bool list → bool list*.*

In both systems the final type assigned to Mycroft’s example is:

int list × bool list*.*

For the full typing derivation under S see appendix A.

* + 1. *Mycroft - Uncoupled*

As before, uncoupling is possible. This is shown in a slightly different form below.

let val rec myMap = fn f => fn l =>

if (null l) then l

else cons (f(hd l)) (myMap f (tl l)) val rec sqList = fn l => myMap (fn x => x \* x) l and compList = fn l => myMap not l

in (sqList [2,4], compList [true,false]) end

*SML Type Checker Reports:*

No Errors

* 1. *Sum List*

The example below finds the sum of the elements of a list, but also applied the polymorphic identity function to each element and sublist in the process. The idea here is that we may want to record some information about each element and its corresponding sublist (possibly via side effects).

let val rec id = fn x => x and sumList = fn l =>

if (null l) then 0

else (id (hd l)) + (sumList (id (tl l))) in sumList [1,2,3] end

*SML Type Checker Reports:*

Error: operator and operand don’t agree [circularity] operator domain: ’Z

operand: ’Z list in expression:

id (tl l)

Using HM∀ we assign the following types:

id : ∀*α.α* → *α*

sumList : int list → int*.*

Using S we assign the following types:

id : (int → int) ∧ (int list → int list)

sumList : int list → int*.*

The final type assigned to this example is: int. This example can be uncoupled in the same way as the previous two examples. A natural question at this point would be to ask why id needs to be defined mutually recursive to sumList. To avoid such a question we could pass id as an argument to sumList and then motivate this move by demonstrating a need to pass two different functions to sumList. We show this for the Matrix Transpose example so we do not show it here.

* 1. *Isomorphic Compositions*

This example uses the composition function as the polymorphic recursive func- tion. The order of two composed functions are switched and applied to differ- ent arguments. The results of both applications are then compared.

let val createList = fn x => [x] val removeList = fn l => hd l

val rec comp = fn f => fn g => f o g and appComp = fn v1 => fn v2 =>

(comp removeList createList v1) = hd (comp createList removeList v2)

in appComp 5 [5] end

*SML Type Checker Reports:*

Error: operator and operand don’t agree [circularity] operator domain: ’Z list -> ’Z

operand: ’Z list -> ’Z list list in expression:

comp createList

Using HM∀ we assign the following types:

createList : int → int list

removeList : int list → int

comp : ∀*α.*∀*β.*∀*η.*(*β* → *η*) → (*α* → *β*) → *α* → *η*

appComp : int → int list → bool*.*

Using S we assign the following types:

createList : int → int list

removeList : int list → int

comp : ((int → int list) → (int list → int) → int list → int list) ∧

((int list → int) → (int → int list) → int → int)

appComp : int → int list → bool*.*

In both systems the final type assigned to this example is: bool. This example can also be uncoupled.

* 1. *Compiler Pass*

This example is very similar to the previous examples and is due to Simon Peyton Jones [[6](#_bookmark9),[7](#_bookmark10)], who states that this is a program that he “really wanted to write”. The author was writing a compiler pass which made use of two data types and three functions written in continuation passing style. It is presented in Haskell syntax.

data Exp = Let Bind Exp

data Bind = MkBind String Exp

doBinds (b:bs) = doBindAndScope b (\b’ -> b’ : doBinds bs) doExp (Let b e) = doBindAndScope b (\b’ -> Let b’ (doExp e)) doBindAndScope (MkBind s e) cont = cont (MkBind s (doExp e))

*GHC Type Checker Reports:*

Couldn’t match ‘[Bind]’ against ‘Exp’ Expected type: [Bind]

Inferred type: Exp

In the application ‘doBinds bs’

In the second argument of ‘(:)’, namely ‘doBinds bs’

The trouble with this program is that doExp and doBindAndScope are defined mutually recursive to one another. This means that the call to doBindAndScope is a recursive call and can not be polymorphic. However, doBinds and doExp each call doBindAndScope with arguments of different types. The author goes on to describe a way to alleviate this problem by encapsulating the polymor- phism inside a data type structure and adding constructors to the arguments of doBindAndScope. However, he points out that this fix is not only “obscure”, but also “inefficient at runtime”.

This example can be typed by either HM∀ or S. Under system HM∀ we can assign the following types:

doBinds : Bind list → Bind list

doExp : Exp → Exp

doBindAndScope : ∀*α.*Bind → (Bind → *α*) → *α.*

Under system S we can assign these types:

doBinds : Bind list → Bind list

doExp : Exp → Exp

doBindAndScope : (Bind → (Bind → Bind list) → Bind list) ∧

(Bind → (Bind → Exp) → Exp)*.*

Besides the method for alleviating this example already discussed, we can uncouple this program in the usual way.

* 1. *Confusing*
     1. *Confusing - Unalleviated*

The following example is not very intuitive but serves a purpose.

let val rec f = fn n => fn x => fn y =>

if x > y orelse n = 0 then n

else if n >= 100

then if n < 200 then n

else f (n div 2) (x \* y) y else if x < y

then f (n\*n) 0.03 1.0

else f (n\*n) 1 1

in f 3 5 6 end

*SML Type Checker Reports:*

Error: operator and operand don’t agree [literal] operator domain: real

operand: int in expression:

(f (n \* n)) 1

Error: operator and operand don’t agree [literal] operator domain: real

operand: int in expression:

(f 3) 5

This example requires the second and third arguments of f to be of types int and real. The example makes use of the overloaded operators *<*, *>*, and \* which are defined for both these types. Notice that if we give f the appropriate type then this example is well-typed within both HM∀ and S.

Under HM∀ we assign the following type:

f : ∀*α.*int → *α* → *α* → int*.*

Under S we assign the following type:

f : (int → int → int → int) ∧ (int → real → real → int)*.*

In both systems, the final type assigned to the example is: int.

This example differs from all the previous examples. The preceding exam- ples all make use of a polymorphic function that is defined mutually recursive to another function. The polymorphic function is then used twice on argu- ments of different types. This example is designed to show that it is possible to define a polymorphic recursive function that is inherently so, without the aid of an external polymorphic function. As a result, this example is diffi- cult to alleviate. In the next sections we will see other polymorphic recursive functions that share this same property but are impossible to type without extensions to HM∀ and S.

* + 1. *An Aside: SML/NJ vs. GHC*

It is worth noting that when translated into Haskell syntax this example can be typed by the GHC compiler. The reason for this is that the GHC compiler

converts the integers in this example to doubles and assigns the following type:

f : double → double → double → double*.*

* + 1. *Confusing - Alleviated*

We can alleviate this example by duplication. Consider the following program.

let val rec f1 = fn n => fn x => fn y =>

if x > y orelse n = 0 then n

else if n >= 100

then if n < 200 then n

else f1 (n div 2) (x \* y) y else if x < y

then f2 (n\*n) 0.03 1.0

else f1 (n\*n) 1 1 and f2 = fn n => fn x => fn y =>

if x > y orelse n = 0 then n

else if n >= 100

then if n < 200 then n

else f2 (n div 2) (x \* y) y else if x < y

then f2 (n\*n) 0.03 1.0

else f1 (n\*n) 1 1

in f1 3 5 6 end

This program is now typable under HM. We can assign the following types:

f1 : int → int → int → int

f2 : int → real → real → int*.*

However, alleviating the example in this way differs from all the previous attempts in that we must duplicate the entire program. Since duplication defeats the purpose of polymorphism this alteration cannot be recommended.

