Available online at [www.sciencedirect.com](http://www.sciencedirect.com/)

[Electronic Notes in Theoretical Computer Science 324 (2016) 151–164](http://dx.doi.org/10.1016/j.entcs.2016.09.013)

[www.elsevier.com/locate/entcs](http://www.elsevier.com/locate/entcs)

Robustness of *f* - and *g*-generated Fuzzy (Co)Implications:

The Yager’s (Co)Implication Case Study

Renata Reiser[1](#_bookmark0), Rosana Zanotelli[1](#_bookmark0), Simone Costa[1](#_bookmark0), Luciana Foss[1](#_bookmark0)

*Centro de Desenvolvimento Tecnol´ogico, CDTEC Universidade Federal de Pelotas*

*Pelotas, RS, Brazil*

Benjamin Bedregal[2](#_bookmark0)

*Departamento de Inform´atica e Matem´atica Aplicada, DIMAP Universidade Federal Rio Grande do Norte*

*Natal, RN, Brazil*

**Abstract**

This paper studies the robustness of intuitionistic fuzzy implications in fuzzy reasoning based on Atanassov’s intuitionistic fuzzy logic. Starting with an evaluation of the sensitivity in representable fuzzy negations, we apply the results in the Yager’s classes of fuzzy implications called the *f* - and *g*-generated fuzzy implications. The paper formally states that the robustness preserves the projection functions in such class and also discusses their corresponding dual operators.

*Keywords:* Robustness analysis, Intuitionistic fuzzy logic, Yager’s implications, *f* - and *g*-generated implications.

# Introduction

Since Yager’s classes of fuzzy implications called the *f* - and *g*-generated implica- tions [[24](#_bookmark49)] have been used in common sense reasoning, there is a practical need for intuitionistic fuzzy versions of these operations, i.e., an operation *If* (*x, y*) that uses the membership degrees *μA*(*u*) = *a* and *μB*(*u*) = *b* of two intuitionistic fuzzy sets *A* and *B* to estimate the uncertainty degree of confidence in the statement

1 Email: *{*reiser,rzanotelli,simone.costa,lfoss*}*@inf.ufpel.edu.br

2 Email: [bedregal@dimap.ufrn.br](mailto:bedregal@dimap.ufrn.br)

<http://dx.doi.org/10.1016/j.entcs.2016.09.013>

1571-0661/© 2016 The Author(s). Published by Elsevier B.V.

This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

*A →f B*. These operations are also extensions of the corresponding crisp opera- tion: *If* (0*,* 1)=*If* (0*,* 0)=1, *If* (1*,* 0)=0 and *If* (1*,* 0) = 1.

The concepts of maximum and average perturbations of fuzzy sets [[25](#_bookmark50)], estimat- ing the maximum and average perturbation parameters for various methods of fuzzy reasoning is relevant for systems based on fuzzy logic (FL) and as a consequence for intuitionistic fuzzy logic (IFL).

* 1. *Main related works*

In [[15](#_bookmark41)], Li et al. study properties of some measures of robustness (or sensitivity) of fuzzy connectives and implication operators and discuss their relationships with perturbation properties of fuzzy sets. Many other works have discussed the robust- ness analysis also including the *δ*-sensitive approach, see e.g. [[14](#_bookmark40)], [[15](#_bookmark41)], [[16](#_bookmark42)], [[17](#_bookmark43)] and [[18](#_bookmark44)].

This paper extends the *δ*-sensitivity study of some intuitionistic fuzzy con- nectives (IFCs) according with results previously presented in [[15](#_bookmark41)], based on Atanassov’s Intuitionistic Fuzzy Logic (A-IFL), as presented in [[1](#_bookmark27)].

In [[24](#_bookmark49)], some properties of the Yager’s classes of fuzzy implications including the *h*-generated implications are discussed, describing their relationships amongst themselves and with the well established strong and residual implication classes [[9](#_bookmark35)]. Additionally, in [[19](#_bookmark45)], the Bandler-Kohout subproduct relational inference sys- tem with the fuzzy implication interpreted as the Yager’s classes of implications are reported, studying many of the desirable properties as interpolativity, conti- nuity, robustness and computational efficiency, expanding the choice of operations

available to practitioners.

A semantic behaviour of a fuzzy rule model is proposed in [[13](#_bookmark39)] as a pair of fuzzy implication and modus ponens generating function used for inference. Such methodology is applied to Yager’s models which are obtained from Yager implication function. By Yager’s implicative implication, it is shown to be midway between the usual residual and strong implications generated from the product t-norm. In fact, Yager’s implication belongs to a more general family of implications that can also be generated from the t-norms.

Such analysis can improve the study of the stability of systems based on intu- itionistic fuzzy rules. The notion of *δ*-sensitivity of fuzzy connectives in the fuzzy intuitionistic approach, which is characterized by the non-complementary relation- ship between the membership and non-membership functions, as proposed in [[2](#_bookmark28)], is considered in this work.

* 1. *Main contribution of the paper*

Following preliminary studies introduced in [[21](#_bookmark47)], this paper considers the robustness analysis defined on *δ*-sensitivity of the Atanassov intuitionistic fuzzy approach of the Yager’s implication classes, the *f* - and *g*-generated implications [[24](#_bookmark49)], focusing on their pointwise components obtained by the projections related to membership and non-membership functions.

Extending previous work in [[20](#_bookmark46)], [[21](#_bookmark47)] and [[27](#_bookmark52)], the paper provides an interpreta- tion of IFCs based on *δ*-sensitivity which is closely related to truth and non-truth in conditional fuzzy rules. Thus, not only the robustness of representable intuitionistic fuzzy negations are proposed but also the analyse of their perturbations based on *δ*-sensitive of the Yager’s implication classes and their dual construction, the *f* - and *g*-generated coimplications.

As the main result, the robustness of intuitionistic fuzzy *f* - and *g*-generated (co)implications can be expressed by the robustness of their arguments, by corre- sponding fuzzy *f* - and *g*-generated (co)implications. These results are summarized in commutative diagrams showing that the *δ*-sensitivity operator commutes with the *NS*-dual operator, by considering both approaches of FL and IFL.

* 1. *Outline of the paper*

The paper is organized as follows.

Firstly, the preliminaries describe the basic concepts of FCs and IFCs.

General results of robustness of FCs are stated in Sections [3](#_bookmark7) including the study of *δ*-sensitivity of fuzzy negations and fuzzy implications, mainly related to *f* - and *g*-generated implications.

In Section [4](#_bookmark16), we consider *δ*˜ = (*δ*1*, δ*2) *∈ U* 2, in the *δ*-sensitivity of an intuitionistic

operator *fI* at point **x**˜

*∈ U*˜*n* in terms of its left-projection (*lU*˜*n* (**x**˜)) and right-

projection (*rU*˜*n* (**x**˜)), which are related to the *δ*-sensitive of the membership and

non-membership degrees of an element *x ∈ χ* associated with the IFS *fI* (*U*˜*n*). Thus,

the study of *δ*-sensitivity of intuitionistic fuzzy negations and intuitionistic fuzzy implications, mainly related to *f* - and *g*-generated implications are considered.

