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Representation of FS-domains Based on Closure Spaces

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**Abstract**

In this paper, we propose the notion of FS-closure spaces by incorporating an additional structure into a given closure space, which provides a concrete representation of FS-domains. Furthermore, we prove that the category of FS-closure spaces with approximable mappings as morphisms is equivalent to that of FS-domains with Scott-continuous functions as morphisms.

*Keywords:* FS-domain, FS-closure space, Closure operator, Categorical equivalence.

# Introduction

A *closure space* is a pair (*X, γ*) consisting of a set *X* and a closure operator *γ* on *X*, where the closure operator is an isotone, extensive and idempotent map on the powerset of *X*. Closure spaces have played an important role in restructuring lat- tices and various order structures. The technique by adding a special structure into a given closure space may be traced back to the early works of Birkhoff’s famous representation theorem for finite distributive lattices [[3](#_bookmark12)] and Stone’s duality theorem for Boolean algebras [[9](#_bookmark19)]. These famous results also encourage researchers to investi- gate the interrelation between lattices and closure spaces. In [[4](#_bookmark13)], Edelman obtained that a lattice is meet-distributive if and only if it is a lattice of closed sets of closure space with the anti-exchange property. Ern´e [[5](#_bookmark14)] developed a uniform approach to representing various complete lattices by closure spaces from the categorical view- point. Guo and Li [[7](#_bookmark17)] proposed the notion of F-augmented closure spaces by adding a family of finite subsets into the closure space, which essentially establishes the

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representation of algebraic domains. Recently, Wang and Li [[8](#_bookmark18)] discuss the relation- ship between continuous domains and closure spaces. They introduce the notion of F-augmented generalized closure spaces by adding a map into a given closure space, and give a concrete representation of continuous domains.

*FS-domains* were introduced by A. Jung in [[1,](#_bookmark11)[2](#_bookmark15)], and proved that the category of FS-domains is a maximal Cartesian closed full subcategory of continuous domains. As is well known, a Cartesian closed category is of great significance that as a formal system with the same expressive power as a typed *λ*-calculus. Based on the basic fact, in this paper, we focus on the representation of FS-domains.

The paper is organized as follows: In Section 2, we recall some basic notions in domain theory. In Section 3, we introduce the concept of FS-closure spaces. Moreover, we prove that every FS-domain arises as the set of F-regular open sets of some FS-closure space. In Section 4, we obtain the main result that the category of FS-closure spaces with approximable mappings as morphisms is equivalent to that of FS-domains with Scott-continuous functions as morphisms.

# Preliminaries

For any set *X*, we write *F ± X* to mean that *F* is a finite subset of *X*. *P*(*X*) and *7* (*X*) are always used to denote the powerset of *X* and the family of all finite subsets of *X*, respectively. Let (*L, ≤*) be a poset. A subset *D* of *L* is called *directed*, if it is nonempty and every finite subset of *D* has an upper bound in *D*. We use *HD* to denote the *least upper bound* of a directed subset *D*. A poset is called a *dcpo* if every directed subset has a least upper bound. Given *x, y ∈ L*, we say *x* is *way below y* (in symbol *x y*) if for any directed subset *D ⊆ L* with *HD* exists, *y ≤ HD* always implies *x ≤ d* for some *d ∈ D*. For any *x ∈ L*, we use ***↓****x* to denote the set

*{y ∈ L | y x}*. A subset *B ⊆ L* is called a *basis* of *L* if for every *x ∈ L*, ***↓****x ∩ B* is a directed subset and *x* = *H*(***↓****x ∩ B*). A dcpo is a *continuous domain* if it has a basis.

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**Definition 2.1** [[6](#_bookmark16)] A function *f* : *L → L* between dcpos is said to be *Scott-*

*continuous* if for any directed subset *D* of *L*, *f* (*HD*)= *Hf* (*D*).

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We denote by [*L → L* ] the set of all Scott-continuous functions from *L* to *L* .