# Typable in S Only

* 1. *Matrix Transpose*
     1. *Matrix Transpose - Unalleviated*

This examples shows a concise and elegant formulation of the matrix transpose operation.

let val map1 = map

val rec map2 = fn f => fn l =>

if (null l) then nil

else if (null (hd l)) then nil

else cons (f hd l) (map2 f (f tl l)) in map2 map1 [[1,2],[3,4]] end

*SML Type Checker Reports:*

Error: operator and operand don’t agree [circularity] operator domain: ’Z list -> ’Z

operand: ’Z list -> ’Z list in expression:

f tl

This example, unlike the previous examples, cannot be typed by polymor- phic recursion with ∀-types. The problem arises when trying to type the first argument to map2, f. To see this, we need only look at the else-branch of the nested if-expression.

Notice that from cons (f hd l) (map2 f (f tl l)) the type of the first occurrence of f must be of the form:

f : (*α* list → *α*) → *α* list list → *α* list*.*

Yet, the second occurrence of f requires the form:

f : (*α* list → *α* list) → *α* list list → *α* list list*.*

Thus, f must have a polymorphic type. However, since we restrict ∀-quantifiers to be only on the outer most portion of the type, a ∀-type for map2 is impos- sible.

Fortunately, using S we are able to assign this example a rank-2 type:

map1 : ((int list → int) → int list list → int list) ∧

((int list → int list) → int list list → int list list)

map2 : (((int list → int) → int list list → int list) ∧

((int list → int list) → int list list → int list list))

→ int list list → int list list*.*

The final type assigned to this example is: int list list.

An objection made in a preliminary presentation of this work is that this example (Matrix Transpose) and the next (Vector Addition) are not cases of truly polymorphic recursive functions, because the polymorphism is not at the outermost position of the type expression, as in the previous examples. However, such a definition of polymorphic recursion is arguably too restrictive, as it disallows function types whose argument type (i.e., expressions to the left of the arrow constructor) are polymorphic.

* + 1. *Matrix Transpose - Alleviated*

Similar to the previous example, uncoupling is impossible. However, we can side-step this dilemma with another crafty trick.

let val map1 = map

val rec map2 = fn f1 => fn f2 => fn l => if (null l)

then nil

else if (null (hd l)) then nil

else cons (f1 hd l)

(map2 f1 f2 (f2 tl l)) in map2 map1 map1 [[1,2],[3,4]] end

*SML Type Checker Reports:*

No Errors

By simply passing the map2 function two different map1 functions so that each one is used with only one type, our example becomes typable. Although, this technique yields a well-typed program the process for transforming unty- pable polymorphic recursive programs has become ad-hoc. No longer, can the programmer use a simple uncoupling scheme. Instead, the programmer must come up with, possibly very complex, fixes for each circumstance. A better programming language would not require these efforts from the programmer,

but rather allow the program to be typed as the programmer wrote it. With this as our goal we reject the alleviated example as our ultimate solution and determine to type the original, unalleviated example.

As an alternative alleviation, one could simply remove the first argument, f, of map2 and replace each f in the body of map2 with the standard map function. However, there may be cases where passing map1 as an argument is advantageous. For example, consider the following. Suppose that given a matrix *M* , one wishes to compare the transpose of *M* with the transpose

of *M*¯ , where *M*¯

is defined as follows: if *m*¯ *ij* is an element (*i, j*) of *M*¯

then

*m*¯ *ij* = *mij* + *i* where *mij* is an element (*i, j*) of *M* . Then we can compute the pair (*M* T*, M*¯ T) as follows.

let val map1 = fn f => fn l => map f (map (map (fn x => x+1)) l) val rec map2 = fn f => fn l =>

if (null l) then nil

else if (null (hd l)) then nil

else cons (f hd l) (map2 f (f tl l)) in (map2 map [[1,2],[3,4]], map2 map1 [[1,2],[3,4]]) end

Otherwise, the programmer would have to compute the transpose of *M* and separately compute *M*¯ T from *M* T. A program that was implemented in this way would require significant code duplication.

* 1. *Vector Addition*

This example computes the addition of equal-length vectors represented as list.

let val addList = fn l => foldr (op +) 0 l val rec addVecs = fn f => fn l =>

if (null (hd l)) then nil

else cons (addList (f hd l)) (addVecs f (f tl l))

in addVecs map [[1,2,3],[4,5,6]] end

*SML Type Checker Reports:*

Error: operator and operand don’t agree [circularity] operator domain: ’Z list -> ’Z

operand: ’Z list -> ’Z list in expression:

f tl

This example is very similar to the Matrix Transpose example. Just has before, the f argument of addVecs requires a polymorphic type. However, since we disallow ∀-quantification within a function type, system HM∀ is not sufficient to type addVecs.

Again using S we are able to assign this example a rank-2 type:

addList : int list → int

addVecs : (((int list → int) → int list list → int list) ∧

((int list → int list) → int list list → int list list))

→ int list list → int list*.*

The final type assigned to this example is: int list.

And again, we can alleviate this example using the alternative techniques to uncoupling described for the Matrix Transpose example alleviation.

# Typable in HM∀ Only

* 1. *Collect*

This example from the ML mailing list was already discussed by Trevor Jim [[5](#_bookmark8)]. This function collects all the data from the defined data type and stores them in a list.

datatype ’a T = EMPTY

| NODE of ’a \* (’a T) T

let val rec collect = fn t =>

case t of

EMPTY = nil

| NODE(n,t) =

cons n

(flatmap collect (collect t))

in collect EMPTY end

*SML Type Checker Reports:*

Error: operator and operand don’t agree [circularity] operator domain: ’Z T

operand: ’Z T T in expression:

collect t

Here flatmap is a function similar to the map function. The type of

flatmap is:

flatmap : (*α* → *β* list) → *α* list → *β* list*.*

Obviously this example is not typable in HM, however, using system HM∀

we can give this example the following types:

flatmap : ∀*α.*∀*β.*(*α* → *β* list) → *α* list → *β* list

collect : ∀*α.α* T → *α* list*.*

Under system S this example is not typable. To see why let’s try to assign

collect the following reasonable type:

collect : *α* T → *α* list*.*

We have no trouble deriving this type for the Empty-branch of the case- expression. However, from the program fragment: collect t, of the Node- branch, collect must have the following type:

collect : *α* TT → *α* T list*,*

since t has the following type:

t : *α* T T*.*

Therefore collect must have a polymorphic type. Unfortunately, using inter- section types, it is not possible to assign the type:

collect : (*α* T → *α* list) ∧ (*α* TT → *α* T list)*,*

because when deriving the type *α* TT → *α* T list for collect we will require:

collect : *α* TTT → *α* T T list*.*

This cyclic dilemma will continue indefinitely.

