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Semantics for a Basic Relevant Logic with Intensional Conjunction and Disjunction

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Abstract

This paper proposes a basic relevant logic B+

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with intensional conjunction H and disjunction H, which

are more primitive than those defined by *A* ○ *B* =*df* ч(*A* → ч*B*) and *A* + *B* =*df* ч*A* → *B*. B+

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is a

conservative extension of the basic relevant logic B+. Stronger logics can be obtained by adding axioms or

rules to B+ . Kripke style semantics for B+

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is given. Three ternary relations *R, S*1*, S*2 are used to deal

with →, H, and H, respectively. We also consider negation-extensions of B+ . The ‘∗’ operator is used to

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model negation.

*Keywords:* Relevant logics, Intensional conjunction, Intensional disjunction

# Introduction

In the literature on relevant logics, two binary connectives, fusion ◦ and fission +, can be defined by *A*◦*B* =*df* ¬(*A* → ¬*B*) and *A*+*B* =*df* ¬*A* → *B*, respectively. They are also called intensional conjunction and disjunction, and share many features classically attributed to extensional analogues, ∧ and ∨. The connective ◦ can be introduced by way of the rule *A* ◦ *B* → *C* ⇔ *A* → (*B* → *C*), having implication as its residual, which makes it important to algebraic treatment and proof theory of relevant logics. Defined by the above methods, ◦ and + are highly related with

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implication. However, two more primitive connectives can be defined: H and H, which we call intensional conjunction and disjunction also, since further schemes can be added to make them share features of ◦ and +.

Our idea is inspired by the work of establishing a non-normal conjunction rule in [[3](#_bookmark4)] (Chapter 8) for systems denying the law of identity. The rule is as following: *I*(*A.B, a*) = 1 iff for some *b, c* ∈ *W* , *b* g *c* ≤ *a* and *I*(*A, b*) = 1 = *I*(*B, c*), where ‘*.*’ represents the non-normal conjunction, *W* is the set of worlds, *a, b* and *c* are members of *W* , *I* is the assignment function, and g is a two-place operation on *W* . If we use the expression *S*1*bca* to represent the inequality *b* g *c* ≤ *a*, where *S*1 is a ternary relation, then this evaluation rule is very similar to that for ◦ in relational semantics of relevant logics. So, it seems that some intensional conjunction can be defined independent of implication. Parallel to the above work in [[3](#_bookmark4)], we design a non-normal disjunction rule, and start our work from these two rules.

In this paper, we propose a base system B+

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, obtained by adding intenstional

conjunction and disjunction, denoted by H and H respectively, to the minimal pos-

itive relevant logic B+. B+ is a conservative extension of B+. Stronger logics can

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be obtained by adding axiom or rule schemes to B+

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. As to semantics, we use the

ternary relation *R* to model implication →. *S*1 and *S*2 are the other two ternary relations in our semantics. The former is used to deal with H, and the latter is for H. To construct a suitable canonical model, we define dualtheory and anti-dualtheory, and prove priming lemma for dualtheories. In addition, the basic negative system

BMHH is obtained by adding contraposition and De Morgan Laws to B+

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with

negation modeled by the Routley ∗-operation.

We concentrate in this paper on the semantics of the basic systems with H and

H. Extensions will be considered in a subsequent draft.

# The Basic System B+

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* 1. *An Axiom System for* B+

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+ is expressed in a language L, which has the two-place connectives →*,* Λ*,* V*,* H

B

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and H, parentheses ( and ), and a stock of propositional variables *p, q, r, ...* Formulas are defined recursively in the usual manner. Some scope conventions are in force, that is, two-place connectives are ranked H, H, Λ, V, → in order of increasing scope (i.e. H binds more strongly than H, H than Λ, etc.), and otherwise association is to the left. *A, B, C, ...* will be used to range over arbitrary formulas.

To define B+ , let us give an axiom system for B+ first, which has the following

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axioms and rules [3](#_bookmark2) :

Axioms

A1 *A* → *A*

A2 *A* → *A* V *B*, *B* → *A* V *B*

A3 *A* Λ *B* → *A*, *A* Λ *B* → *B*

3 Note that this axiom system is as same as that in [[4](#_bookmark5)] and [[5](#_bookmark6)] except that disjunctive forms of rules are not given distinctivly.

A4 *A* Λ (*B* V *C*) → (*A* Λ *B*) V *C*

A5 (*A* → *B*) Λ (*A* → *C*) → (*A* → *B* Λ *C*)

A6 (*A* → *C*) Λ (*B* → *C*) → (*A* V *B* → *C*)

Rules

R1 *A, A* → *B* ⇒ *B* (Modus Ponens)

R2 *A, B* ⇒ *A* Λ *B* (Ajunction)

R3 *A* → *B, C* → *D* ⇒ (*B* → *C*) → (*A* → *D*) (Affixing).