Final remarks are reported in the conclusion.

# Preliminaries

By recalling some basic concepts of FL and IFL, we firstly report notions of FL as conceived by Zadeh [[26](#_bookmark51)] concerning negations and (co)implications [[10](#_bookmark36)]. Relevant papers studied different classes of fuzzy implications, see [[5](#_bookmark30),[11](#_bookmark37),[12](#_bookmark38)] and [[24](#_bookmark49)].

Let *U* = [0*,* 1] be the unit interval of real numbers. Recall that a function

*N* : *U→U* is a **fuzzy negation** if it satisfies, for all*∈U* the properties:

**N1**: *N* (0)=1 and *N* (1)=0; **N2**: If *x≥y* then *N* (*x*)*≤N* (*y*).

A fuzzy negation satisfying the involutive property:

**N3**: *N* (*N* (*x*)) = *x*, *∀x ∈ U* ; is called a **strong fuzzy negation** (SFN), e.g. the standard negation *NS*(*x*)=1 *− x*.

When **x** = (*x*1*, x*2*,..., xn*) *∈ Un* and *N* is a fuzzy negation, the following nota- tion is considered: *N* (**x**)= (*N* (*x*1)*,N* (*x*2)*,...,N* (*xn*)).

Let *N* be a negation. The *N* **-dual function** of *f* : *Un → U* is given by:

*fN* (**x**)= *N* (*f* (*N* (**x**)))*, ∀***x** *∈ Un.* (1)

An **implicator operator** *I* : *U* 2 *→ U* extends the classical implication:

**I0**: *I*(1*,* 1)=*I*(0*,* 1)=*I*(0*,* 0)=1, *I*(1*,* 0)=0.

**Definition 2.1** [[10](#_bookmark36)] In the sense of J. Fodor and M. Roubens, when *x, y, z ∈*

*U* 2, a fuzzy implication *I* : *U* 2 *→ U* is an implicator also verifying:

**I1**: *I*(*x, y*) *≥ I*(*z, y*) if *x ≤ z* (first place antitonicity); **I2**: *I*(*x, y*) *≤ I*(*x, z*) if *y ≤ z* (second place isotonicity); **I3**: *I*(0*, y*) = 1 (dominance of falsity);

**I4**: *I*(*x,* 1) = 1 (boundary condition);

Analogously, a coimplicator *J* : *U* 2 *→ U* verifies the conditions:

***J* 0**: *J* (0*,* 0)=*J* (1*,* 0)=*J* (1*,* 1)=0, *J* (0*,* 1)=1.

It is immediate that a fuzzy coimplication is a coimplicator analogously defined as a fuzzy implication, replacing **I3** and **I4** in Definition [2.1](#_bookmark1) by **J3** :*J* (*x,* 0) = 0 and **J4** :*J* (1*, y*) = 0, respectively.

Among many classes of fuzzy (co)implication functions (see, e.g., [[6](#_bookmark31)] and [[7](#_bookmark32)]), the class of **axiomatic representation** of fuzzy implications, named *A*-implications, is described in [[23](#_bookmark48)] in terms of non-commutativity property related to t-norms. The *A*-implications are based on a subset of the axioms listed in [[10](#_bookmark36)]. In this paper, we focus on Yager’s implication.

In [[24](#_bookmark49)], Yager proposed two new classes of fuzzy implications, called *f* -generated implications and *g*-generated implications, which can not be fulfilled in the above presented classes.

By [[24](#_bookmark49), Sect.3], let *f* : [0*,* 1] *→* [0*, ∞*] be an *f* -generator, which means, a strictly decreasing and continuous function such that *f* (0) = 1 and its pseudo-inverse *f* (*−*1) : [0*, ∞*] *→* [0*,* 1] is defined by: *f* (*−*1)(*x*) = *f−*1(*x*)*,* if *x ≤ f* (0); and 0*,* otherwise. When 0*·∞*=0, an *f* -generated fuzzy implication *If* :*U* 2*→U* is given by

*If* (*x, y*)= *f* (*−*1) (*x · f* (*y*)) *.*

Moreover, let *g* : [0*,* 1] *→* [0*, ∞*] bea *g*-generator, which means, a strictly increasing and continuous function such that *g*(0) = 0 and its pseudo-inverse *g*(*−*1) : [0*, ∞*] *→* [0*,* 1] is defined by: *g*(*−*1)(*x*) = *g−*1(*x*)*,* if *x ≤ g*(1); and 1*,* otherwise. When 1 = *∞*

0

and *∞·* 0= *∞*, a *g*-generated fuzzy implication *Ig* :*U* 2*→U* is given by

*I* (*x, y*)= *g*(*−*1) 1 *· g*(*y*) *.*

*g*

*x*

**Proposition 2.2** *[*[*24*](#_bookmark49)*] The binary function IY ,* (*JY* ):*U* 2*→U given by*

*IYf* (*x, y*)= 1*, if x* = *y* = 0*; and IYf* (*x, y*)= *yx, otherwise.* (2)

*JYf* (*x, y*)= 0*, if x* = *y* = 1*; and JYf* (*x, y*)=1 *−* (1 *− y*)1*−x, otherwise.* (3) *is an f-generated fuzzy (co)implication called Yager(co)implication. Additionally, the function Ig,* (*Jg*):*U* 2*→U given by*

1

*IYg* (*x, y*)= 1*, if x* = *y* = 0*; and IYg* (*x, y*)=1 *−* (1 *− y*) *x , otherwise.* (4)

1

*JYg* (*x, y*)= 0*, if x* = *y* = 1*; and JYg* (*x, y*)= *y* 1*−x , otherwise.* (5)

*is an g-generated fuzzy (co)implication.*

**Example 2.3** [[4](#_bookmark33), Examples 3 and 4] Eqs. ([2](#_bookmark2)) and Eq. ([4](#_bookmark2)) (Eqs. ([3](#_bookmark2)) and Eq. ([5](#_bookmark2))) define fuzzy (co)implications, according to Definition [2.1](#_bookmark1).

**Corollary 2.4** *Both functions* (*IYg , JYg* ) *and* (*IYf , JYf* ) *deﬁne pairs of NS-dual fuzzy (co)implications.*

**Proof.** Straightforward. *2*

* 1. *Intuitionistic fuzzy connectives*

This section briefly study intuitionistic fuzzy connectives. For further references, see [[1](#_bookmark27),[2](#_bookmark28),[3](#_bookmark29),[8](#_bookmark34)] and [[9](#_bookmark35)].

According to [[1](#_bookmark27)], an intuitionistic fuzzy set (IFS) *AI* in a non-empty, universe *χ*, is expressed as *AI* = *{*(*x, μA*(*x*)*, νA*(*x*)) : *x ∈ χ, μA*(*x*)+ *νA*(*x*)) *≤* 1*}*. Thus, an intuitionistic fuzzy truth value of an element *x* in an IFS *AI* is related to the ordered pair (*μA*(*x*)*, νA*(*x*)). Moreover, an IFS *AI* generalizes a FS *A* = *{*(*x, μA*(*x*)) : *x ∈ χ}*, since *νA*(*x*), which means that the non-membership degree of an element *x*, is less than or equal to the complement of its membership degree *μA*(*x*), and therefore *νA*(*x*) is not necessarily equal to its complement 1 *− μA*(*x*).