If we were to extend system S with infinite width intersection types such as the following:

collect : ∧*i*∈*N τi*+1 → *τi* list*,*

where

*α* if *i* = 0*,*

*τ* =

*i*

*τi*−1T otherwise*,*

then we could derive a typing derivation for this example. However, we since we do not know how to deal with infinite width intersection types we reject this idea and resort to system HM∀ and polymorphic recursion with ∀-types.

Uncoupling this examples is impossible.

* 1. *BAR*

This example is a bit contrived but displays an interesting form of polymorphic recursion that is impossible to alleviate by uncoupling. Assuming the second argument to BAR is the f defined in the example, BAR can be understood by the following mathematical formula:

BAR *x* (*λx.x* × 2) *Z* = *Z* ∗ 22# of recursive calls = *Z* ∗ 22log2 (4*/x*) *.*

Below we show the example program.

let val r = fn i => i >= 4 val f = fn i => i \* 2 val a = 5

val rec BAR = fn x => fn F => fn Z =>

if r x then F Z

else BAR (f x) (fn v => fn w => v (v w)) F Z

in BAR 1 f a end

*SML Type Checker Reports:*

Error: right-hand-side of clause doesn’t agree with function result type [circularity]

expression: ((’Z -> ’Z) -> ’Z -> ’Z) ->

(’Z -> ’Z) -> ’Z -> ’Z

result type: ((’Z -> ’Z) -> ’Z -> ’Z) ->

((’Z -> ’Z) -> ’Z -> ’Z) ->

(’Z -> ’Z) -> ’Z -> ’Z

in declaration:

BAR = (fn x => (fn <pat> => <exp>))

Error: operator and operand don’t agree [literal] operator domain: (’Z -> ’Z) -> ’Z -> ’Z operand: int -> int

in expression: (BAR 1) f

This example, much like the previous, requires an infinite width intersec- tion type. To see why, observe that both sides of the if-expression in the body of BAR are required to of the same type by the rule (If). Assume, without a loss of generality, that the arguments to BAR have the following types:

x : int

F : int → int

Z : int*.*

then the then-branch has type: int. As a result, BAR must have the following type:

BAR : int → (int → int) → int → int*.*

Also as a result, the else-branch must have type: int. If this is to occur then the result of BAR applied to its three arguments in the else-branch must be type: int → int. This can only happen if the occurrence of BAR within the else-branch has the following type:

BAR : int → ((int → int) → (int → int)) → (int → int) → (int → int)*.*

Just as we saw in the last example this issue can be resolved if we give BAR

the type:

BAR :(int → (int → int) → int → int) ∧

(int → ((int → int) → (int → int)) → (int → int) → (int → int))*.*

However, now the rule (∧) requires us to type BAR as both components of the above intersection. Typing it as the second component will require us to expand the type of BAR even more. This cycle makes an infinite intersection type for BAR imperative.

We will now show such an infinite intersection type. Consider the following type:

*τ* = *α* if *i* = 0*, τi*−1 → *τi*−1 otherwise*.*

*i*

We can use this to define an infinite intersection type for BAR as follows: BAR : ∧*i*∈*N* int → *τi*+1 → *τi* → *τi.*

However, for the same reasons as before we choose to use ∀-types for this example. Under HM∀ we assign the following type:

BAR : ∀*α.*int → (*α* → *α*) → *α* → *α.*

The final type assigned to this example is: int.

# Typable in S∀ Only

* 1. *Construct List*

The following example presents a function, constList, that takes an input x and a number n. constList then constructs a list of 22 elements, all equal to x. Here is the program.

n

let val rec constList = fn x => fn n =>

if (n = 0) then [x,x] else cons x

(tl (concat

(constList (constList x (n-1)) (n-1))))

val applyCL = fn l1 => fn l2 => fn f =>

((constList l1 (f l1)), (constList l2 (f l2)))

in applyCL [1,2,3] [true,false,true] length end

*SML Type Checker Reports:*

Error: operator and operand don’t agree [circularity] operator domain: ’Z list list \* ’Z list list list operand: ’Z list list \* ’Z list

in expression:

x :: tl (concat ((constList <exp>) (<exp> - <exp>)))

Error: operator and operand don’t agree [literal] operator domain: \_ list list

operand: int list in expression:

applyCL (1 :: 2 :: 3 :: nil)

The above program is composed of one main function (constList), and one auxillary function (applyCL). The applyCM function makes two calls to constList (one for each input list) after applying an input function to each input list.

constList is a simple formulation of a function that constructs a list of the length described above without the use of arithmetical operations to explic- itly calculate 22 . Notice that a more concise formulation is not immediately

n

evident.

This example is unique in that it requires both ∀-types and intersection types. The need for ∀-types stems from the clause:

constList (constList x (n-1)) (n-1)

This statement requires that the result of constList be the same type as the first argument to constList. Suppose the first argument to constList is of type *α*. We also know that the return type of constList must be of type *α* list from the then-branch of the conditional. If we try to assign constList the type: (*α* → int → *α* list) ∧(*α* list → int → *α* list list) then we run into the same cyclic dilemma that was described in the Collect example. Therefore the type of constList must be: ∀*α.α* → int → *α* list (infinite width intersection types

are another option but, again, we choose ∀-types). Note that this example uses the same mechanism to require ∀-types as the Collect example. Yet, this example does not involve a recursive data type as the Collect example does. Instead this example uses only lists.

The need for intersection types arises when we inspect applyCL. Notice that we would like f to be a polymorphic argument to applyCL (this because we apply applyCL to two lists of different types). Since f is an argument it is impossible assign it a ∀-type since we have restricted our ∀-types such that quantifiers are not allowed inside a type. Therefore our only option is to assign f an intersection type.

Under S∀ the following types can be assigned:

constMatrix : ∀*α.α* → int → *α* list

applyCL : int list → bool list →

((int list → int list) ∧ (bool list → bool list)) →

(int list list × bool list list)

Uncoupling is not immediately evident for this example due to the fragment of the constList function that requires a ∀-type.

* + 1. *An Aside: SML/NJ vs. GHC*

We now return to our comparison of SML/NJ and GHC error reporting. The BAR example, this example, and the following example (Delay) all demon- strate a difference between the error reporting of the two compilers that we have not yet seen. Here we show the Haskell translation and GHC error mes- sage of this example.

constList x 0 = [x,x]

constList x n = (x:(tail (concat (constList

(constList x (n-1)) (n-1)))))

applyCL l1 l2 f = ((constList l1 (f l1)), (constList l2 (f l2)))

*GHC Type Checker Reports:*

Occurs check: cannot construct the infinite type: a = [a] Expected type: [[a]]

Inferred type: [a]

In the application ‘constList (constList x (n - 1)) (n - 1)’

In the first argument of ‘concat’, namely ‘(constList (constList x (n - 1)) (n - 1))’

Notice that the error message reported by GHC consists of only one message while SML/NJ reports two messages. This suggests that GHC may get to the heart of the error while SML/NJ reports numerous superfluous messages. On the other hand, perhaps SML/NJ error reporting is more precise, exposing every relevant error location. Since this is not the main objective of this report we leave this issue for future inquiry. However, the interested reader is advised to see [[3](#_bookmark5)] for more discussion.