Thus, B+ is obtained by adding the following axioms and rules to B+:

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A7 (*A* V *B*) H *C* — (*A* H *C*) V (*B* H *C*),

*C* H (*A* V *B*) — (*C* H *A*) V (*C* H *B*)

A8 (*A* H *C*) Λ (*B* H *C*) — (*A* Λ *B*) H *C*,

(*C* H *A*) Λ (*C* H *B*) — *C* H (*A* Λ *B*) R4 *A* → *B, C* → *D* ⇒ *A* H *C* → *B* H *D* R5 *A* → *B, C* → *D* ⇒ *A* H *C* → *B* H *D*.

It can be noted that the special cases of R3 are:

*A* → *B* ⇒ (*C* → *A*) → (*C* → *B*) (Prefixing) *A* → *B* ⇒ (*B* → *C*) → (*A* → *C*) (Suffixing) *A* → *B, B* → *C* ⇒ *A* → *C* (Transitivity).

And the special cases of R4 and R5 are, respectively:

*A* → *B* ⇒ *C* H *A* → *C* H *B* *A* → *B* ⇒ *A* H *C* → *B* H *C A* → *B* ⇒ *C* H *A* → *C* H *B*

*A* → *B* ⇒ *A* H *C* → *B* H *C*.

We note that this axiomatisation contains slight redundancies. R4 and R5, together with axioms and rules of B+, suffice to prove each of A7 and A8 in one direction.

* 1. *Semantics for* B+

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Now we define interpretations for B+

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. The semantics is an extension of semantics

for B+ in [[6](#_bookmark7)].

+ -*frame* (or *model structure*) is a 6-tuple *< g, O, W, R, S*1*, S*2 *>*, where *W*

A B

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is a set (of worlds); *g* ∈ *W* (the base world); *O* is a unary relation on *W* ; and *R, S*1*,* and *S*2 are ternary relations on *W* , such that the following definitions apply and postulates hold for all *a, b, c, d* ∈ *W* .

d1. *a* ≤ *b* =*df* E*x*(*Ox* and *Rxab*);

p1. *Og*;

p2. *a* ≤ *a*;

p3. if *Rdbc* and *a* ≤ *d* then *Rabc*;

p4. if *S*1*abd* and *d* ≤ *c* then *S*1*abc*;

p5. if *S*2*abd* and *c* ≤ *d* then *S*2*abc*.

+ -*model* (or *interpretation*) is a 7-tuple *< g, O, W, R, S*1*, S*2*,I >*, where

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*< g, O, W, R, S*1*, S*2 *>* is a B+

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-frame, and *I* is a function which assigns to each

pair of propositional parameter, *p*, and world, *a*, a truth value *I*(*p, a*) ∈ {1*,* 0}, satisfying the Atomic Hereditary Condition. Truth values of all formulas at worlds are assigned by the following evaluation rules.

Atomic Hereditary Condition. For a propositional variable *p*, if *I*(*p, a*)=1 and

*a* ≤ *a*', then *I*(*p, a*')= 1.

Evaluation Rules.

* *I*(*A* Λ *B, a*)=1 iff *I*(*A, a*)= 1 and *I*(*B, a*)= 1;
* *I*(*A* V *B, a*)=1 iff *I*(*A, a*)= 1 or *I*(*B, a*)= 1;
* *I*(*A* H *B, a*)=1 iff E*b, c* ∈ *W* , *S*1*bca*, *I*(*A, b*)=1 and *I*(*B, c*)= 1;
* *I*(*A* H *B, a*)=1 iff 6*b, c* ∈ *W* , if *S*2*bca* then *I*(*A, b*)=1 or *I*(*B, c*)= 1;
* *I*(*A* → *B, a*)= 1 iff 6*b, c* ∈ *W* , if *Rabc* and *I*(*A, b*)= 1 then *I*(*B, c*)= 1.

+ -model is indeed an extension of a B+-model in [[6](#_bookmark7)] by adding *S*1, *S*2, and

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evaluation rules for H, H.

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We give the following definitions, with B+

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-frame(s) and B+

-model(s) short-

ened as frame(s) and model(s), respectively. *A* is *valid* on a model if *I*(*A, g*) = 1; *A implies B* on a model if for all *a* ∈ *W* : if *I*(*A, a*) = 1 then *I*(*B, a*) = 1; *A* is *valid* on a frame if *A* is valid on all models based on this frame; *A implies B* on

a frame if *A* implies *B* on all models based on this frame. At last, *A* is B+

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-*valid*

if *A* is valid on all frames; *A* B+

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-*implies B* if *A* implies *B* on all frames. For any

extension of B+

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, similar definitions can also be given.

The following lemmas will simplify the proof for soundness.