**Definition 2.2** [[6](#_bookmark16)] Let *L* be a dcpo.

* 1. An *approximate identity* for a dcpo *L* is a directed set *Ð⊆* [*L → L*] satisfying sup *Ð* = 1*L*, the identity on *L*.
  2. A Scott-continuous function *f* : *L → L* is *ﬁnitely separating* if there exists a finite set *Mf* such that for each *x ∈ L*, there exists *m ∈ Mf* such that *f* (*x*) *≤ m ≤ x*.
  3. *L* is called an *FS-domain* if there is an approximate identity for *L* consisting of finitely separating functions.

**Lemma 2.3** [[6](#_bookmark16)] Let *L* be a dcpo.

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* + 1. If *Ð⊆* [*L → L*] is an approximate identity for *L*, then *Ð* = *{f* = *f◦f* : *f ∈ Ð}*

is also an approximate identity.

* + 1. If *f ∈* [*L → L*] is finitely separating, then *f* (*x*) *x* for all *x ∈ L*.

**Definition 2.4** [[8](#_bookmark18)] Let (*X, γ*) be a closure space. A pair (*X, τ ◦ γ*) is called a *generalized closure space*, if *τ ◦ γ* is the composition map of *γ* and *τ* , where *τ* is a map on *P*(*X*) satisfies the following conditions, for any *A, B ⊆ X*:

1. *τ* (*γ*(*A*)) *⊆ γ*(*A*);
2. *τ* (*τ* (*γ*(*A*))) = *τ* (*γ*(*A*));
3. *τ* (*γ*(*A*)) *⊆ τ* (*γ*(*B*)) whenever *A ⊆ B*. For simplicity, we write *⟨A⟩* for *τ* (*γ*(*A*)).

**Definition 2.5** [[8](#_bookmark18)] Let (*X, τ ◦ γ*) be a generalized closure space and *7* a nonempty family of finite subsets of *X*. The triplet (*X, τ ◦ γ, 7* ) is called an *F-augmented generalized closure space* if, for any *F ∈7* and *M ± ⟨F⟩*, there exists *F*1 *∈7* such that *M ⊆ ⟨F*1*⟩* and *F*1 *⊆ ⟨F⟩*.

**Definition 2.6** [[8](#_bookmark18)] Let (*X, τ ◦ γ, 7* ) be an F-augmented generalized closure space. A nonempty subset *U* of *X* is called an *F-regular open set* of (*X, τ ◦ γ, 7* ) if, for any *M ± U* , there exists some *F ∈7* such that *M ⊆ ⟨F⟩⊆ U* .

For convenience, we use *R*(*X*) to denote the family of all F-regular open sets of (*X, τ ◦ γ, 7* ).

**Proposition 2.7** *[*[*8*](#_bookmark18)*] Let* (*X, τ ◦γ, 7* ) *be an F-augmented generalized closure space.*

1. For any *F ∈ 7*, *⟨F⟩* is an F-regular open set of (*X, τ ◦ γ, 7* ).
2. *U* is an F-regular open set of (*X, τ ◦ γ, 7* ) if and only if *{⟨F⟩| F ∈ 7,F ⊆ U}*

is directed and *U* = S*{⟨F⟩ | F ∈ 7,F ⊆ U}*.

1. If *{Uj}j∈J* is a directed family of F-regular open sets of (*X, τ ◦ γ, 7* ), then

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*j∈J Uj* is an F-regular open set of (*X, τ ◦ γ, 7* ).