# Untypable

* 1. *Delay Evaluation*

The following example shows some of the limitations of polymorphic recursion using intersection types and ∀-types.

let val delay = fn x => fn () => x val rec nDelays = fn n => fn x =>

if n=0 then x

else nDelays (n-1) (delay x) in nDelays 3 (fn x => x + 1) end

*SML Type Checker Reports:*

Error: right-hand-side of clause doesn’t agree with function result type [circularity]

expression: ’Z -> ’Z

result type: (unit -> ’Z) -> ’Z in declaration:

nDelays = (fn n => (fn <pat> => <exp>))

Error: operator and operand don’t agree [literal] operator domain: unit -> ’Z

operand: int -> int in expression:

(nDelays 3) (fn x => x + 1)

Polymorphic recursion with ∀-types is not powerful enough to type this example. To see why there is no ∀-type let us inspect the example. First, it is easy to see that the type of delay is:

delay : ∀*α.α* → unit → *α.*

It is apparent that n has type int. Suppose next, that we give x type *α*. From the then-branch we see that the return type of the function must be of type *α*. So far we have assigned nDelays the following type:

nDelays : ∀*α.*int → *α* → *α.*

Next, according to the rule (If), we will make sure that the else-branch also has type *α*. This is where the problem manifests. The first argument to nDelays, n-1, clearly has type int. However, the second argument to nDelays, delay x, has type unit → *α* which, according to the type previously assigned to nDelays, means the else-branch has type unit → *α*.

Polymorphic recursion with intersection types is also not sufficient to type this example. To see why, first observe that to type:

fn n => fn x => if n=0

then x

else nDelays (n-1) (delay x),

we require x to have an intersection type. In the then-branch, x must have the same type as the result of nDelays which we will call *τ* . In the else-branch, we require x to have a type with strictly fewer units than *τ* has, since the call to delay will add one unit and nDelays does not accept arguments with a greater number of units than its return type. Therefore by assigning the following type to x:

x : *α* ∧ (unit → *α*)*,*

we are able to derive the same type in both branches of the if-expression.

However, this presents a different problem. In order to type:

nDelays 3 (fn x => x + 1),

we require nDelays to have the type:

int → (int → int) → *τ.*

but as a result of the subtyping relation rules, this type is not attainable if we require the second argument of nDelays to have an intersection type. We can see this from the following failed subtype derivation (where the boxed judgement is the failure point).

int ≤ int (S-Refl)

(S-Refl)

(S-Fun)

*σ* ≤ *σ*

int → int ≤ ((int → int) ∧ *.. .*)

((int → int) ∧ *.. .*) → *σ* ≤ (int → int) → *σ*

int → (((int → int) ∧ *.. .*) → *σ*) ≤ int → ((int → int) → *σ*) (S-Fun)

Therefore we cannot derive an intersection type for this example using our system.

# High-Order ∀-Polymorphism

In this section we describe the difference between our construction of S and HM∀. We have chosen to disallow ∀-quantifiers anywhere inside a type. How- ever, we allow ∧ to occur freely inside a type. At first glance, these choices may seem biased toward the intersection type system. The rationalization behind these choices was a decision to investigate the typability of programs for which there is a known type inference algorithm that does not rely on any type annotations. It is known how to infer types for high-rank uses of inter- section types [[14](#_bookmark17)], but this is not the case for high-rank uses of ∀-types. This being said, if we were to consider a system of ∀-types that allowed arbitrary rank uses of ∀-types, then under this system we could type every example in this report that S types.

# Conclusion

In summary, we have shown several examples of programs that require poly- morphic recursion. Each program is not typable in the traditional Hindley- Milner system (HM). Some of the examples require ∧-types and others require

∀-types. Still others are not typable even with a combination of the two. We have seen that an intersection type system (S) can type many of our examples including Mycroft’s example. To the best of our knowledge System S is the first type system that has been able to achieve this. Therefore, although a finite width intersection type system is not able to type all possible polymor-

phic recursive programs, it can type a significant subset with the possibility of decidable type inference.

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# Mycroft Typing Derivation in System S

Suppose we have the following types:

*τ*int = int → int

*τ*bool = bool → bool

*τ*int list = int list → int list

*τ*bool list = bool list → bool list

*τ*× = int list × bool list

*τ*∧ = (*τ*int → *τ*int list) ∧ (*τ*bool → *τ*bool list)*.*

Also suppose we have the following context:

Γ = myMap : *τ*∧*,* sqList : *τ*int list*,* compList : *τ*bool list*.*

Finally, suppose we have the following terms:

M = fn *f* =*>* fn *l* =*>* if(null *l*) then *l* else cons (*f* (hd *l*)) (myMap *f* (tl *l*))

S = fn *l* =*>* myMap (fn *x* =*> x* ∗ *x*) *l*

C = fn *l* =*>* myMap not *l*

E = (sqList (cons 2 (cons 4 nil))*,* compList (cons true (cons false nil)))*.*

Then we have the following typing derivation:

98. *τ*bool → *τ*bool list ≤ *τ*bool → *τ*bool list (S-Refl)

97. *τ*∧ ≤ *τ*bool → *τ*bool list (S-∧) from 98

96. Γ*,f* : *τ*bool*,l* : bool list ▶ myMap : *τ*∧ (∧-Var)

95. Γ*,f* : *τ*bool*,l* : bool list ▶ *l* : bool list (∧-Var)

94. Γ*,f* : *τ*bool*,l* : bool list ▶ hd : bool list → bool (∧-Const)

93. *τ*int → *τ*int list ≤ *τ*int → *τ*int list (S-Refl)

92. *τ*∧ ≤ *τ*int → *τ*int list (S-∧) from 93

91. Γ*,f* : *τ*int*,l* : int list ▶ myMap : *τ*∧ (∧-Var)

90. Γ*,f* : *τ*int*,l* : int list ▶ *l* : int list (∧-Var)