Lemma 2.1 (Hereditary Condition) *For an arbitrary formula A, if I*(*A, a*)= 1

*and a* ≤ *a*'*, then I*(*A, a*')= 1*.*

Proof. The proof is by an induction on the length of *A* with Atomic Hereditary Condition as induction basis. Here we give proofs for H and H.

H. *A* is of the form *B* H *C*. Suppose *I*(*B* H *C, a*) = 1 and *a* ≤ *a*', to show *I*(*B* H *C, a*') = 1. For some *b, c* ∈ *W* , *S*1*bca*, and *I*(*B, b*) = 1 = *I*(*C, c*). By p4, *S*1*bca*'. So, *I*(*B* H *C, a*') = 1 as required.

H. *A* is of the form *B* H *C*. Suppose *I*(*B* H *C, a*) = 1 and *a* ≤ *a*', to show *I*(*B* H *C, a*') = 1. Suppose further *b, c* ∈ *W* and *S*2*bca*', to show *I*(*B, b*) = 1 or *I*(*C, c*)= 1. By p5, *S*2*bca*. So, *I*(*B, b*)=1 or *I*(*C, c*) = 1 as required.

Lemma 2.2 (Verification Lemma) • *If A implies B on a* B+

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*-model, then*

*A* → *B is valid on this model.*

* *If A implies B on a* B+

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*-frame, then A* → *B is valid on this frame.*

* *A* B+

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*-implies B iff A* → *B is* B+

*-valid.*

Proof. For details of proof, please consult [[6](#_bookmark7)] (pp. 302-303).

* 1. *Soundness*

The soundness of the semantics is demonstrated in this section.

Theorem 2.3 *If A is a theorem of* B+

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*then A is* B+

*-valid.*

Proof. The proof is by a simple induction over the length of proofs. It suffices to

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prove that all axioms are B+

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-valid and all rules preserve validity. We give proofs

for one of A8 (in one direction) and R4.

For A8, suppose for an arbitrary model, *a* ∈ *W* and *I*((*A* Λ *B*) H *C, a*) /= 1. Hence for some *b, c* ∈ *W* , *S*2*bca*, *I*(*A* Λ *B, b*) /=1 and *I*(*C, c*) /= 1. So, *I*(*A, b*) /=1 or *I*(*B, b*) /= 1. Hence, *I*(*A*H*C, a*) /=1 or *I*(*B*H*C, a*) /= 1, i.e. *I*((*A*H*C*)Λ(*B*H*C*)*, a*) /= 1.

By Lemma 2.2, (*A* H *C*) Λ (*B* H *C*) → (*A* Λ *B*) H *C* is B+

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-valid.

For R4, suppose *A* → *B* and *C* → *D* are B+

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-valid, to show that *A*H*C* → *B* H*D*

is B+

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-valid. Suppose further for an arbitrary model, *a* ∈ *W* and *I*(*A* H *C, a*)= 1.

Hence for some *b, c* ∈ *W* , *S*1*bca* and *I*(*A, b*)= 1 = *I*(*C, c*). By Lemma 2.2, *I*(*B, b*)= 1= *I*(*D, c*). So, *I*(*B* H *D, a*) = 1. Then, the result follows by Lemma 2.2.

* 1. *Key Notions for Completeness*

Completeness is established by the usual way. For any non-theorem *A*, we design a canonical interpretation which refutes *A*. Most of techniques come from [[6](#_bookmark7)] (Chapter 4) and [[3](#_bookmark4)] (Chapter 8). In this section, we give some notions for any logic L in this paper.

First, where *V* and *U* are sets of formulas:

(1) ▶*L A* iff *A* is a theorem of L.

1. *U* is L-*derivable* from *V* , written *V* ▶*L U* , iff for some *A*1*, ..., An* in *V* and some *B*1*, ..., Bm* in *U* , ▶*L A*1 Λ *...* Λ *An* → *B*1 V *...* V *Bm*.
2. An L-*derivation* of *A* from *V* , written *V* ▶*L A*, is a finite sequence of formulas

*A*1*, ..., An*, with *An* = *A* such that each member of the sequence either belongs to

*V* or is obtained from predecessors in the sequence by adjunction or a provable

L-implication (i.e. in the latter case *Ai* is obtained from *Aj* since ▶*L Aj* → *Ai*).

1. An L-*derivation* of *U* from *V* is an L-derivation of some disjunction *B*1 V

*...* V *Bm* of formulas *B*1*, ..., Bm* of *U* from *V* . Hence, *U* is L-derivable from *V* iff there is an L-derivation of *U* from *V* .

1. If Σ is the set of all formulas of the language L, *< V, U >* is an L-*maximal pair* iff:
   * *V* ∪ *U* = Σ;
   * *V bL U* .