Let *U*˜ = *{*(*x*1*, x*2) *∈ U* 2*|x*1 *≤ NS*(*x*2)*}* be the set of all intuitionistic fuzzy

values and *lU*˜ *, rU*˜ : *U*˜ *→ U* be the projection functions on *U*˜, which are given by

*lU*˜ (*x*˜)= *lU*˜ (*x*1*, x*2)= *x*1 and *rU*˜ (*x*˜)= *rU*˜ (*x*1*, x*2)= *x*2, respectively.

Thus, for all **x**˜ = (*x*˜1*,..., x*˜*n*) *∈ U*˜*n*, such that *x*˜*i* = (*xi*1*, xi*2) and *xi*1 *≤ NS*(*xi*2) when 1 *≤ i ≤ n*, considering *lU*˜*n , rU*˜*n* : *U*˜*n → Un* as the projections given by:

*lU*˜*n* (**x**˜)= (*lU*˜ (*x*˜1)*, lU*˜ (*x*˜2)*,..., lU*˜ (*x*˜*n*)) = (*x*11*, x*21*,... xn*1); (6)

*rU*˜*n* (**x**˜)= (*rU*˜ (*x*˜1)*, rU*˜ (*x*˜2)*,... rU*˜ (*x*˜*n*)) = (*x*12*, x*22*,... xn*2)*.* (7)

By [[2](#_bookmark28)], for *x*˜*, y*˜ *∈ U*˜, the order relation *≤U*˜ is given as

*x*˜ *≤U*˜ *y*˜ *⇔ x*1 *≤ y*1 and *x*2 *≥ y*2*, suchthat*˜0= (0*,* 1) *≤U*˜ *x*˜ and ˜1= (1*,* 0) *≥U*˜ *x*˜(*.*8) Moreover, the folowing expression is known:

*x*˜ *≤U*˜ *y*˜ *⇔ x*1 *≤ y*1 and *x*2 *≤ y*2*,* (9)

An *intuitionistic fuzzy negation* (IFN shortly) *NI* : *U*˜

*x*˜*, y*˜*∈ U*˜, the following properties:

**N***I* **1** : *NI* (˜0) = *NI* (0*,* 1) = ˜1 and *NI* (˜1) = *NI* (1*,* 0) = ˜0;

**N***I* **2**: If *x*˜ *≥ y*˜ then *NI* (*x*˜) *≤ NI* (*y*˜).

*→ U*˜

satisfies, for all

Additionally, *NI* is a **strong intuitionistic fuzzy negation** (SIFN) verifying the condition:

**N***I* **3**: *NI* (*NI* (*x*˜)) = *x*˜, *∀x*˜ *∈ U*˜.

Consider *NI* as IFN in *U*˜ and *f*˜ : *U*˜*n → U*˜. For all **x**˜ = (*x*˜1*,..., x*˜*n*) *∈ U*˜*n*, the

*NI* **-dual intuitionistic function of** *f*˜, denoted by *f*˜*NI*

: *U*˜*n → U*˜, is given by:

*f*˜*NI* (**x**˜)= *NI* (*f*˜(*NI* (*x*˜1)*,..., NI* (*x*˜*n*)))*.* (10)

When *N*˜*I* is a SIFN, *f*˜ is a self-dual intuitionistic function. Additionally, by [[3](#_bookmark29),

Theorem 1] [[8](#_bookmark34)], a SIFN *NI* : *U*˜ *→ U*˜

such that:

is a SIFN iff there exists a SFN *N* : *U → U*

*NI* (*x*˜)= (*N* (*NS*(*x*2))*, NS*(*N* (*x*1)))*,* (11)

Additionally, if *N* = *NS*, Eq. [11](#_bookmark5) can be reduced to *NI* (*x*˜)= (*x*2*, x*1).

According with [[7](#_bookmark32), Definition 3], an Atanassov intuitionistic fuzzy implication *II* : *U* 2 *→ U* is an intuitionistic fuzzy implicator such that, the analogous conditions from **I***I* **1** to **I***I* **4** in Definition [2.1](#_bookmark1) are verified with the additional property:

˜ ˜

**I***I* **5:** If *x*˜ = (*x*1*, x*2) such that *x*1 + *x*2=1 it holds that *NS*(*x*1 + *x*2)=0

Thus, recovering Definition [2.1](#_bookmark1) of a fuzzy implication in the sense of J. Fodor and M. Roubens’ work [[10](#_bookmark36)], an intuitionistic fuzzy implication also reproduces fuzzy (co)implications if, for all *x*˜ = (*x*1*, x*2), *y*˜ = (*y*1*, y*2) *∈ U* we have *x*1 = *NS*(*x*2) and *y*1 = *NS*(*y*2).

˜

According to [[2](#_bookmark28)], another way of defining an operator *II* is to consider boundary conditions in **I***I* 0 and properties **I***I* 1 and **I***I* 2. In [[7](#_bookmark32)], Bustince et al. constructed fuzzy implications for intuitionistic fuzzy logic, in the sense of [[7](#_bookmark32), Definition 3], based on aggregation operators and SFNs.

Considering the above results, the functions *II* (*JI* ) : *U*˜2 *→ U*˜ are **repre-**

˜ ˜

**sentable fuzzy (co)implications** based on SFN *NS* : *U → U* if there exist fuzzy (co)implications *Ia, Ib*(*Ja, Jb*) : *U* 2 *→ U* such that, for all *x*˜*, y*˜ *∈ U*˜, the following holds:

*II* (*x*˜*, y*˜)= (*Ia*(*NS*(*x*2)*, y*1)*, NS*(*Ib*(*x*1*, NS*(*y*2)))); (12)

*JI* (*x*˜*, y*˜)= (*Ja*(*NS*(*x*2)*, y*1)*, NS*(*Jb*(*x*1*, NS*(*y*2))))*.* (13)

**Proposition 2.5** *Let If , Ig,* (*Jf , Jg*) : *U* 2 *→ U be f- (g-)generated fuzzy (co)implications deﬁned in Proposition* [*2.2*](#_bookmark2)*. The functions IYf I, IYg I* (*JYf I, JYg I* ) : *U*˜2 *→ U*˜ *are representable fuzzy (co)implications expressed as:*

*IYf I* (*x*˜*, y*˜)=  *y ,* 1 *−* (1 *− y*2) ; *JY* (*x*˜*, y*˜)= 1 *−* (1 *− y*1) *, y*1)*, y*  (*.*14)

1*−x*2 *x*1 *x*2 1*−x*1

1

*f I*

2

1 1 1 1

*IY*

(*x*˜*, y*˜)=

1 *−* (1 *− y*1) 1*−x*2 *,y x*1

; *JY*

(*x*˜*, y*˜)=

*y x*2 *,* 1 *−* (1 *− y*2) 1*−x*1

*.* (15)

*g I*

2

*g I*

1

**Proof.** Eq.([14](#_bookmark6)) follows from Eq.([12](#_bookmark5)) by taking *Ia* = *Ib* = *IYf* and *Ja* = *Jb* = *JYf* . And, Eq.([15](#_bookmark6)) follows from Eq.([13](#_bookmark5)) by taking *Ia* = *Ib* = *IYg* and *Ja* = *Jb* = *JYg* . *2*

# Pointwise sensitivity of fuzzy connectives

Based on [[15](#_bookmark41)] and [[20](#_bookmark46)], the study of a *δ*-sensitivity of *n−*order function *f* at point **x**

on the domain *U* is considered, in the context of robustness of fuzzy logic, mainly

related to the class of (*S, N* )-implications.