89. Γ*,f* : *τ*int*,l* : int list ▶ hd : int list → int (∧-Const)

|  |  |  |
| --- | --- | --- |
| 88. Γ*,f* : *τ*bool*,l* : bool list ▶ | *l* : bool list | (∧-Var) |
| 87. Γ*,f* : *τ*bool*,l* : bool list ▶ | tl : *τ*bool list | (∧-Const) |
| 86. Γ*,f* : *τ*bool*,l* : bool list ▶ | *f* : *τ*bool | (∧-Var) |
| 85. Γ*,f* : *τ*bool*,l* : bool list ▶ | myMap : *τ*bool → *τ*bool list | (Sub) from 96, 97 |
| 84. Γ*,f* : *τ*bool*,l* : bool list ▶ | hd *l* : bool | (App) from 94, 95 |
| 83. Γ*,f* : *τ*bool*,l* : bool list ▶ | *f* : *τ*bool | (∧-Var) |
| 82. Γ*,f* : *τ*int*,l* : int list ▶ | *l* : int list | (∧-Var) |
| 81. Γ*,f* : *τ*int*,l* : int list ▶ | tl : *τ*int list | (∧-Const) |
| 80. Γ*,f* : *τ*int*,l* : int list ▶ | *f* : *τ*int | (∧-Var) |
| 79. Γ*,f* : *τ*int*,l* : int list ▶ | myMap : *τ*int → *τ*int list | (Sub) from 91, 92 |
| 78. Γ*,f* : *τ*int*,l* : int list ▶ | hd *l* : int | (App) from 89, 90 |
| 77. Γ*,f* : *τ*int*,l* : int list ▶ | *f* : *τ*int | (∧-Var) |
| 76. Γ*,f* : *τ*bool*,l* : bool list ▶ | tl *l* : bool list | (App) from 87, 88 |
| 75. Γ*,f* : *τ*bool*,l* : bool list ▶ | myMap *f* : *τ*bool list | (App) from 85, 86 |
| 74. Γ*,f* : *τ*bool*,l* : bool list ▶ | *f* (hd *l*) : bool | (App) from 83, 84 |
| 73. Γ*,f* : *τ*bool*,l* : bool list ▶ | cons : bool → *τ*bool list | (∧-Const) |
| 72. Γ*,f* : *τ*int*,l* : int list ▶ | tl *l* : int list | (App) from 81, 82 |
| 71. Γ*,f* : *τ*int*,l* : int list ▶ | myMap *f* : *τ*int list | (App) from 79, 80 |
| 70. Γ*,f* : *τ*int*,l* : int list ▶ | *f* (hd *l*) : int | (App) from 77, 78 |
| 69. Γ*,f* : *τ*int*,l* : int list ▶ | cons : int → *τ*int list | (∧-Const) |
| 68. Γ*,l* : int list*,x* : int ▶ | *x* : int | (∧-Var) |
| 67. Γ*,l* : int list*,x* : int ▶ | ∗ : int → *τ*int | (∧-Const) |
| 66. Γ*,f* : *τ*bool*,l* : bool list ▶ | myMap *f* (tl *l*) : bool list | (App) from 75, 76 |
| 65. Γ*,f* : *τ*bool*,l* : bool list ▶ | cons (*f* (hd *l*)) : *τ*bool list | (App) from 73, 74 |

64. Γ*,f* : *τ*bool*,l* : bool list ▶ *l* : bool list (∧-Var)

63. Γ*,f* : *τ*bool*,l* : bool list ▶ null : bool list → bool (∧-Const)

62. Γ*,f* : *τ*int*,l* : int list ▶ myMap *f* (tl *l*) : int list (App) from 71, 72

61. Γ*,f* : *τ*int*,l* : int list ▶ cons (*f* (hd *l*)) : *τ*int list (App) from 69, 70

60. Γ*,f* : *τ*int*,l* : int list ▶ *l* : int list (∧-Var)

59. Γ*,f* : *τ*int*,l* : int list ▶ null : int list → bool (∧-Const)

58. Γ*,l* : int list*,x* : int ▶ *x* : int (∧-Var)

57. Γ*,l* : int list*,x* : int ▶ ∗*x* : *τ*int (App) from 67, 68

56. Γ ▶ 4 : int (∧-Const)

55. Γ ▶ cons : (∧-Const)

int → int list → int list

54. Γ ▶ false : bool (∧-Const)

53. Γ ▶ cons : (∧-Const)

bool → bool list → bool list

52. Γ*,f* : *τ*bool*,l* : bool list ▶ cons (*f* (hd *l*))

(myMap *f* (tl *l*)) : bool list (App) from 65, 66

51. Γ*,f* : *τ*bool*,l* : bool list ▶ *l* : bool list (∧-Var)

50. Γ*,f* : *τ*bool*,l* : bool list ▶ (null *l*) : bool (App) from 63, 64

49. Γ*,f* : *τ*int*,l* : int list ▶ cons (*f* (hd *l*)) (App) from 61, 62

(myMap *f* (tl *l*)) : int list

48. Γ*,f* : *τ*int*,l* : int list ▶ *l* : int list (∧-Var)

47. Γ*,f* : *τ*int*,l* : int list ▶ (null *l*) : bool (App) from 59, 60

46. Γ*,l* : int list*,x* : int ▶ *x* ∗ *x* : int (App) from 57, 58

45. *τ*int → *τ*int list ≤ (S-Refl)

*τ*int → *τ*int list

44. *τ*∧ ≤ *τ*int → *τ*int list (S-∧) from 45

43. Γ*,l* : int list ▶ myMap : *τ*∧ (∧-Var)

42. *τ*bool → *τ*bool list ≤ (S-Refl)

*τ*bool → *τ*bool list

41. *τ*∧ ≤ *τ*bool → *τ*bool list (S-∧) from 42

40. Γ*,l* : bool list ▶ myMap : *τ*∧ (∧-Var)

39. Γ ▶ nil : int list (∧-Const)

38. Γ ▶ cons 4 : (App) from 55, 56

int list → int list

37. Γ ▶ 2 : int (∧-Const)

36. Γ ▶ cons : (∧-Const)

int → int list → int list

35. Γ ▶ nil : bool list (∧-Const)

34. Γ ▶ cons false : (App) from 53, 54

bool list → bool list

33. Γ ▶ true : bool (∧-Const)

32. Γ ▶ cons : (∧-Const)

bool → bool list → bool list

31. Γ*,f* : *τ*bool*,l* : bool list ▶ if(null *l*) then *l* (If) from 50, 51, 52

else cons (*f* (hd *l*)) (myMap *f* (tl *l*)) : bool list

30. Γ*,f* : *τ*int*,l* : int list ▶ if(null *l*) then *l* (If) from 47, 48, 49

else cons (*f* (hd *l*))

(myMap *f* (tl *l*)) : int list

29. Γ*,l* : int list ▶ fn *x* =*> x* ∗ *x* : *τ*int (Abs) from 46

28. Γ*,l* : int list ▶ myMap : *τ*int → *τ*int list (Sub) from 43, 44

27. Γ*,l* : bool list ▶ not : *τ*bool (∧-Const)

26. Γ*,l* : bool list ▶ myMap : *τ*bool → *τ*bool list (Sub) from 40, 41

25. Γ ▶ cons 4 nil : int list (App) from 38, 39

24. Γ ▶ cons 2 : int list → int list (App) from 36, 37

23. Γ ▶ cons false nil : bool list (App) from 34, 35

22. Γ ▶ cons true : (App) from 32, 33

bool list → bool list

21. Γ*,f* : *τbool* ▶ fn *l* =*>* if(null *l*) then *l* (Abs) from 31

else cons (*f* (hd *l*)) (myMap *f* (tl *l*)) : *τ*bool list

20. Γ*,f* : *τint* ▶ fn *l* =*>* if(null *l*) then *l* (Abs) from 30

else cons (*f* (hd *l*)) (myMap *f* (tl *l*)) : *τ*int list

19. Γ*,l* : int list ▶ *l* : int list (∧-Var)