Please note that if *< V, U >* is an L-maximal pair, then *V* ∩ *U* = ∅.

Next, it can be noted that if *a* is a set of formulas, and *b* = Σ — *a*, where Σ is the set of all formulas of the language L, then *a* satisfies the following a1, a2, a3 separately iff *b* satisfies b1, b2, b3 separately.

a1. If ▶*L A* → *B*, then if *A* ∈ *a* then *B* ∈ *a*;

a2. if *A* ∈ *a* and *B* ∈ *a* then *A* Λ *B* ∈ *a*;

a3. if *A* V *B* ∈ *a* then *A* ∈ *a* or *B* ∈ *a*;

b1. if ▶*L A* → *B*, then if *B* ∈ *b* then *A* ∈ *b*;

b2. if *A* Λ *B* ∈ *b* then *A* ∈ *b* or *B* ∈ *b*;

b3. if *A* ∈ *b* and *B* ∈ *b* then *A* V *B* ∈ *b*.

Then we define, for arbitrary sets of formulas *a, b*:

1. *a* is an L-*theory* iff it satisfies a1 and a2;
2. an L-theory *a* is *prime* iff it satisfies a3;
3. an L-theory *a* is *regular* iff whenever ▶*L A*, *A* ∈ *a*;
4. *a* is an L-*anti-dualtheory* iff it satisfies a1 and a3;
5. an L-anti-dualtheory *a* is *prime* iff it satisfies a2;
6. *b* is an L-*dualtheory* iff it satisfies b1 and b3;
7. an L-dualtheory *b* is *prime* iff it satisfies b2.

So, let *a* be a set of formulas and *b* =Σ — *a*, then: *a* is a prime L-theory iff *a* is a prime L-anti-dualtheory iff *b* is a prime L-dualtheory; *a* is an L-anti-dualtheory iff *b* is an L-dualtheory.

In following text, if system L is obvious, then the subscript ‘*L*’ and the prefix ‘L-’ will simply be omitted.

Now, we define four operations as follows. For arbitrary sets of formulas *a, b*:

* + *a* ⊕ *b* = {*B* : E*A* ∈ *b, A* → *B* ∈ *a*};
  + *a* g *b* = {*C* : E*A* ∈ *a,* E*B* ∈ *b,* ▶*L A* H *B* → *C*};
  + *a* *b* =Σ — {*C* : E*A* ∈*/ a,* E*B* ∈*/ b,* ▶*L C* → *A* H *B*}.

Based on the above definitions, we define three ternary relations *R*, *S*1, *S*2 on any set of sets of formulas:

* + *Rabc* iff *a* ⊕ *b* ⊆ *c*, i.e., for all *A, B*, if *A* → *B* ∈ *a* and *A* ∈ *b*, then *B* ∈ *c*;
  + *S*1*abc* iff *a* g *b* ⊆ *c*, i.e., for all *A, B*, if *A* ∈ *a, B* ∈ *b*, and ▶*L A* H *B* → *C*, then

*C* ∈ *c*;

* + *S*2*abc* iff *c* ⊆ *a* *b*, i.e., for all *A, B*, if *A* ∈*/ a*, *B* ∈*/ b*, and ▶*L C* → *A* H *B*, then

*C* ∈*/ c*.

* 1. *Lemmas about Prime Theories*

Our results are based on some lemmas in [[6](#_bookmark7)]. First, we list several lemmas, which are proved in [[6](#_bookmark7)] (pp. 307-308), or easy to get.

Lemma 2.4 *If < V, U > is an L-maximal pair, then V is a prime L-theory, and*

*U is a prime L-dualtheory.*

Lemma 2.5 (Extension Lemma) *Let V and U be sets of formulas such that*

*V bL U. Then there is an L-maximal pair < V* '*,U* ' *> with V* ⊆ *V* ' *and U* ⊆ *U* '*.*

Lemma 2.6 (Priming Lemma 1) *Let V be an L-theory, U be closed under dis- junction, and V* ∩ *U* = ∅*. Then there is an L-theory V* ' *such that (1) V* ⊆ *V* '*; (2) V* ' ∩ *U* = ∅*; and (3) V* ' *is prime.*

Similar to the Priming Lemma 1, we have the Priming Lemma 2.

Lemma 2.7 (Priming Lemma 2) *Let V be closed under conjunction, U be an L-dualtheory, and V* ∩ *U* = ∅*. Then there is an L-dualtheory U* ' *such that (1) U* ⊆ *U* '*; (2) V* ∩ *U* ' = ∅*; and (3) U* ' *is prime.*

Proof. First, *V bL U* . Otherwise there would be *A*1*, ..., An* ∈ *V* such that *A*1 Λ

*...* Λ *An* ∈ *V* ∩ *U* , since *U* is an L-dualtheory. By Lemma 2.5, there are *V* ' ⊇ *V*

and *U* ' ⊇ *U* such that *< V* '*,U* ' *>* is an L-maximal pair. So, the result follows by Lemma 2.4.