**Definition 3.1** [[15](#_bookmark41), Definition 1] Let *f* : *Un → U* be an *n−*order function, *δ ∈ U* and **x** = (*x*1*, x*2*,... xn*), **y** = (*y*1*, y*2*,... yn*) *∈ Un*. The *δ***-sensitivity of** *f* **at point x**, denoted by Δ*f* (**x***, δ*), is given by

Δ*f* (**x***, δ*)= sup*{|f* (**x**) *− f* (**y**)*|* : **y** *∈ Un* and  (**x***,* **y**) *≤ δ}* (16) wherever (**x***,* **y**) = max*{|xi − yi|* : *i* = 1*,..., n}*. Additionally, **the maximum** *δ*

**sensitivity of** *f* , denoted as Δ*f* (*δ*), is defined as follows:

Δ*f* (*δ*)=

**x***∈Un*

Δ*f* (**x***, δ*)*.* (17)

**Proposition 3.2** *[*[*20*](#_bookmark46)*, Theorem 1] If N* = *NS and fN is the N-dual function of f*

*then* ***the sensitivity of*** *fN* ***at point* x** *is given by*

Δ*fN* (**x***, δ*)= Δ*f* (*N* (**x**)*, δ*)*.* (18)

* 1. *δ sensitivity of f− and g−generated fuzzy (co)implications*

Now, we investigate the *δ*-sensitivity in FCs, in terms of Definition [3.1](#_bookmark8) based on results previously presented in [[15](#_bookmark41)]. In order to provide an easier notation, when *f* : *U* 2 *→ U* and **x** = (*x, y*) *∈ U* 2, consider the following notations:

*f[***x***|≡ f* ((*x − δ*) *∨* 0*,* (*y* + *δ*) *∧* 1); *f[***x***♩≡ f* ((*x* + *δ*) *∧* 1*,* (*y − δ*) *∨* 0)*.*

**Proposition 3.3** *[*[*15*](#_bookmark41)*, Theorem 1] Consider f* : *U* 2 *→U, δ ∈ U and* **x** = (*x, y*) *∈ U* 2*. If f veriﬁes both properties, ﬁrst place antitonicity (***I1***) and second place isotonicity (***I2***), then:*

Δ*f* (**x***, δ*)= (*f* (**x**) *− f[***x***♩*) *∨* (*f[***x***|− f* (**x**))*.* (19)

**Proposition 3.4** *The δ-sensitivity of the functions IYf , JYf , IYg , JYg deﬁned in Proposition* [*2.2*](#_bookmark2) *by Eqs. (*[*2*](#_bookmark2)*)-(*[*5*](#_bookmark2)*) is given by Eq. (*[*19*](#_bookmark12)*).*

**Proof.** Straightforward Proposition [3.3](#_bookmark11) and Corollary [2.3](#_bookmark3). *2*

* 1. *Maximum sensitivity of f− and g−generated fuzzy (co)implications*

In the following, we consider the maximum *δ* sensitivity of the *f* -and *g*-generated fuzzy implications *IYf* and *IYg* , showing that they coincide with their *NS*-dual con-

structions, (*JYf*

) and (*JYf*

), by considering the endpoints of unitary interval in *U* 2.

**Remark 3.5** Based on Eqs. ([19](#_bookmark12)) and ([18](#_bookmark10)), also including results in Proposition[18](#_bookmark10), we obtain the following:

Δ*IYf* ((0*,* 0)*, δ*)= (1 *−* 0) *∨* (1 *−* 1) = 1 = Δ*JYf* ((1*,* 1)*, δ*) ;

Δ*IYf* ((0*,* 1)*, δ*)= (1 *−* (1 *− δ*)*δ*) *∨* (1 *−* 1) = 1 *−* (1 *− δ*)*δ* = Δ*JYf* ((1*,* 0)*, δ*) ;

Δ*IYf* ((1*,* 1)*, δ*)= (1 *−* (1 *− δ*)) *∨* (1 *−* 1) = *δ* = Δ*JY* ((0*,* 0)*, δ*);

Δ*IYf* ((1*,* 0)*, δ*)= (0 *−* 0) *∨* (*δ*1 *− δ −* 0) = *δ*(1*−δ*) = Δ*JYf* ((0*,* 1)*, δ*).

**Remark 3.6** Based on Eqs. ([19](#_bookmark12)) and ([18](#_bookmark10)), also including results in Proposition[18](#_bookmark10), we obtain the following:

1

*∞*

Δ*IYg* ((0*,* 0)*, δ*)= 1 *δ ∨* (*−*1 *− δ*)

=1= Δ*JYg* ((1*,* 1)*, δ*) ;

1 1

Δ*IYg* ((0*,* 1)*, δ*)= *δ δ ∨* 0= *δ δ* = Δ*JYg* ((1*,* 0)*, δ*) ;

Δ*IYg* ((1*,* 1)*, δ*)= *δ ∨* 0= *δ* = Δ*JYg* ((0*,* 0)*, δ*);

1*−δ*  1

*g*

Δ*IYg*

((1*,* 0)*, δ*)=0 *∨* 1 *−* (1 *− δ*)

=1 *−* (1 *− δ*) 1*−δ* = Δ*JY*

((0*,* 1)*, δ*).

**Proposition 3.7** *The maximum sensitivity of the f- and g-generated fuzzy (co)implications IYf , JYf , IYg , JYg , as deﬁned in Proposition* [*2.2*](#_bookmark2) *by Eqs. (*[*2*](#_bookmark2)*)-(*[*5*](#_bookmark2)*), is given as follows:*

Δ*IYf* (*δ*)=1= Δ*IYg* (*δ*) *and* Δ*JYg* (*δ*)=1= Δ*JYf* ((*δ*); (20)

Δ*IYg* (*δ*)=1= Δ*IYg* (*δ*) *and* Δ*JYg* (*δ*)=1= Δ*JYg* (*δ*)*.* (21)

**Proof.** Straightforward from Remarks [3.5](#_bookmark13) and [3.6](#_bookmark14). *2*

Based on Proposition [3.7](#_bookmark15), the maximum *δ* sensitivity of the *f* - and *g*-generated fuzzy (co)implications *IYf* and *IYg* is related to the endpoints (0*,* 0) and (1*,* 1), re- spectively. Such results are closely related to their corresponding definition.