**Theorem 2.8** *[*[*8*](#_bookmark18)*] Let* (*X, τ ◦ γ, 7* ) *be an F-augmented generalized closure space. Then* (*R*(*X*)*, ⊆*) *is a continuous domain.*

Given a continuous domain (*L, ≤*) with a basis *BL*, for any *A ⊆ BL*, define

*γ*(*A*)= *↓A ∩ BL,τ* (*A*)= ***↓****A ∩ BL.*

Let *7L* be the family of all finite subsets of *BL* with a greatest element under the induced order *≤*. Then for any *F ∈ 7L*, we have *∨F ∈ F* and

*⟨F⟩* = (*↓ ∨ F* ) *∩ BL.*

**Theorem 2.9** *[*[*8*](#_bookmark18)*] Let* (*L, ≤*) *be a continuous domain with a basis BL. Then* (*BL,τ◦ γ, 7L*) *is an F-augmented generalized closure space. And* (*L, ≤*) *is isomorphic to* (*R*(*BL*)*, ⊆*)*.*

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**Definition 2.10** [[8](#_bookmark18)] Let (*X, τ ◦ γ, 7* ) and (*X ,τ ◦ γ , 7* ) be two F-augmented

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generalized closure spaces. A relation Θ *⊆ 7 × X* is an *approximable mapping*

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from (*X, τ ◦ γ, 7* ) to (*X ,τ ◦ γ , 7* ), if the following hold:

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1. *F* Θ*F ⇒ F* Θ*⟨F ⟩*,

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1. *F ± ⟨F*1*⟩,F* Θ*M ⇒ F*1Θ*M* ,

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1. *F* Θ*M*

*⇒* (*∃G ∈ 7,G ∈7* )(*G ⊆ ⟨F⟩,M ⊆ ⟨G ⟩, G*Θ*G* ),

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for any *F, F*1 *∈ 7,F*

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*∈ 7* and *M*

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*± X* , where *F* Θ*M*

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means that *F* Θ*x*

for any

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*x ∈ M* .

Given an F-augmented generalized closure space (*X, τ ◦ γ, 7* ), define a relation

id*X ⊆7 × X* by

id*X* = *{*(*F, x*) *∈7 × X | x ∈ ⟨F⟩}.*

It is obvious that id*X* is an approximable mapping from (*X, τ ◦ γ, 7* ) to itself.

**Theorem 2.11** *[*[*8*](#_bookmark18)*] The category* **DOM** *of continuous domains with Scott- continuous functions is equivalent to the category* **FGC** *of F-augmented generalized closure spaces with approximable mappings.*

# FS-closure spaces

In this section, we give a special type of F-augmented generalized closure space which is called FS-closure space, and use this notion to obtain the representation of FS-domains.

**Definition 3.1** An *FS-closure space* is an F-augmented generalized closure space (*X, τ ◦ γ, 7* ) which satisfies: there exists a directed family *{*Θ*j}j∈J* of approximable mappings for (*X, τ ◦ γ, 7* ) such that :

* 1. S*j∈J* Θ*j* = id*X* ,
  2. For every Θ*j*, we have a finite subset family *Mj ⊆7* such that for each *F ∈ 7*, there exists *M ∈ Mj*, *F* Θ*jx* implies *M ⊆ ⟨F⟩* and *x ∈ ⟨M⟩*.

Throughout this paper, we use *R*(*X*) to denote the family of all F-regular open sets (Definition [2.6](#_bookmark1)) of FS-closure space (*X, τ ◦ γ, 7* ).

**Theorem 3.2** *Let* (*X, τ ◦ γ, 7* ) *be an FS-closure space. Then* (*R*(*X*)*, ⊆*) *is an FS-domain.*

**Proof.** Theorem [2.8](#_bookmark3) has shown that (*R*(*X*)*, ⊆*) is a continuous domain. To finish the proof, it is sufficient to show that there is an approximate identity for *R*(*X*) consisting of finitely separating functions. By hypothesis, there exists a directed family *{*Θ*j}j∈J* of approximable mappings for (*X, τ ◦ γ, 7* ) satisfies the conditions in Definition [3.1.](#_bookmark6) For every Θ*j*, define *φ*Θ*j* : *R*(*X*) *→ R*(*X*) by *φ*Θ*j* (*U* ) = *{x ∈ X |* (*∃F ∈ 7*)*F ⊆ U* & *F* Θ*jx}*. From [[8](#_bookmark18), Theorem 4.4], we know that *φ*Θ*j* is a Scott-continuous function.