The following corollary is proved in [[6](#_bookmark7)] (pp. 309).

Corollary 2.8 (Corollaries of Priming Lemma 1) *1. If A is a non-theorem of L then, there is a prime regular L-theory c such that A* ∈*/ c.*

*2. For all L-theories a, b*'*, c*' *if Rab*'*c*' *and C* ∈*/ c*' *then, there are prime L-theories*

*b, c such that Rabc, b*' ⊆ *b and C* ∈*/ c.*

Corollary 2.9 (Corollaries of Priming Lemma 1) *1. For all L-theories a*'*,b and prime L-theory c, if S*1*a*'*bc then, there is a prime L-theory a such that a*' ⊆ *a and S*1*abc.*

*2. For all L-theories a, b*' *and prime L-theory c, if S*1*ab*'*c then, there is a prime*

*L-theory b such that b*' ⊆ *b and S*1*abc.*

Proof. We only give proof for 1. The proof for 2 is similar.

1. Set *U* = {*A* : E*B* ∈ *b,* E*C* ∈*/ c,* ▶*L A* H *B* → *C*}. Then:
   1. *U* is closed under disjunction;
   2. *a*' is disjoint from *U* .

For (1), suppose *A*1*, A*2 ∈ *U* , then E*B*1*, B*2 ∈ *b*, and E*C*1*, C*2 ∈*/ c,* ▶*L A*1H*B*1 → *C*1

and ▶*L A*2 H*B*2 → *C*2. Since ▶*L B*1 Λ*B*2 → *B*1, by R4, ▶*L A*1 H(*B*1 Λ*B*2) → *A*1 H*B*1.

Then by R3, ▶*L A*1 H (*B*1 Λ *B*2) → *C*1. Similarly, ▶*L A*2 H (*B*1 Λ *B*2) → *C*2. So,

▶*L* (*A*1 H(*B*1 Λ *B*2))V (*A*2 H(*B*1 Λ *B*2)) → *C*1 V *C*2. Then, by A7, ▶*L* (*A*1 V *A*2)H(*B*1 Λ

*B*2) → (*A*1 H (*B*1 Λ *B*2)) V (*A*2 H (*B*1 Λ *B*2)). So, ▶*L* (*A*1 V *A*2) H (*B*1 Λ *B*2) → *C*1 V *C*2.

Since *c* is prime, *C*1V*C*2 ∈*/ c*. Since *b* is an L-theory, *B*1Λ*B*2 ∈ *b*. Hence, *A*1V*A*2 ∈ *U* ,

i.e. *U* is closed under disjunction.

For (2), suppose otherwise *A* ∈ *U* and *A* ∈ *a*'. Then for some *B* ∈ *b*, *C* ∈*/ c*,

▶*L A* H *B* → *C*. But *S*1*a*'*bc*, whence *C* ∈ *c*, giving a contradiction.

Hence by (1) and (2), Lemma 2.6 applies to provide a prime L-theory *a* disjoint from *U* with *a*' ⊆ *a*. Next, we prove *S*1*abc*. Suppose *A* ∈ *a*, *B* ∈ *b* and ▶*L A*H*B* → *C*. Since *a* is disjoint from *U* , *C* ∈ *c*, i.e., whenever *A* ∈ *a*, *B* ∈ *b* and ▶*L A* H *B* → *C*, then *C* ∈ *c*. Hence *S*1*abc*.

Corollary 2.10 (Corollaries of Priming Lemma 2) *1. For all L-anti- dualtheories a*'*,b and prime L-anti-dualtheory c, if S*2*a*'*bc then, there is a prime L-anti-dualtheory a such that a* ⊆ *a*' *and S*2*abc.*

1. *For all L-anti-dualtheories a, b*' *and prime L-anti-dualtheory c, if S*2*ab*'*c then, there is a prime L-anti-dualtheory b such that b* ⊆ *b*' *and S*2*abc.*

Proof. We only give proof for 1. The proof for 2 is similar.

1. Set *V* = {*A* : E*B* ∈*/ b,* E*C* ∈ *c,* ▶*L C* → *A* H *B*}. Then:

1. *V* is closed under conjunction;
2. Σ — *a*' is disjoint from *V* .

For (1), suppose *A*1*, A*2 ∈ *V* , then E*B*1*, B*2 ∈*/ b*, and E*C*1*, C*2 ∈ *c,* ▶*L C*1 → *A*1H*B*1

and ▶*L C*2 → *A*2 H*B*2. Since ▶*L B*1 → *B*1 V*B*2, by R5, ▶*L A*1 H*B*1 → *A*1 H(*B*1 V*B*2).