Moreover, based on Remarks [3.5](#_bookmark13) and [3.6](#_bookmark14), one can observe that *IYg* is more robust than *IYf* at point (0*,* 1), since Δ*IYf* ((0*,* 1)*, δ*) *>* Δ*IYg* ((0*,* 1)*, δ*). In the converse, *IYf* is more robust than *IYg* at point (1*,* 0), since Δ*IYf* ((1*,* 0)*, δ*) *<* Δ*IYg* ((1*,* 0)*, δ*).

# Robustness of intuitionistic fuzzy connectives

In this section, we consider *YI ∈ {Yf I, YgI}* which means, *YI* denotes either *f−* or

*g−*generated intuitionistic fuzzy implication.

In order to provide a formal definition of robustness which can be applied to

*n*-order intuitionistic fuzzy *f* - and *g*-generated implication operators, consider def-

inition of the *δ*-sensitivity of an *n−*order fuzzy negation *fI* :

**x**˜ = (**x**˜1*,* **x**˜2*,...,* **x**˜*n*) *∈ U*˜*n*.

*U*˜*n → U*˜

at point

Thus, when *δ*˜ = (*δ*1*, δ*2) *∈ U* 2, the *δ*-sensitivity of an intuitionistic operator

*fI* at point **x**˜

*∈ U*˜*n* is defined in terms of its left-projection (*lU*˜*n* (**x**˜)) and right-

projection (*rU*˜*n* (**x**˜)), which are related to the *δ*-sensitive of the membership and

non-membership degrees of an element *x ∈ χ* associated with the IFS *fI* (*U*˜*n*).

**Definition 4.1** For **y**˜ *∈ U*˜*n*, the *δ***-sensitivity of** *fI* **at point x**˜ is defined by Δ*fI* (**x**˜*, δ*˜)=sup*{|fI* (**x**˜)*−fI* (**y**˜)*|* : (*lU*˜*n* (**x**˜)*, lU*˜*n* (**y**˜))*≤δ*1 and (*rU*˜*n* (**x**˜)*, rU*˜*n* (**y**˜))*≤δ*2*},*

when (**x***,* **y**) = max*{|xi*1 *− yi*1*|* : *i* = 1*,..., n}*, (**x***,* **y**) = min*{|xi*2 *− yi*2*|* : *i* = 1*,..., n}*.

* 1. *Preserving the robustness of representable negations*

The next proposition states that the pointwise sensitivity is preserved by the pro- jection functions applied to an intuitionistic fuzzy negation (IFN) which is repre- sentable in the same sense of [[3](#_bookmark29)] and [[8](#_bookmark34)].

**Proposition 4.2** *[*[*27*](#_bookmark52)*] Let NI* : *U*˜*n → U*˜

*be a SIFN as deﬁned by Eq.(*[*11*](#_bookmark5)*). When*

*δ*˜ = (*δ*1*, δ*2) *∈ U* 2 *and* **x**˜ *∈ U*˜*n, the δ****-sensitivity of*** *NI* ***at point* x**˜*, is given by*

Δ*NI* (**x**˜*, δ*˜)= (Δ*N◦NS* (*rU*˜*n* (**x**˜)*, δ*2)*,* Δ*NS◦N* (*lU*˜*n* (**x**˜)*, δ*1))*.* (22)

**Corollary 4.3** *[*[*27*](#_bookmark52)*] When*

*δ*˜ = (*δ*1*, δ*2) *∈ U* 2*, NI* = *NSI and* **x**˜

*∈ U*˜*n, the δ****-***

***sensitivity of*** *NI* ***at point* x**˜*, can also be expressed as*

*lU*˜ (Δ*NI* (**x**˜*, δ*˜))=Δ*l* ˜ *◦NI* (*rU*˜*n* (**x**˜)*, rU*˜ (*δ*˜)); (23)

*U*

*rU*˜ (Δ*NI* (**x**˜*, δ*˜))=Δ*r* ˜ *◦NI* (*lU*˜*n* (**x**˜)*, lU*˜ (*δ*˜)))*.* (24)

*U*

*In particular, we have that* Δ*NSI* (**x**˜*, δ*˜)= (*δ*2*, δ*1)*.*

The diagram below summarizes the main results of Proposition [4.2](#_bookmark17) and Corol- lary [4.3](#_bookmark18): the robustness of representable IFNs can be expressed by robustness of their arguments:

(**x**˜*, δ*) *Eq.* ([22](#_bookmark19)) ) Δ (**x**˜*, δ*)

*NSI*

*Eqs.*([6](#_bookmark4))([7](#_bookmark4)) *Eq.*([6](#_bookmark4))([7](#_bookmark4))

v *Eqs.* ([23](#_bookmark18))*,* ([24)](#_bookmark18)) v

(*rU*˜2 (**x**˜*, δ*)*, lU*˜2 (**x**˜*, δ*)) (Δ*rU*˜ *◦NI* (**x**˜*, δ*)*,* Δ*rU*˜ *◦NI* (**x**˜*, δ*))

Fig. 1. Robustness operator on the class of representable IFNs.

* 1. *Robustness of intuitionistic fuzzy functions and NSI-dual constructions*

From the *δ*-sensitivity of *fI* : *U*˜*n → U*˜ at point **x**˜ one can obtain the *δ*-sensitivity of corresponding dual construction, as described in the following proposition:

**Proposition 4.4** *Let fI* : *U*˜*n → U*˜ *be a representable fuzzy (co)implication deﬁned*

*by Eq.(*[*12*](#_bookmark5)*) (Eq.(*[*13*](#_bookmark5)*)) and* Δ*fI* (**x**˜*, δ*) *be the δ-sensitivity of fI at point* **x**˜*. When*

*δ*˜ = (*δ*1*, δ*2) *∈ U* 2 *, NI* = *NSI and fIN is the NI-dual function of fI,* ***the*** *δ****-***

*I*

***sensitivity of*** *fINI* ***at point* x**˜ *is given by*

Δ(*fI* )*NSI* (**x**˜*, δ*˜)= Δ*fI* (*NSI* (**x**˜)*, δ*˜) *.* (25)

**Proof.** For all **x**˜*,* **y**˜ *∈ U*˜*n*, it holds that:

Δ(*fI* )*NS* (**x**˜*, δ*˜)= (Δ*fN* (*NS*(*x*12)*, x*21)*, δ*1)*,* Δ*NS◦fN* (*x*11*, NS*(*x*22))*, δ*2)) by Eqs. ([6](#_bookmark4)),([7](#_bookmark4))

= (Δ*f* ((*x*12*, NS*(*x*21))*, δ*1)*,* Δ*NS◦f* ((*NS*(*x*11)*, x*22)*, δ*2)) by Eq.([18](#_bookmark10))

= Δ*fI* (*lU*˜ (*NI* (**x**˜)*, δ*˜)*,* (Δ*fI* (*rU*˜ (*NI* (**x**˜)*, δ*˜)) by Eqs.([23](#_bookmark18)),([24](#_bookmark18))

= (Δ*fI* (*NSI* (**x**˜)*, NSI* (*δ*))) by Eqs.([6](#_bookmark4)),([7](#_bookmark4))