We firstly prove that *{φ*Θ*j }j∈J* is an approximate identity for *R*(*X*). The di- rectivity of *{φ*Θ*j }j∈J* just follows immediately from the definition of *φ*Θ*j* . Suppose

*U ∈ R*(*X*), we have

(*supj∈J φ*Θ*j* )(*U* )= *supj∈J φ*Θ*j* (*U* )

= *φ*Θ*j* (*U* )

*j∈J*

= *{x ∈ X |* (*∃F ∈ 7*) *F ⊆ U* & *F* Θ*jx}*

*j∈J*

= *{x ∈ X |* (*∃F ∈ 7*) *F ⊆ U* & (*F, x*) *∈* Θ*j}*

*j∈J*

= *{x ∈ X |* (*∃F ∈ 7*) *F ⊆ U* & *F* id*Xx}*

= *{x ∈ X |* (*∃F ∈ 7*) *F ⊆ U* & *x ∈ ⟨F⟩}*

= *U.*

This means that *supj∈J φ*Θ*j* = id*R*(*X*).

We now prove that *φ*Θ*j* is finitely separating for every *j*. By Definition [3.1](#_bookmark6), for every Θ*j*, we have a finite subset family *Mj ⊆ 7* such that for each *F ∈ 7*, there exists *M ∈ Mj*, *F* Θ*jx* implies *M ⊆ ⟨F⟩* and *x ∈ ⟨M⟩*. Suppose *U ∈ R*(*X*), from Proposition [2.7](#_bookmark2), we obtain that *U* = *{⟨F⟩ | F ∈ 7,F ⊆ U}* and *{⟨F⟩ | F ∈ 7,F ⊆ U}* is directed. Set *Ð* = *{⟨F⟩ | F ∈ 7,F ⊆ U, M ∈ M , ∀F* Θ *x ⇒ M ⊆*

S

*M j j*

*⟨F⟩* & *x ∈ ⟨M⟩}*. It follows that *U* = S S*M∈Mj ÐM* and S*M∈Mj ÐM* is directed.

Since *Mj* is a finite subset family of *7* , there is a *M*0 *∈ Mj* such that *ÐM*0 is a

cofinal subset of S *′*

*M∈Mj ÐM* . We denote *Mj* = *{⟨M⟩ | M ∈ Mj}*. It is clear

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that *Mj* is a finite subset family of *R*(*X*). We finish the proof by checking that

*φ*Θ*j* (*U* ) *⊆ ⟨M⟩ ⊆ U* for some *M ∈ Mj*. In fact, suppose *x ∈ φ*Θ*j* (*U* ), then there exists *F ∈ 7* such that *F ⊆ U* and *F* Θ*jx*. It follows that *⟨F⟩ ⊆ ⟨F*0*⟩* for some *F*0 *∈7* and *⟨F*0*⟩∈ ÐM*0 . Therefore, *φ*Θ*j* (*U* ) *⊆ ⟨M*0*⟩⊆ U* . *2*

Given an FS-domain *L*, and its approximate identity *{φj}j∈J* for *L* consisting of finitely separating functions. For every *φj*, define a relation Θ*φj ⊆ 7L × L* by

*F* Θ*φ x ⇔ x φ*2(*∨F* )*.*

*j j*

**Lemma 3.3** *Let L be an FS-domain. Then* Θ*φj is an approximable mapping on*

(*L, τ ◦ γ, 7L*) *for every j ∈ J.*

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**Proof.** From Definition [2.10](#_bookmark5), suppose that *F, F*

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*∈ 7L* and *F* Θ*φj F* . Then by the

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definition of Θ*φj* , we have *x φj* (*∨F* ) for every *x ∈ F* . Since *F*

is finite and

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*∨F ∈ F* , it follows that *∨F φj* (*∨F* ). Thus *F* Θ*φj ⟨F ⟩*.