Then by R3, ▶*L C*1 → *A*1 H (*B*1 V *B*2). Similarly, ▶*L C*2 → *A*2 H (*B*1 V *B*2). So,

▶*L C*1 Λ*C*2 → (*A*1 H(*B*1 V*B*2))Λ(*A*2 H(*B*1 V*B*2)). Then, by A8, ▶*L* (*A*1 H(*B*1 V*B*2))Λ

(*A*2 H (*B*1 V *B*2)) → (*A*1 Λ *A*2) H (*B*1 V *B*2). So, ▶*L C*1 Λ *C*2 → (*A*1 Λ *A*2) H (*B*1 V *B*2).

Since *c* is a prime L-anti-dualtheory, *C*1 Λ *C*2 ∈ *c*. Since *b* is an L-anti-dualtheory,

*B*1 V *B*2 ∈*/ b*. Hence, *A*1 Λ *A*2 ∈ *V* , i.e. *V* is closed under conjunction.

For (2), suppose otherwise *A* ∈ *V* and *A* ∈ Σ — *a*', i.e. *A* ∈*/ a*'. Then for some

*B* ∈*/ b*, *C* ∈ *c*, ▶*L C* → *A* H *B*. But *S*2*a*'*bc*, whence *C* ∈*/ c*, giving a contradiction.

Since *a*' is an L-anti-dualtheory, Σ — *a*' is an L-dualtheory. Hence by (1) and (2), Lemma 2.7 applies to provide a prime L-dualtheory *a*'' disjoint from *V* with Σ — *a*' ⊆ *a*''. Let *a* = Σ — *a*'', then *a* ⊆ *a*'. Since *a*'' is a prime L-dualtheory, *a* is a prime L-theory, i.e. prime L-anti-dualtheory. Next, we prove *S*2*abc*. Suppose *A* ∈*/ a*, i.e *A* ∈ *a*'', *B* ∈*/ b* and ▶*L C* → *A* H *B*. Since *a*'' is disjoint from *V* , *C* ∈*/ c*, i.e., whenever *A* ∈*/ a*, *B* ∈*/ b* and ▶*L C* → *A* H *B*, then *C* ∈*/ c*. Hence *S*2*abc*.

* 1. *Completeness*

For any non-theorem *A* of B+

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, by 1 of Corollary 2.8, there is a a prime reg-

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ular B+

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-theory *gc* such that *A* ∈*/*

*gc*. Design a *canonical model* for B+ ,

*< g*c*, O*c*, W*c*, R, S*1*, S*2*,I >*, where *W*c is the class of all prime theories, i.e. the class of all prime anti-dualtheories; *O*c is defined as the subset of *W*c such that *a* ∈ *O*c if *a* is regular; *R*, *S*1 and *S*2 are defined as the above; for every prime theory *a* in *W*c and propositional parameter *p*, *I*(*p, a*)=1 iff *p* ∈ *a*.

Theorem 2.11 *If A is* B+ *-valid, then A is a theorem of* B+ *.*

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Proof. We prove the contrapositive. Given a non-theorem *A*, there is a canonical

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model *< g*c*, O*c*, W*c*, R, S*1*, S*2*,I >* for B+

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. We show it is really a B+

-model. It

suffices to show that p1-5 hold, and *I* satisfies the Atomic Hereditary Condition.

Now, p1 and Atomic Hereditary Condition are immediate by definitions. By the same proof in [[6](#_bookmark7)] (pp. 312), it can be proved *a* ≤ *b* iff *a* ⊆ *b*. So, we get p2. And, p3-5 are established by definitions of *R*, *S*1 and *S*2. Hence, the canonical model is

+ -model.

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Next, we show that for every world *a* and formula *A*, *I*(*A, a*) = 1 iff *A* ∈ *a*. It follows that *A* is not valid on *< g*c*, O*c*, W*c*, R, S*1*, S*2*,I >*, and hence that *A* is not

+ -valid. The proof is by induction on the complexity of the formulas. The cases

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for Λ and V are proved by definitions of theory and prime theory. Here, we give proofs for →, H and H.

→. Suppose *A* → *B* ∈ *a*, to show *I*(*A* → *B, a*) = 1. Using the induction hypothesis and the definition of *R*, it follows that for all *b, c* ∈ *W*c, if *Rabc* and

*I*(*A, b*)= 1 then *I*(*B, c*) = 1. Hence, *I*(*A* → *B, a*) = 1 by the evaluation rule for →.