18. Γ*,l* : int list ▶ myMap (fn *x* =*> x* ∗ *x*) : *τ*int list (App) from 28, 29

17. Γ*,l* : bool list ▶ *l* : bool list (∧-Var)

16. Γ*,l* : bool list ▶ myMap not : *τ*bool list (App) from 26, 27

15. Γ ▶ cons 2 (cons 4 nil) : int list (App) from 24, 25

14. Γ ▶ sqList : *τ*int list (∧-Var)

13. Γ ▶ cons true (cons false nil) : (App) from 22, 23

bool list

|  |  |  |  |
| --- | --- | --- | --- |
| 12. | Γ ▶ | compList : *τ*bool list | (∧-Var) |
| 11.  10. | Γ ▶  Γ ▶ | fn *f* =*>* fn *l* =*>* if(null *l*) then *l* else cons (*f* (hd *l*))  (myMap *f* (tl *l*)) : *τ*bool → *τ*bool list  fn *f* =*>* fn *l* =*>* | (Abs) from 21  (Abs) from 20 |
|  |  | if(null *l*) then *l*  (myMap *f* (tl *l*)) : *τ*int → *τ*int list | else cons (*f* (hd *l*)) |
| 9. | Γ*,l* : int list ▶ | myMap (fn *x* =*> x* ∗ *x*) *l* : int list | (App) from 18, 19 |
| 8. | Γ*,l* : bool list ▶ | myMap not *l* : bool list | (App) from 16, 17 |
| 7. | Γ ▶ | sqList  (cons 2 (cons 4 nil)) : int list | (App) from 14, 15 |
| 6. | Γ ▶ | compList  (cons true (cons false nil)) :  bool list | (App) from 12, 13 |
| 5. | Γ ▶ | M : *τ*∧ | (∧) from 10, 11 |
| 4. | Γ ▶ | S : *τ*int list | (Abs) from 9 |
| 3. | Γ ▶ | C : *τ*bool list | (Abs) from 8 |
| 2. | Γ ▶ | E : *τ*× | (Pair) from 6, 7 |
| 1. | ▶ | let val rec myMap = M and sqList = S  and compList = C inE end : *τ*× | (∧-Rec) from 2, 3, 4, 5 |

# Proof of Soundness of a Subsystem of S

In this section our goal is to show the soundness of a subsystem of S. We choose to eliminate the pair and conditional rules of S for the simplicity of the proof. We do not anticipate any difficulties in the proof of soundness if

these additional rules were included. To achieve soundness we first define the operational semantics of our system. After this we prove the Inversion and Substitution Lemmas which allow us to show Subject Reduction holds.

Before we define the operational semantics of our system let us define the expressions and values of our system.

*M, N* ∈ Expressions ::= *x* | *c* | fn *x* =*> M* | *M N* | let val *x* = *M* in *N* end |

let val rec *x*1 = *M*1 and *...* and *xn* = *Mn* in *N* end

*V* ∈ Values ::= *x* | fn *x* =*> M*

Now we review the static semantics of our system.

*Subsystem of System* S *Typing Rules:*

type(*c*) = *τ*

∆ ▶ *c* : *τ*

(∧-Const)

∆(*x*) = *τ*

(*τ* closed)

∆ ▶ *x* : *τ* (∧-Var)

∆ ▶ fn *x* =*> M* : *τ* → *τ* ' (Abs)

∆*,x* : *τ* ▶ *M* : *τ* '

∆ ▶ *M* : *τ* → *τ* ' ∆ ▶ *N* : *τ*

∆ ▶ *MN* : *τ* '

(App)

∆ ▶ *M* : *τ* ' ∆*,x* : *τ* ' ▶ *N* : *τ*

∆ ▶ let *x* = *M* in *N* end : *τ* (∧-Let)

∆*, x*1 : *τ*1*,... , xn* : *τn* ▶ *N* : *τ*

∆*, x*1 : *τ*1*,... , xn* : *τn* ▶ *Mp* : *τp*

∆ ▶ let val rec *x*1 = *M*1 and *...*

and *xn* = *Mn* in *N* end : *τ*

(∧-Rec)

(1 ≤ *p* ≤ *n*)

∆ ▶ *M* : *τi i* ∈ *I* (∧)

∆ ▶ *M* : ∧*i*∈*I τi*

(size(*I*) ≥ 2), (size(*I*) is finite)

∆ ▶ *M* : *τ τ* ≤ *τ* '

∆ ▶ *M* : *τ* '

(Sub)

*τ* ≤ *τ*

(S-Refl)

*τ*1 ≤ *τ*2 *τ*2 ≤ *τ*3 (S-Trans)

*τ*1 ≤ *τ*3

*τ*1 ≤ *τ* ' *τ* ' ≤ *τ*2

*τ* ' → *τ* ' ≤ *τ*1 → *τ*2

1 2 (S-Fun)

*τi* ≤ *τ* ' *i* ∈ *I I* ⊆ *J*

*i* (S-∧)

1 2

∧*i*∈*J τi* ≤ ∧*i*∈*I τ* '

*i*

Below are the dynamic semantics our subsystem.

*Subsystem of System* S *Operational Semantics:*

*M N* ⇒ *M* ' *N* (E-App1)

*M* ⇒ *M* '

*V N* ⇒ *V N* ' (E-App2)

*N* ⇒ *N* '

(fn *x* =*> M* ) *V* ⇒ *M* [*x* := *V* ] (E-AppAbs)

let val *x* = *M* in *N* end ⇒ let val *x* = *M* ' in *N* end (E-Let1)

*M* ⇒ *M* '

let val *x* = *V* in *N* end ⇒ *N* [*x* := *V* ] (E-Let2)

*Mp*[*x*1 := *M*1] *...* [*xn* := *Mn*] ⇒ *M* '

*p*

(E-Rec1)

let val rec *x*1 = *V*1 and *...* and *xp* = *Mp* and *...*

and *xn* = *Mn* in *N* end ⇒

let val rec *x*1 = *V*1 and *...* and *xp* = *M* ' and *...*

(1 ≤ *p* ≤ *n*)

*p*

and *xn* = *Mn* in *N* end

(E-Rec2)

let val rec *x*1 = *V*1 and *...* and *xn* = *Vn* in *N* end ⇒

*N* [*x*1 := *V* '] *...* [*xn* := *V* ']

1

*n*

Lemma B.1 (Inversion of the Subtype Relation)

*If τ*1 → *τ*2 ≤ *τ* ' → *τ* ' *, then τ* ' ≤ *τ*1 *and τ*2 ≤ *τ* ' *.*

1 2 1 2

Proof. There are three possible subtyping rules which may have been the last rule applied in the subtyping derivation of the judgement *τ*1 → *τ*2 ≤

' → *τ* ' . If the rule (S-Fun) was last applied then the result is obvious. If

*τ*

2

1

the rule (S-Refl) rule was last applied then the result can be obtained by straightforward induction on the premise of the rule. If the rule (S-Trans) was last applied then again we proceed by induction on the premises of the rule, but we must also apply the (S-Trans) rule to these results.