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Let *F ± ⟨F ⟩* and *F* Θ*φj M* , then *x ∨F* and *m φj* (*∨F* ) for every *x ∈*

2 2 *′′* 2 *′′*

*F, m ∈ M* . As *φj* is order-preserving, *φj* (*∨F* ) *≤ φj* (*∨F*

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). Thus *m φj* (*∨F*

) for

any *m ∈ M* , which implies *F* Θ*φj M* .

Assume that *F* Θ*φ M* , then *m φ*2(*∨F* ) for all *m ∈ M* . By the interpolation

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property, there exists *a ∈ L* such that *m a φ*2(*∨F* ). Notice that *φj*(*∨F* ) =

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*φj*(*∨*(*↓ ∨ F* )) = *∨φj*(*↓ ∨ F* ). Moreover, *φ*2(*∨F* ) = *∨φ*2(*↓ ∨ F* ), then there exists

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*b ∈ ↓ ∨ F* such that *a φ*2(*b*). Set *G* = *{b}* and *G′*

*j*

= *{a}*. It is clear that

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*G, G ∈ 7L* such that *G ⊆ ⟨F⟩,M ⊆ ⟨G ⟩* and *G*Θ*φj G* . *2*

**Theorem 3.4** *Let* (*L, ≤*) *be an FS-domain. Then* (*L, τ ◦ γ, 7L*) *is an FS-closure space. Moreover,* (*L, ≤*) *is order isomorphic to* (*R*(*L*)*, ⊆*)*.*

**Proof.** By Theorem [2.9](#_bookmark4), (*L, τ ◦ γ, 7L*) is an F-augmented generalized closure space and (*L, ≤*) is order isomorphic to (*R*(*L*)*, ⊆*). Then it suffices to prove that *{*Θ*φj }j∈J* satisfies the conditions in Definition [3.1.](#_bookmark6) From Lemma [3.3](#_bookmark8), we know that Θ*φj* is an approximable mapping on the F-augmented generalized closure space (*L, τ ◦ γ, 7L*) for ever *j ∈ J* . We claim that *{*Θ*φj }j∈J* is a directed family with respect to inclusion

order. For any Θ*φj ,* Θ*φj* where *j*1*, j*2 *∈ J* , by definition of Θ*φj* , for any (*F, x*) *∈* Θ*φj*

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if and only if *x φ*2 (*∨F* ) for *i* = 1*,* 2*.* Because *{φj}j∈J* is directed, there exists

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*j ∈ J* such that *φj*1 *, φj*2 *≤ φj*. Now we prove that Θ*φj ,* Θ*φj ⊆* Θ*φj* for this *j*.

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Indeed, suppose (*F, x*) *∈* Θ*φ* , *i* = 1*,* 2*,* then *x φ*2 (*∨F* ) *≤ φ*2(*∨F* ), which implies

*ji*

(*F, x*) *∈* Θ*φj* . Thus Θ*φj ,* Θ*φj*

*i* S 1 2

We claim that

*j∈J* Θ*φj* = id*L*. Assume that (*F, x*) *∈* id*L*, then *x ∈ ⟨F⟩* = *↓∨F* .

*⊆* Θ*φj* .

*ji j*

Since *∨F* = *supj∈J φ*2(*∨F* ), there exists *j ∈ J* such that *x φ*2(*∨F* ). Thus

*j j*

(*F, x*) *∈* S*j∈J* Θ*φj* . Conversely, if (*F, x*) *∈* S*j∈J* Θ*φj* , then (*F, x*) *∈* Θ*φj* for some

*j ∈ J* . It follows that *x φ*2(*∨F* ) *∨F* . Hence *x ∈ ↓ ∨ F* = *⟨F⟩*. Therefore,

*j*

(*F, x*) *∈* id*L*.