For the converse, suppose *A* → *B* ∈*/ a*, to show *I*(*A* → *B, a*) /= 1. By the

induction hypothesis and the evaluation rule for →, it suffices to find *b, c* ∈ *W*c such that *Rabc*, *A* ∈ *b* and *B* ∈*/ c*. Define *b*' = {*C* :▶ *A* → *C*} and *c*' = {*D* : E*C* ∈ *b*'*,C* → *D* ∈ *a*}. Then *b*' is a theory by R1, R2, R3, and A5. To show *c*' is a theory, suppose ▶ *D* → *E* and *D* ∈ *c*'. So, ▶ (*C* → *D*) → (*C* → *E*). Since *C* → *D* ∈ *a*, *C* → *E* ∈ *a*. Hence, *E* ∈ *c*'. Suppose further *D*1*, D*2 ∈ *c*', i.e. for some *C*1*, C*2 ∈ *b*', *C*1 → *D*1*, C*2 → *D*2 ∈ *a*. Since ▶ *A* → *C*1 and ▶ *A* → *C*2, using R3 it follows that *A* → *D*1*,A* → *D*2 ∈ *a*. So *A* → *D*1 Λ *D*2 ∈ *a* by A5, i.e. *D*1 Λ *D*2 ∈ *c*'. It follows

that *c*' is a theory. By the definition of *R*, *Rab*'*c*'. Further *A* ∈ *b*', but *B* ∈*/ c*'.

Suppose otherwise, *B* ∈ *c*', then for some *C* ∈ *b*', i.e. ▶ *A* → *C*, *C* → *B* ∈ *a*. Hence

▶ (*C* → *B*) → (*A* → *B*). So, *A* → *B* ∈ *a*, contradicting assumptions. Applying Corollary 2.8, there are prime theories *b, c* such that *Rabc*, *A* ∈ *b*' ⊆ *b* and *B* ∈*/ c*.

H. Suppose *A* H *B* ∈ *a*, to show *I*(*A* H *B, a*) = 1, i.e. for some *b, c* ∈ *W*c, *S*1*bca*, *I*(*A, b*)= 1 = *I*(*B, c*). By the induction hypothesis, it suffices to find *b, c* ∈ *W*c such that *b*g*c* ⊆ *a*, *A* ∈ *b* and *B* ∈ *c*. Define *b*' = {*C* :▶ *A* → *C*} and *c*' = {*D* :▶ *B* → *D*}. Then *b*' and *c*' are theories by R1, R2, R3, and A5. It is immediate that *A* ∈ *b*' and *B* ∈ *c*'. To show *b*' g *c*' ⊆ *a*, suppose *E* ∈ *b*' g *c*', then for some *C* ∈ *b*' and *D* ∈ *c*', ▶ *C* H *D* → *E*. Since ▶ *A* → *C* and ▶ *B* → *D*, by R4, ▶ *A* H *B* → *C* H *D*. By R3, ▶ *A* H *B* → *E*. Since *A* H *B* ∈ *a*, *E* ∈ *a*. Accordingly, *b*' g *c*' ⊆ *a*, i.e. *S*1*b*'*c*'*a*. So, by Corollary 2.9, *b*' can be primed to *b* with *b*' ⊆ *b*, and *c*' can be primed to *c* with *c*' ⊆ *c* such that *S*1*bca*.

For the converse, suppose *I*(*A* H *B, a*) = 1, i.e. for some *b, c* ∈ *W*c, *S*1*bca*, *I*(*A, b*) = 1 = *I*(*B, c*). Then using the induction hypothesis, it follows that *A* ∈ *b* and *B* ∈ *c*. By A1, and the definition of g, *A* H *B* ∈ *b* g *c*. Hence *A* H *B* ∈ *a*.

H. Suppose *I*(*A* H *B, a*) /= 1, i.e. for some *b, c* ∈ *W*c, *S*2*bca*, *I*(*A, b*) /= 1 and *I*(*B, c*) /= 1. Then using the induction hypothesis, it follows that *A* ∈*/ b* and *B* ∈*/ c*. By A1, and the definition of , *A* H *B* ∈*/ b* *c*. Hence, *A* H *B* ∈*/ a*.

For the converse, suppose *A* H *B* ∈*/ a*, to show *I*(*A* H *B, a*) /= 1, i.e. for some

*b, c* ∈ *W*c, *S*2*bca*, *I*(*A, b*) /= 1 and *I*(*B, c*) /= 1. By the induction hypothesis,

it suffices to find *b, c* ∈ *W*c such that *a* ⊆ *b* *c*, *A* ∈*/ b* and *B* ∈*/ c*. Define

*b*' = Σ — {*C* :▶ *C* → *A*} and *c*' = Σ — {*D* :▶ *D* → *B*}. Then {*C* :▶ *C* → *A*} and