Lemma B.2 (Inversion) *If* ∆ ▶ fn *x* =*> M* : *τ*1 → *τ*2*, then*

∆*,x* : *τ* ' ▶ *M* : *τ*2 *and τ*1 ≤ *τ* ' *.*

1 1

Proof. By inspection of the inference rules we observe that the last rule applied in the typing derivation of the judgement ∆ ▶ fn *x* =*> M* : *τ*1 → *τ*2 can only be one of two possibilities. We proceed by case analysis.

∆*, x* : *τ*1 ▶ *M* : *τ*2

*case: D* = ∆ ▶ fn *x* =*> M* : *τ*1 → *τ*2 (Abs)

Then we have ∆*,x* : *τ*1 ▶ *M* : *τ*2 where *τ* ' = *τ*1 and *τ*1 ≤ *τ*1 by (S-Refl).

1

∆ ▶ fn *x* =*> M* : *τ* ' → *τ* ' *τ* ' → *τ* ' ≤ *τ*1 → *τ*2

*case: D* =

1 2 1 2 (Sub)

∆ ▶ fn *x* =*> M* : *τ*1 → *τ*2

*τ*1 ≤ *τ* ' and *τ* ' ≤ *τ*2 Subtype Inversion Lemma on

1 2

*τ* ' → *τ* ' ≤ *τ*1 → *τ*2

1 2

∆*,x* : *τ* '' ▶ *M* : *τ* ' and *τ* ' ≤ *τ* '' I.H. on ∆ ▶ fn *x* =*> M* : *τ* ' → *τ* '

1

*τ*1 ≤ *τ* ''

1

2 1 1

1 2

(S-Trans) applied to *τ*1 ≤ *τ* ' and

1

*τ* ' ≤ *τ* ''

1 1

∆*,x* : *τ* '' ▶ *M* : *τ*2 (Sub) applied to ∆*,x* : *τ* '' ▶ *M* : *τ* '

1 1 2

and *τ* ' ≤ *τ*2

2

Lemma B.3 (Weakening) *If* ∆ ▶ *M* : *τ, then* ∆*,* ∆' ▶ *M* : *τ, provided that*

∆*,* ∆' *is a valid context.*

Proof. The proof proceeds by straightforward induction on the structure of the derivation *D* :: ∆ ▶ *M* : *τ* . The only case in which the context is examined is when the rule (Var) is the last rule applied in the derivation. It should be

clear that (Var) is only applicable if the context ∆ contains the assignment *x* :

*τ* . And by extending the context with additional, non-conflicting assignments we do not alter this property.

Lemma B.4 (Substitution) *If* ∆ ▶ *N* : *τ and* ∆*,x* : *τ,* ∆' ▶ *M* : *τ* '*, then*

∆*,* ∆' ▶ *M* [*x* := *N* ] : *τ* '*.*

Proof. By structural induction on the derivation *D* :: ∆*,x* : *τ,* ∆' ▶ *M* : *τ* '.

We show only a few cases, as the rest follow the same pattern.

∆*,x* : *τ,* ∆'(*y*) = *τ* '

*case: D* = ∆*,x* : *τ,* ∆' ▶ *y* : *τ* ' (∧-Var)

Depending on whether *x* = *y* we have two subcases.

*subcase: x* = *y* and *τ* = *τ* '

*x*[*x* := *N* ] = *N* Definition of Substitution

∆*,* ∆' ▶ *N* : *τ* Weakening Lemma on assumption ∆ ▶ *N* : *τ*

*subcase: x* /= *y*

*y*[*x* := *N* ] = *y* Definition of Substitution

∆*,* ∆' ▶ *y* : *τ* ' Assumptions ∆*,x* : *τ,* ∆' ▶ *y* : *τ* ' and *x* /= *y*

∆*,x* : *τ,* ∆' ▶ *M*1 : *τ* ' → *τ* ' ∆*,x* : *τ,* ∆' ▶ *M*2 : *τ* '

*case: D* =

1 1

∆*,x* : *τ,* ∆' ▶ *M*1*M*2 : *τ* '

(App)

Depending on whether *x* ∈ FV(*M*1) we have two subcases.

*subcase: x* ∈ FV(*M*1)

Depending on whether *x* ∈ FV(*M*2) we have two subsubcases.

*subsubcase: x* ∈ FV(*M*2)

∆*,* ∆' ▶ *M*1[*x* := *N* ] : *τ* ' → *τ* ' I.H. on

1

∆*,* ∆' ▶ *M*2[*x* := *N* ] : *τ* '

1

∆*,x* : *τ,* ∆' ▶ *M*1 : *τ* ' → *τ* '

I.H. on ∆*,x* : *τ,* ∆' ▶ *M*2 : *τ* '

1

1

∆*,* ∆' ▶ *M*1[*x* := *N* ]*M*2[*x* := *N* ] : *τ* ' (App) applied to

∆*,* ∆' ▶ *M*1[*x* := *N* ] : *τ* ' → *τ* ' and

1

∆*,* ∆' ▶ *M*2[*x* := *N* ] : *τ* '

1

∆*,* ∆' ▶ (*M*1*M*2)[*x* := *N* ] : *τ* ' Definition of Substitution

*subsubcase: x* /∈ FV(*M*2) and ∆*,* ∆' ▶ *M*2 : *τ* '

1

∆*,* ∆' ▶ *M*1[*x* := *N* ] : *τ* ' → *τ* ' I.H. on

1

∆*,x* : *τ,* ∆' ▶ *M*1 : *τ* ' → *τ* '

1

∆*,* ∆' ▶ (*M*1[*x* := *N* ])*M*2 : *τ* ' (App) applied to

∆*,* ∆' ▶ *M*1[*x* := *N* ] : *τ* ' → *τ* ' and

1

∆*,* ∆' ▶ *M*2 : *τ* '

1

∆*,* ∆' ▶ (*M*1*M*2)[*x* := *N* ] : *τ* ' Definition of Substitution

*subcase: x* /∈ FV(*M*1) and ∆*,* ∆' ▶ *M*1 : *τ* ' → *τ* '

1

Depending on whether *x* ∈ FV(*M*2) we have two subsubcases.

*subsubcase: x* ∈ FV(*M*2)

∆*,* ∆' ▶ *M*2[*x* := *N* ] : *τ* ' I.H. on ∆*,x* : *τ,* ∆' ▶ *M*2 : *τ* '

1

1

∆*,* ∆' ▶ *M*1(*M*2[*x* := *N* ]) : *τ* ' (App) applied to

∆*,* ∆' ▶ *M*1 : *τ* ' → *τ* ' and

1

∆*,* ∆' ▶ *M*2[*x* := *N* ] : *τ* '

1

∆*,* ∆' ▶ (*M*1*M*2)[*x* := *N* ] : *τ* ' Definition of Substitution

*subsubcase: x* /∈ FV(*M*2) and ∆*,* ∆' ▶ *M*2 : *τ* '

1

(*M*1*M*2)[*x* := *N* ] = *M*1*M*2 Definition of Substitution

∆*,* ∆' ▶ *M*1*M*2 : *τ* ' Assumptions ∆*,x* : *τ,* ∆' ▶ *M*1*M*2 : *τ* ', *x* /∈ FV(*M*1),

and *x* /∈ FV(*M*2)

The remaining cases are similar.