Therefore Eq.([25](#_bookmark21)) holds. *2*

* 1. *δ-sensitivity of f− and g−generated intuitionistic fuzzy (co)implications*

In this section, we study the robustness of the Atanassov intuitionistic fuzzy ap- proach related to the *f−* and *g−*generated intuitionistic fuzzy (co)implications *IYf I*

2 2 ˜ ˜

(*JYg I* ) at point **x**˜ *∈ U*˜ . For that, when *fI* : *U*˜ *→ U*˜, *δ* = (*δ*1*, δ*2) *∈ U* and

**x**˜ = (*x*˜*, y*˜) *∈ U*˜2, we follow the notations below:

*fI[***x**˜*|≡ fI* ((*x*˜ *− δ*˜) *∨* ˜0*,* (*y*˜ + *δ*˜) *∧* ˜1); *f[***x**˜*♩≡ fI* ((*x*˜ + *δ*˜) *∧* ˜1*,* (*y*˜ *− δ*˜) *∨* ˜0)*.*

**Proposition 4.5** *Consider fI* :

*U*˜2 *→ U*˜*,*

*δ*˜ = (*δ*1*, δ*2) *∈ U*˜

*and* **x**˜

*∈ U*˜2*. If fI*

*veriﬁes both properties, ﬁrst place antitonicity and second place isotonicity, then:*

Δ*fI* (**x**˜*, δ*˜)= (*fI* (**x**˜) *− f[***x**˜*♩*) *∨* (*f[***x**˜*|− f* (**x**˜)) (26)

**Proof.** Straightforward Proposition [3.3](#_bookmark11). *2*

**Proposition 4.6** *Let IYf I* (*JYf I* )*, IYg I* (*JYg I* ) : *U*˜ *→ U*˜ *be a representable f- and*

2

*g-generated (co)implications as given by Eqs.(*[*14*](#_bookmark6)*) and (*[*15*](#_bookmark6)*). If δ*˜ = (*δ*1*, δ*2) *∈ U* 2

*and* **x**˜ = (**x**1*,* **x**2) *∈ U*˜2 *the δ-sensitivity of both IY*

*f I*

*deﬁned by Eq.(*[*26*](#_bookmark22)*).*

(*JYf I*

)*, IYg I*

(*JYg I*

) *at point* **x**˜ *is*

**Proof.** Straightforward, since they verify both Properties **I1***I* and **I2***I* , by Propo- sition[2.5](#_bookmark6). *2*

**Proposition 4.7** *Let IYf I* (*JYf I* )*, IYg I* (*JYg I* ) : *U*˜ *→ U*˜ *be a representable f- and*

2

* 1. *enerated (co)implications as given by Eqs.(*[*14*](#_bookmark6)*) and (*[*15*](#_bookmark6)*). If δ*˜ = (*δ*1*, δ*2) *∈ U* 2

*and* **x**˜ = (**x**1*,* **x**2) *∈ U*˜2 *the δ****-sensitivity of both*** *IY*

*f I*

*expressed as follows:*

***and*** *IYg I*

***at point* x**˜ *can be*

Δ*I* (**x**˜*, δ*˜)= Δ*I*

*Y*

*f*

*Y*

*f I*

(*lU*˜ (*NS*(**x**1)*,* **x**2)*, δ*1)*,* Δ*IYf*

(*rU*˜ (*NS*(**x**1)*,* **x**2))*, δ*2) ; (27)

Δ*IYg*

*I*

(**x**˜*, δ*˜)= Δ*IYg* (*lU*˜ (*NS*(**x**1)*,* **x**2)*, δ*1)*,* Δ*IYg* (*rU*˜ (*NS*(**x**1)*,* **x**2))*, δ*2) *.* (28)

*Analogously, the δ-sensitivity of JYf I and JYg I at point* **x**˜ *is deﬁned by*

Δ*J* (**x**˜*, δ*˜)= Δ*J*

*Y*

*f*

*Y*

*f I*

(*lU*˜ (**x**1*, NS*(**x**2))*, δ*1)*,* Δ*JYf*

(*rU*˜ (**x**1*, NS*(**x**2)))*, δ*2) ; (29)

Δ*JYg*

*I*

(**x**˜*, δ*˜)= Δ*JYg* (*lU*˜ (**x**1*, NS*(**x**2))*, δ*1)*,* Δ*JYg* (*rU*˜ (**x**1*, NS*(**x**2)))*, δ*2) *.* (30)

**Proof.** Let *IYf I* be an intuitionistic fuzzy Yager’s implication which is representable by the Yager fuzzy implication *IYf* and the standard negation *NS*, as defined by Eq.( [12](#_bookmark5)), then:

Δ*IY* (**x**˜*, δ*˜)=

=sup*{|IY* (**x**˜)*−IY* (**y**˜)*|* : **y**˜ *∈ U*˜2*,* (*l* ˜2 (**x**˜)*,l* ˜2 (**y**˜)) *≤ δ*1 and (*r* ˜2 (**x**˜)*,r* ˜2 (**y**˜)) *≤ δ*2*}*

*U*

*U*

*U*

*U*

=sup*{|IY* ((*x*11*, x*12)*,* (*x*21*, x*22))*−IY* ((*y*11*, y*12)*,* (*y*21*, y*22))*|* : **y**˜ *∈ U*˜2 and

(*lU*˜2 (**x**˜)*, lU*˜2 (**y**˜)) *≤ δ*1 and (*rU*˜2 (**x**˜)*, rU*˜2 (**y**˜)) *≤ δ*2*}*

=sup*{|*(*IY* (*NS*(*x*12)*, x*21)*, NS*(*IY* (*x*11*, NS*(*x*22))*−*(*IY* (*NS*(*y*12)*, y*21)*, NS*(*IY* (*y*11*, NS*(*y*22))*|* :

**y**˜ *∈ U*˜2 and (*l* ˜*n* (**x**˜)*,l* ˜2 (**y**˜)) *≤ δ*1 and (*r* ˜2 (**x**˜)*,r* ˜2 (**y**˜)) *≤ δ*2*} by Eq.*( [12](#_bookmark5))

*U*

*U*

*U*

*U*

=sup*{|l* ˜2 (*IY* (**x**1*,* **x**2))*−l* ˜(*IY* (**y**1*,* **y**2))*|* : **y**˜ *∈ U*˜2 and (*l* ˜2 (**x**˜)*,l* ˜2 (**y**˜)) *≤ δ*1*},*

*U*

*U*

*U*

*U*

sup*{|r* ˜2 (*IY* (**x**1*,* **x**2))*−r* ˜(*IY* (**y**1*,* **y**2))*|* : **y**˜ *∈ U*˜2 and (*r* ˜2 (**x**˜)*,l* ˜2 (**y**˜)) *≤ δ*2*}*)

*U*

*U*

*U*

*U*

=(Δ*IY* (*lU*˜ (*NS*(**x**1)*,* **x**2)*, δ*1)*,* Δ*IY* (*rU*˜ (*NS*(**x**1)*,* **x**2)*, δ*2)) *by Eq.*( [16](#_bookmark9))*.*