{*D* :▶ *D* → *B*} are dualtheories by R1, R2, R3, and A6. So, *b*' and *c*' are anti- dualtheories. It is immediate that *A* ∈*/ b*' and *B* ∈*/ c*'. To show *a* ⊆ *b*' *c*', suppose *F* ∈*/ b*' *c*', then for some *C* ∈*/ b*', *D* ∈*/ c*', ▶ *F* → *C* H *D*. Since *C* ∈*/ b*', ▶ *C* → *A*, and since *D* ∈*/ c*', ▶ *D* → *B*. By R5, ▶ *C* H *D* → *A* H *B*. Since ▶ *F* → *C* H *D*, by R3, ▶ *F* → *A* H *B*. *A* H *B* ∈*/ a*, so *F* ∈*/ a* as required. Hence, *a* ⊆ *b*' *c*', i.e. *S*2*b*'*c*'*a*. So, by Corollary 2.10, *b*' can be primed to *b* with *b* ⊆ *b*', and *c*' can be primed to *c* with *c* ⊆ *c*' such that *S*2*bca*.

Now, we can see that B+ is a conservative extension of B+, in the sense that it

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is an extension by adding new notations (H,H), axioms (A7,A8) and rules (R4,R5), which has the following feature: let *A* be a formula in the notation of B+; then if *A*

is provable in B+ , then *A* is also provable in B+ [[1](#_bookmark3)], since every B+ -valid formula

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*A* involving only connectives →, Λ and V is also B+-valid.

Actually, further features of H and H such as commutativity and associativity

can be obtained by adding axiom or rule schemes to B+

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. Also, we can introduce

*A* ◦ *B* → *C* e *A* → (*B* → *C*), which makes → a residual of H, such that *S*1 collapses to *R*. We leave this topic for further discussion.

# Negation

For basic negation-extension of B+ A9 ч(*A* Λ *B*) — ч*A* V ч*B*

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A10 ч*A* Λ ч*B* — ч(*A* V *B*)

R6 *A* → *B* ⇒ ч*B* → ч*A*.

, we add De Morgan Laws and contraposition:

We call this system BMHH. Please note that A9 and A10 also contain redundan- cies. By contraposition and positive axioms, we can prove each of A9 and A10 in one direction.

A BMHH-frame is a 7-tuple *< g, O, W, R, S*1*, S*2*,* ∗ *>*, where ∗ is a one-place function from *W* to *W* , and the other elements are as before, such that postulate p6 holds for all *a, b* ∈ *W* :

p6. If *a* ≤ *b* then *b*∗ ≤ *a*∗,

which is necessary for the Hereditary Condition.

A BMHH-model is a 8-tuple *< g, O, W, R, S*1*, S*2*,* ∗*,I >*, where *< g, O, W, R, S*1*, S*2*,* ∗ *>* is a BMHH-frame, and *I* is as before, with the following evaluation rule for negation:

* *I*(ч*A, a*)=1 iff *I*(*A, a*∗) /= 1.

BMHH is sound with respect to the evaluation rule. For completeness, define ∗ on a set of formulas *a* as: *a*∗ = {*A*|ч*A* ∈*/ a*}. Given a non-theorem *A*, the canonical interpretation for BMHH is now *< g*c*, O*c*, W*c*, R, S*1*, S*2*,* ∗*,I >*. By De Morgan Laws and contraposition, it can be shown that: if *a* is an anti-dualtheory, then *a*∗ is a theory; if *a* is a theory, then *a*∗ is an anti-dualtheory. (For details of proof, please consult [[4](#_bookmark5)].) Hence, if *a* is a prime theory, so is *a*∗, i.e. ∗ is well-defined. Also, by the definition of *a*∗, p6 is easy to verify.

The system BHH is obtained by adding Double Negation, *A* — чч*A* to BMHH. A BHH-model is a BMHH-model satisfying: for all *a* ∈ *W* , *a*∗∗ = *a*.

¿From the definitions of BMHH-model and BHH-model, we can see that BMHH

is a conservative extension of BM, and BHH is a conservative extension of B.

If *A* → (*B* → *C*) ⇒ (*B* → ч*A*H*C*) and *B* → *A*H*C* ⇒ ч*A* → (*B* → *C*) are added to BHH, we can get that *S*2 is dependent on *R*. Further, *A* H *B* — ч(*A* → ч*B*) and *A* H *B* — (ч*A* → *B*) can be established with some further axiom or rule schemes added.

# Concluding Remarks

This paper considered a basic relevant logic B+

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with intensional conjunction H and

disjunction H , and some of its negation-extensions. Kripke style semantics were given for these systems. Our semantics extend the traditional relational semantics for relevant logics in [[6](#_bookmark7)] by introducing ternary relations *S*1 and *S*2.

In fact, a wealth of stronger systems can be obtained by adding axioms or rules to these basic systems in this paper. We will consider a range of axiom and rule schemes, and give the corresponding semantical postulates for these schemes in another draft.

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