Theorem B.5 (Subject Reduction) *If* ∆ ▶ *M* : *τ and M* ⇒ *M* '*, then*

∆ ▶ *M* ' : *τ.*

Proof. By structural induction on the derivation of *D* :: ∆ ▶ *M* : *τ* .

type(*c*) = *τ*

*case: D* =

∆ ▶ *c* : *τ* (∧-Const)

Can’t happen because there are no evaluation rules for constants.

∆(*x*) = *τ*

*case: D* = ∆ ▶ *x* : *τ* (∧-Var)

Can’t happen because there are no evaluation rules for variables.

∆*,x* : *τ* ▶ *M* : *τ* '

*case: D* = ∆ ▶ fn *x* =*> M* : *τ* → *τ* ' (Abs)

Can’t happen because there are no evaluation rules for abstractions.

∆ ▶ *M* : *τ* → *τ* ' ∆ ▶ *N* : *τ*

*case: D* =

∆ ▶ *MN* : *τ* ' (App)

From the operational semantics there are three ways we can derive *M* ⇒ *M* '. We proceed by cases.

*subcase: M* ⇒ *M* '

*M N* ⇒ *M* ' *N* (E-App1)

∆ ▶ *M* ' : *τ* → *τ* ' I.H. on ∆ ▶ *M* : *τ* → *τ* ' and *M* ⇒ *M* '

∆ ▶ *M* '*N* : *τ* ' (App) applied to ∆ ▶ *M* ' : *τ* → *τ* ' and ∆ ▶ *N* : *τ*

*subcase: M* is a value and *N* ⇒ *N* '

*M N* ⇒ *M N* ' (E-App2)

∆ ▶ *N* ' : *τ* I.H. on ∆ ▶ *N* : *τ* and *N* ⇒ *N* '

∆ ▶ *MN* ' : *τ* ' (App) applied to ∆ ▶ *M* : *τ* → *τ* ' and ∆ ▶ *N* ' : *τ*

*subcase: M* = fn *x* =*> M* ' and *N* is a value

(fn *x* =*> M* ') *N* ⇒ *M* '[*x* := *N* ] (E-AppAbs)

∆*,x* : *τ* '' ▶ *M* ' : *τ* ', where *τ* ≤ *τ* '' Inversion Lemma on

∆ ▶ fn *x* =*> M* ' : *τ* → *τ* '

∆ ▶ *N* : *τ* '' (Sub) applied to ∆ ▶ *N* : *τ*

and *τ* ≤ *τ* ''

∆ ▶ *M* '[*x* := *N* ] : *τ* ' Substitution Lemma on

∆*,x* : *τ* '' ▶ *M* ' : *τ* ' and ∆ ▶ *N* : *τ* ''

∆ ▶ *M* : *τ* ' ∆*,x* : *τ* ' ▶ *N* : *τ*

*case: D* = ∆ ▶ let *x* = *M* in *N* end : *τ* (∧-Let)

From the operational semantics there are two ways we can derive *M* ⇒ *M* '. We proceed by cases.

*subcase: M* ⇒ *M* '

let val *x* = *M* in *N* end ⇒

let val *x* = *M* ' in *N* end (E-Let1)

∆ ▶ *M* ' : *τ* ' I.H. on ∆ ▶ *M* : *τ* ' and *M* ⇒ *M* '

∆ ▶ let val *x* = *M* ' in *N* end : *τ* (∧-Let) applied to ∆ ▶ *M* ' : *τ* ' and

∆*,x* : *τ* ' ▶ *N* : *τ*

*subcase: M* is a value

let val *x* = *M* in *N* end ⇒ *N* [*x* := *M* ] (E-Let2)

∆ ▶ *N* [*x* := *M* ] : *τ* Substitution Lemma on

∆*,x* : *τ* ' ▶ *N* : *τ* and

∆ ▶ *M* : *τ* '

∆*, x*1 : *τ*1*,... , xn* : *τn* ▶ *N* : *τ* ∆*, x*1 : *τ*1*,... , xn* : *τn* ▶ *Mp* : *τp*

Rec)

*case: D* = ∆ ▶ let val rec *x*1 = *M*1 and *...* and *xn* = *Mn* in *N* end : *τ* (∧-

From the operational semantics there are two ways we can derive *M* ⇒ *M* '.

We proceed by cases.

*subcase: Mp* ⇒ *M* ' , where 1 ≤ *p* ≤ *n*

*p*

let val rec *x*1 = *M*1 and *...* (E-Rec1)

and *xp* = *Mp* and *...*

and *xn* = *Mn* in *N* end ⇒ let val rec *x*1 = *M*1 and *...* and *xp* = *M* ' and *...*

*p*

and *xn* = *Mn* in *N* end

∆*, x*1 : *τ*1*,... , xn* : *τn* ▶ *M* ' : *τp* I.H. on

*p*

∆*, x*1 : *τ*1*,... , xn* : *τn* ▶ *Mp* : *τp*

and *Mp* ⇒ *M* '

*p*

∆ ▶ let val rec *x*1 = *M*1 and *...* (∧-Rec) applied to

∆*, x*1 : *τ*1*,... , xn* : *τn* ▶ *M* ' : *τp* and

*p*

and *xp* = *M* ' and *...* ∆*, x*1 : *τ*1*,... , xn* : *τn* ▶ *N* : *τ*

*p*

and *xn* = *Mn* in *N* end : *τ*

*subcase: M*1 *... Mn* are all values.

let val rec *x*1 = *M*1 and *...* (E-Rec2)

and *xn* = *Mn* in *N* end ⇒

*N* [*x* := *M*1] *...* [*x* := *Mn*]

∆ ▶ *N* [*x* := *M*1] *...* [*x* := *Mn*] : *τ* By *n* applications of the

Substitution Lemma

*case: D* =

∆ ▶ *M* : *τi i* ∈ *I*

∆ ▶ *M* : ∧*i*∈*I τi*

(∧)

∆ ▶ *M* ' : *τi i* ∈ *I* I.H. on ∆ ▶ *M* : *τi i* ∈ *I* and *M* ⇒ *M* '

∆ ▶ *M* ' : ∧*i*∈*I τi* (∧) applied to ∆ ▶ *M* ' : *τi i* ∈ *I*

∆ ▶ *M* : *τ τ* ≤ *τ* '

*case: D* =

∆ ▶ *M* : *τ* ' (Sub)

∆ ▶ *M* ' : *τ* I.H. on ∆ ▶ *M* : *τ* and *M* ⇒ *M* '

∆ ▶ *M* ' : *τ* ' (Sub) applied to ∆ ▶ *M* ' : *τ* and *τ* ≤ *τ* '