For every *j ∈ J* , since *φj* is finitely separating function of *L*, there exists a finite subset *Mj ⊆ L* such that for each *x ∈ L*, there is *m ∈ Mj* such that *φj*(*x*) *≤ m ≤ x*. We set *Mj* = *{{φj*(*m*)*} | m ∈ Mj}*, then *Mj ⊆ 7L* is a finite subset family. For every *F ∈ 7L*, there is *m ∈ Mj* such that *φj*(*∨F* ) *≤ m ≤ ∨F* . If *F* Θ*φj x*, which implies *x φ*2(*∨F* ) *≤ φj*(*m*) *≤ φj*(*∨F* ) *∨F* . Therefore, *{φj*(*m*)*} ⊆ ⟨F⟩* and *x ∈ ⟨{φj*(*m*)*}⟩*. *2*

*j*

# The categorical equivalence between FS-closure spaces and FS-domains

In this section, we investigate the connection between approximable mappings and Scott-continuous functions. Moreover, we establish the equivalence between FS- closure spaces and FS-domains from the categorial point of view.

Given an FS-closure space (*X, τ ◦ γ, 7* ), define a relation id*X ⊆7 × X* by (*F, x*) *∈* id*X ⇔ x ∈ ⟨F⟩.*

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Given two approximable mappings Θ : (*X, τ ◦ γ, 7* ) *→* (*X ,τ ◦ γ , 7* ) and

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Θ : (*X ,τ ◦ γ , 7* ) *→* (*X ,τ*

* *γ , 7*

), define a relation Θ

* Θ *⊆7 × X* by

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*F* (Θ *◦* Θ)*x ⇔* (*∃G ∈7* ) (*F* Θ*G* & *G*Θ *x* )*.*

**Proposition 4.1** *FS-closure spaces and approximable mappings form a category that is denoted as* **FSC***.*

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**Proof.** Routine checks verify that Θ *◦* Θ is an approximable mapping from (*X, τ ◦*

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*γ, 7* ) to (*X ,τ*

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* *γ , 7*

*′′*

) and id*X* is an approximable mapping from (*X, τ ◦ γ, 7* )

to itself. *2*

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**Lemma 4.2** *Let* (*L, ≤*) *and* (*L , ≤* ) *be FS-domains. For any Scott-continuous*

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*function φ* : *L → L , deﬁne a relation* Θ*φ ⊆ 7L × L by*

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(*F, x* ) *∈* Θ*φ ⇔ x φ* (*∨F* )*.*

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*Then* Θ*φ is an approximable mapping from* (*L, τ ◦ γ, 7L*) *to* (*L ,τ ◦ γ , 7L′* )*.*

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*Conversely, suppose* (*X, τ ◦ γ, 7* ) *and* (*X ,τ ◦ γ , 7* ) *be FS-closure spaces. For*

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*any approximable mapping* Θ *from* (*X, τ ◦ γ, 7* ) *to* (*X ,τ*

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*φ*Θ : *R*(*X*) *→ R*(*X* ) *by*

* *γ , 7* )*, deﬁne a map*

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*φ*Θ(*U* )= *{x ∈ X |* (*∃F ∈ 7*) *F ⊆ U* & *F* Θ*x }.*

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*Then φ*Θ *is a Scott-continuous function from* (*R*(*X*)*, ⊆*) *to* (*R*(*X* )*, ⊆*)*.*

**Proof.** The proof is similar to that of [[8](#_bookmark18), Theorem 4.4, Theorem 4.6]. *2*

**Proposition 4.3** F : **FSC** *→* **FSdom** *is a functor which maps every FS-closure*

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*space* (*X, τ ◦ γ, 7* ) *to R*(*X*) *and approximable mapping* Θ: (*X, τ ◦ γ, 7* ) *→* (*X ,τ ◦*

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*γ , 7* ) *to φ*Θ : *R*(*X*) *→ R*(*X* )*, where φ*Θ *is deﬁned in Lemma* [*4.2*](#_bookmark10)*.*

**Proof.** Based on Theorem [3.2](#_bookmark7) and Lemma [4.2](#_bookmark10), F is well-defined. We check that F

preserves the identity morphism. For any *U ∈ R*(*X*),

F(id*X* )(*U* )= *φ*id*X* (*U* )