Therefore, for all **x**˜*,* **y**˜ *∈ U*˜2, by Eqs. ([6](#_bookmark4)) and ([7](#_bookmark4)), it follows that *lU*˜2 (Δ*I* (**x**˜*, δ*˜)) =

*Y*

Δ*IY* (*lU*˜ (*NS*(**x**1)*,* **x**2)*, δ*1); and *rU*˜2 (Δ*IY* (**x**˜*, δ*)) = Δ*IY* (*rU*˜ (*NS*(**x**1)*,* **x**2))*, δ*2). In analo-

gous manner, Eq. ([28](#_bookmark24)) and corresponding dual constructions can be proved. *2*

The diagram below summarizes the main results of Propositions [4.6](#_bookmark23) and [4.7](#_bookmark24) related to an *f* -generated intuitionistic fuzzy (co)implications: the robustness of intuitionistic fuzzy *f* -generated (co)implication can be expressed by the robustness of their arguments, by corresponding fuzzy *f* -generated (co)implications:

(**x**˜*, δ*˜) *Eq.* ([26](#_bookmark22)) ) Δ (**x**˜*, δ*˜)

*IY*

*I*

*Eqs.*([6](#_bookmark4))([7](#_bookmark4))

v

*Eqs.*([6](#_bookmark4))([7](#_bookmark4))

v

(*rU*˜2

(**x**˜*, δ*˜)*, lU*˜2

(**x**˜*, δ*˜)) *Eqs.* ([27](#_bookmark24))([29)](#_bookmark24)) (Δ

*U* *I*

*l* ˜ *◦IY*

(**x**˜*, δ*˜)*,* Δ

*rU*˜ *◦IYI*

(**x**˜*, δ*˜))

Fig. 2. Robustness operator on the class of representable IFNs.

The following theorem extends the results in [[15](#_bookmark41)]:

**Theorem 4.8** *Consider δ*˜ *∈ U*˜ *e* **x**˜ *∈ U*˜2*. It follows that:*

* + 1. Δ*IY I* (**x**˜*, δ*˜) = Δ*JY I* (*NS*(**x**˜)*, δ*˜) *when IY (JY ) is an f-generated fuzzy (co)implication;*
    2. Δ*IY I* (**x**˜*, δ*˜) = Δ*JY I* (*NS*(**x**˜)*, δ*˜) *when IY (JY ) is an g-generated fuzzy (co)implication.*

**Proof.** Straightforward Proposition [4.4](#_bookmark20). *2*

Table[4.3](#_bookmark25) summarizes the *δ*-sensitivity of the Atanassov intuitionistic approach of the Yager’s (co)implication, in the endpoints of *U*˜.

In the following, we discuss the examples in the first line. Other cases in Table[4.3](#_bookmark25) can be analogously extended:

**x**˜ Δ*I* (**x**˜*, δ*˜) Δ*J* (**x**˜*, δ*˜)

*Y Y*

*f f I*

(˜0*,* ˜0) (1*,* 1) (*δ*1*, δ*2)

1

2

(˜0*,* ˜1) 1 *−* (1 *− δ*1)*δ*1 *,* 1 *−* (1 *− δ*2)*δ*2 *δ*(1*−δ*1)*, δ*(1*−δ*2)

(˜1*,* ˜0) *δ*(1*−δ*1)*, δ*2(1*−δ*2) 1 *−* (1 *− δ*1)*δ*1 *,* 1 *−* (1 *− δ*2)*δ*2

(˜1*,* ˜1) (*δ*1*, δ*2) (1*,* 1)

Table 1

1

Sensitivity analysis for the intuitionistic approach of an *f* -generated (co)implication

Δ*I* ((˜0*,* ˜0)*, δ*˜)= (Δ*I* ((0*,* 0)*, δ*1)*,* Δ*J* ((1*,* 1)*, δ*2)

*Y Y Y*

*f I f f*

=((*IYf* (0*,* 0)*−IYf [*0*,* 0*♩*) *∨* (*IYf [*0*,* 0*|−IYf* (0*,* 0))*,*

(*JYf* (1*,* 1)*−JYf [*1*,* 1*♩*) *∨* (*JYf [*1*,* 1*|−JYf* (1*,* 1)))

= ((1 *−* 0) *∨* (1 *−* 1)*,* (0 *−* 1) *∨* (1 *−* 0)) = (1*,* 1)*.*

Δ*J* ((˜0*,* ˜0)*, δ*˜)= (Δ*J* ((0*,* 0)*, δ*1)*,* Δ*I* ((1*,* 1)*, δ*2)

*Y Y Y*

*f I f f*

= ((*JYf* (0*,* 0)*−JYf [*0*,* 0*♩*) *∨* (*JYf [*0*,* 0*|−JYf* (0*,* 0))*,*

(*IYf* (1*,* 1)*−IYf [*1*,* 1*♩*) *∨* (*IYf [*1*,* 1*|−IYf* (1*,* 1)))

= (*δ*1*, δ*2)*.*

Analogously, Table[4.3](#_bookmark25) summarizes the *δ*-sensitivity of the *g*-generated (co)implication, in the endpoints of *U*˜.

**x**˜ Δ (**x**˜*, δ*˜) Δ (**x**˜*, δ*˜)

*g I g I*

*IY JY*

(˜0*,* ˜0) (1*,* 1) (*δ*1*, δ*2)

˜ ˜

1

1*−δ*1

2

1*−δ*2

1

1*−δ*1

2

1*−δ*2

˜ ˜

1

1*−δ*1

2

1*−δ*2

1

1*−δ*1

2

1*−δ*2

(0*,* 1) 1 *− δ* ) 1 *,* (1 *− δ* ) 1 1 *−* (1 *− δ* ) 1 *,* 1 *−* (1 *− δ* ) 1

(1*,* 0) 1 *−* (1 *− δ* ) 1 *,* 1 *−* (1 *− δ* ) 1 1 *− δ* ) 1 *,* (1 *− δ* ) 1

(˜1*,* ˜1) (*δ*1*, δ*2) (1*,* 1)

Table 2

Sensitivity analysis for the intuitionistic approach of an fuzzy *g*-generated (co)implication

**Proposition 4.9** *The maximum sensitivity of the intuitionistic fuzzy f- and g-generated (co)implications IYf I, JYf I, I, JYg , as deﬁned in Proposition* [*4.7*](#_bookmark24) *by Eqs. (*[*27*](#_bookmark24)*)-(*[*30*](#_bookmark24)*), is given as follows:*

Δ*IYf I* (*δ*˜)= (1*,* 1) = Δ*IYg I* (*δ*˜) *and* Δ*JYg I* (*δ*˜)= (1*,* 1) = Δ*JYf I* (*δ*˜); (31) Δ*IYg I* (*δ*˜)= (1*,* 1) = Δ*IYg I* (*δ*˜) *and* Δ*JYg I* (*δ*˜)= (1*,* 1) = Δ*JYg I* (*δ*˜)*.* (32)

**Proof.** Straightforward from Proposition [4.4](#_bookmark20) and the results reported in Tables [4.3](#_bookmark25) and [4.3](#_bookmark26). *2*