= *{x ∈ X |* (*∃F ∈ 7*) *F ⊆ U* & (*F, x*) *∈* id*X}*

= *{x ∈ X |* (*∃F ∈ 7*) *F ⊆ U* & *x ∈ ⟨F⟩}*

= *U*

= id*R*(*X*)*.*

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Let Θ be an approximable mapping from (*X, τ ◦ γ, 7* ) to (*X ,τ ◦ γ , 7* ) and

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Θ an approximable mapping from (*X ,τ*

* *γ , 7* ) to (*X ,τ*
* *γ , 7*

). For any

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*U ∈ R*(*X*) and *x*

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*∈ X* , we have

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*x ∈* F(Θ *◦* Θ)(*U* ) *⇔ x ∈ φ*Θ*′ ◦*Θ(*U* )

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*⇔* (*∃F ∈ 7*) *F ⊆ U* & *F* (Θ *◦* Θ)*x*

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*⇔* (*∃F ∈ 7,G ∈7* ) *F ⊆ U* & *F* Θ*G* & *G*Θ *x*

*′ ′ ′′*

*⇔* (*∃G ∈7* ) *G ⊆ φ*Θ(*U* ) & *G*Θ *x*

*⇔ x′′ ∈* F(Θ*′* )(F(Θ)(*U* ))*.*

*′ ′*

It implies F(Θ *◦* Θ) = F(Θ ) *◦* F(Θ).

*2*

Now, we obtain the main result of this paper.

**Theorem 4.4 FSC** *and* **FSdom** *are categorically equivalent.*

**Proof.** According to Theorem [3.4](#_bookmark9), it is sufficient to show that the functor F is full and faithful.

*′ ′ ′ ′*

We claim that F is full. Let (*X, τ ◦γ, 7* ) and (*X ,τ ◦γ , 7* ) be FS-closure spaces.

*′ ′*

For any Scott-continuous map *φ* : *R*(*X*) *→ R*(*X* ), define a relation Θ*φ ⊆ 7 × X*

by

*′*

*F* Θ*φx*

*′*

*⇔ x ∈ φ*(*⟨F⟩*)*.*

It is straightforward to check that Θ*φ* is an approximable mapping from (*X, τ ◦γ, 7* )

*′ ′ ′ ′*

to (*X ,τ ◦γ , 7* ). Now we only need to prove that F(Θ*φ*)= *φ*. Suppose *U ∈ R*(*X*),

F(Θ*φ*)(*U* )= *φ*Θ*φ* (*U* )

*′ ′ ′*

= *{x ∈ X |* (*∃F ∈ 7*) *F ⊆ U* & *F* Θ*φx }*

*′ ′*

= *{x ∈ X*

*′*

*|* (*∃F ∈ 7*) *F ⊆ U* & *x*

*∈ φ*(*⟨F⟩*)*}*

= *{φ*(*⟨F⟩*) *| F ∈7* & *F ⊆ U}*

= *φ*( *{⟨F⟩ | F ∈7* & *F ⊆ U}*)

= *φ*(*U* )*.*

This implies that F is full.

*′*

We claim that F is faithful. Suppose that Θ*,* Θ be two approximable mappings

*′ ′ ′ ′*

from (*X, τ ◦ γ, 7* ) to (*X ,τ ◦ γ , 7* ) such that *φ*Θ = *φ*Θ*′* . For any *F ∈ 7* and

*′ ′*

*x ∈ X* , we have

*′ ′*

(*F, x* ) *∈* Θ *⇔* (*∃G ∈ 7*)*G ⊆ ⟨F⟩* & *G*Θ*x*

*′*

*⇔ x ∈ φ*Θ(*⟨F⟩*)

*′*

*⇔ x ∈ φ*Θ*′* (*⟨F⟩*)

*′ ′*

*⇔* (*F, x* ) *∈* Θ *.*

Then Θ=Θ , and hence F is faithful. *2*

*′*

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