**Corollary 4.10** *Consider f- and g-generated (co)implications IYf I, JYf I, I, JYg as deﬁned in Proposition* [*4.7*](#_bookmark24) *by Eqs. (*[*27*](#_bookmark24)*)-(*[*30*](#_bookmark24)*). Then the following holds:*

1. *IYg I is at least as robust as IYf I at point* **x**˜ *∈ {*(˜0*,* ˜0)*,* (˜1*,* ˜1)*};*
2. *JYg I is at least as robust as JYf I at point* **x**˜ *∈ {*(˜0*,* ˜0)*,* (˜1*,* ˜1)*};*
3. *JYf I is more robust than IYf I at point* (˜0*,* ˜1)*; and*
4. *IYf I is more robust than JYf I at point* (˜1*,* ˜0)*.*

**Proof.** Straightforward. *2*

# Conclusion

Estimating the sensitivity to small changes is related to reducing sensitivity in the corresponding pointwise components of such fuzzy connectives. Thus, in this paper, by taking the class of strong fuzzy negation (standard negation), the paper formally states that the sensitivity of an *n*-order intuitionistic fuzzy connective at a point **x** *∈ Un* preserves its projections related to the sensitivity of its fuzzy approach at the same point, when representable fuzzy negations are considered.

The main contribution is concerned with the study of robustness on Atanassov intuitionistic fuzzy approach related to the *f* - and *g*-generated (co)implication. Some additional studies, considering *δ*-sensitivity of *A*-implications and their cor- responding dual construction should be carried out.

Ongoing work, focussing on the sensitivity of fuzzy inference dependent on in- tuitionistic fuzzy rules based on intuitionistic fuzzy connectives, including the ex- tension of the robustness studies of R-(co)implications, will also be investigated.

To sum up, future research aims to contribute with fundamental theoretical results for applications dealing with main results in the robustness analysis con- sidering their principal operators, e.g. erosion, dilation, closing, opening operators used in the mathematical morphology.

# Acknowledgment

This work is partially supported by the Brazilian funding agencies under the processes 309533/2013-9 (CNPq), 309533/2013-9 (FAPERGS) e 448766/2014-0 (MCTI/CNPQ).

# References

1. K. H. Atanassov, Intuitionistic Fuzzy Sets, *Fuzzy Sets and Systems* **20** (1986) 87-96.
2. K. H. Atanassov, and G. Gargov, Elements of Intuitionistic Fuzzy Logic - Part I, *Fuzzy Sets and Systems* **95** (1989) 39-52.
3. M. Baczyn´ski, On some properties of intuitionistic fuzzy implications, *in: Proceedings IFSA/EUSFLAT*

(2003), 168–171.

1. M. Baczyn´ski, and B. Jayaram, Yagers classes of fuzzy implications: some properties and intersections,

*Kybernetika* **43**(2) (2007) 157-182.

1. M. Baczyn´ski, and B. Jayaram, (S,N)- and R-implications: A state-of-the-art survey, *Fuzzy Sets and* *Systems*, **159** (2008) 1836–1859.
2. M. Baczyn´ski, and B. Jayaram, Fuzzy implications, *Studies in Fuzziness and Soft Computing* **231**

(Springer, Berlin-Heidelberg, 2008).

1. H. Bustince, E. Barrenechea, and V. Mohedano, Intuitionistic fuzzy implication operators – an expression and main properties, *International Journal of Uncertainty, Fuzziness and Knowledge-Based* *Systems* **12**(3) (2004) 387–406.
2. G. Cornelis, G. Deschrijver, and E. Kerre, On the representation of intuitionistic fuzzy t-norms and t-conorms, *IEEE Transactions on Fuzzy Systems* **12**(1) (2004) 45–61.
3. G. Deschrijver, and E. E. Kerre, Smets-Magrez axioms for intuitionistic fuzzy R-implicators,

*International Journal of Uncertainty, Fuzziness and Knowledge-Based* Systems **13** (2005) 453–464.

1. J. Fodor, and M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support* (Kluwer, Dordrecht, 1994).
2. J. Fodor, On fuzzy implication operators, *Fuzzy Sets and Systems* **42** (1991) 293–300.
3. J. Fodor, Contrapositive symmetry of fuzzy implications, *Fuzzy Sets and Systems* **69** (1995) 141–156.
4. J. Villar, M. Sanz-Bobi, Semantic analysis of fuzzy models, application to Yager models, in: Nikos Mostorakis (Ed.) *Proceedings Advances in Fuzzy Systems and Evolutionary Computation* (World Scientific and Engineering Society Press, 2001) 82-87.
5. J. Jin, Y. Li, and C. Li, Robustness of fuzzy reasoning via logically equivalence measure. *Information* *Science*, **47**(177) (2007) 5103–5117.
6. Y. Li, D. Li, W. Pedrycz, and J. Wu, An Approach to Measure the Robustness of Fuzzy Reasoning,

*International Journal of Intelligent Systems* **20** (2005) 393–413.

1. Y. Li, Approximation and robustness of fuzzy finite automata. *International Journal of Approximate* *Reasoning*, **47**(2) (2008) 247–257.
2. Y. Li, K. Oin, X. He, Robustness of fuzzy connectives and fuzzy reasoning. *Fuzzy Sets and Systems*,

**225** No. 16 (2013) 93–105.

1. D. Li, Y. Li, and Y. Xie, Robustness of interval-valued fuzzy inference. *Information Science*, **181**(20) (2011) 4754–4764.
2. S. Mandal, and B. Jayaram, Bandler-Kohout Subproduct With Yager’s Classes of Fuzzy Implications,

*IEEE T. Fuzzy Systems* **22**(3) (2014) 469-482.

1. R. Reiser, and B. Bedregal, Robustness of N-dual fuzzy connectives, in: P. Melo-Pinto, P. Couto, C. Serdio, J. Fodor and B. Baets, eds., *Advances in Intelligent and Soft Computing, EUROFUSE 2011* (Springer Heidelberg 2011), 79–90.
2. R. Reiser, and B. Bedregal, Robustness on intuitionistic fuzzy connectives, *Trends in Applied and Computational Mathematics* **15** (2014) 133–149.
3. R. Reiser, and B. Bedregal, R. Santiago, and M. Amaral, Canonical representation of the Yagers classes of fuzzy implications, *Computational & Applied Mathematics* **32** (2013) 401-412.
4. I. Turksen, V. Kreinovich, and R. Yager, A new class of implications. Axioms of fuzzy implications revisited, *Fuzzy Sets and Systems* **100** (1998) 267–272.
5. R. Yager, On some new classes of implication operators and their role in approximate reasoning,

*Information Sciences* **167** (2004), 193–216.

1. M. Ying, Perturbation of Fuzzy Reasoning, *IEEE Transactions on Fuzzy Systems* **7**(5) (1999) 625–629.
2. L. Zadeh, Fuzzy sets, *Information and Control* **8** (1965) 338–353.
3. R. Zanotelli, R. Reiser, S. Costa, L. Foss and B. Bedregal, Towards robustness and duality analysis of intuitionistic fuzzy aggregations, in: Proceedings IEEE International Conference on Fuzzy Systems (2015), 1–8.