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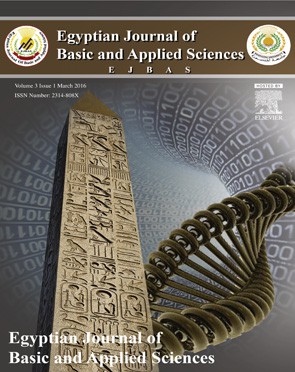
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**Full Length Article**

**Some applications of *Dα*-closed sets in topological spaces**



***O.R. Sayed*** [***a***](#_bookmark0)***,***[***\****](#_bookmark2)***, A.M. Khalil*** [***b***](#_bookmark1)***,***[***\****](#_bookmark2)

a *Department of Mathematics, Faculty of Science, Assiut University, Assiut, 71516, Egypt*

b *Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut, 71524, Egypt*

A R T I C L E I N F O A B S T R A C T

*Article history:*

Received 23 January 2015 Received in revised form 27 July 2015

Accepted 27 July 2015

Available online 18 August 2015

*Keywords:*

*α*-closed

*Dα*-closed

*Dα*-continuous *Dα*-closed graph

Strongly *Dα*-closed graph

In this paper, a new kind of sets called *Dα*-open sets are introduced and studied in a to- pological spaces. The class of all *Dα*-open sets is strictly between the class of all *α*-open sets and g-open sets. Also, as applications we introduce and study *Dα*-continuous, *Dα*-open, and *Dα*-closed functions between topological spaces. Finally, some properties of *Dα*-closed and strongly *Dα*-closed graphs are investigated.

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nc-nd/4.0/).

*2010 Mathematics subject classification:*

54A05

54C08

54C10

54D10

# Introduction and preliminaries

Generalized open sets play a very important role in General Topology, and they are now the research topics of many to- pologies worldwide. Indeed a significant theme in General

Topology and Real Analysis is the study of variously modified forms of continuity, separation axioms, etc. by utilizing gen- eralized open sets. One of the most well-known notions and also inspiration source are the notion of *α*-open [[1]](#_bookmark3) sets intro- duced by Njåstad in 1965 and generalized closed (g-closed) subset of a topological space [[2]](#_bookmark4) introduced by Levine in 1970.

\* *Corresponding authors.* Tel.: +2088 2242873.

*E-mail addresses:* [o\_sayed@aun.edu.eg](mailto:o_sayed@aun.edu.eg) (O.R. Sayed); [a.khalil@azhar.edu.eg](mailto:a.khalil@azhar.edu.eg) (A.M. Khalil). <http://dx.doi.org/10.1016/j.ejbas.2015.07.005>

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Since then, many mathematicians turned their attention to the generalization of various concepts in General Topology by con- sidering *α*-open sets [[3–10]](#_bookmark5) and generalized closed sets [[11–13]](#_bookmark8). In 1982 Dunham [[14]](#_bookmark9) used the generalized closed sets to define a new closure operator, and thus a new topology *τ\**, on the space, and examined some of the properties of this new to- pology. Throughout the present paper (*X*, ** ), (*Y*, ** ) and (*Z*, *ν*) denote topological spaces (briefly *X*, *Y* and *Z*) and no separa- tion axioms are assumed on the spaces unless explicitly stated. For a subset *A* of a space (*X*, *τ*), *Cl*(*A*) and *Int*(*A*) denote the closure and the interior of *A*, respectively. Since we require the following known definitions, notations, and some properties, we recall in this section.

Definition 1.1. Let (*X*, *τ*) be a topological space and *A* ⊆ *X*. Then

1. *A* is *α*-open [[1]](#_bookmark3) if *A*  *Int*(*Cl*(*Int*(*A*)) and *α*-closed [[1]](#_bookmark3) if

*Cl*(*Int*(*Cl*(*A*))  *A* .

1. *A* is generalized closed (briefly g-closed) [[2]](#_bookmark4) if *Cl*(*A*) ⊆ *U*

whenever *A* ⊆ *U* and *U* is open in *X*.

1. *A* is generalized open(briefly g-open) [[2]](#_bookmark4) if *X*\*A* is g-closed.

The *α*-closure of a subset *A* of *X* [[3]](#_bookmark5) is the intersection of all *α*-closed sets containing *A* and is denoted by *Clα*(*A*). The *α*-interior of a subset *A* of *X* [[3]](#_bookmark5) is the union of all *α*-open sets contained in *A* and is denoted by *Intα*(*A*). The intersection of all g-closed sets containing *A* [[14]](#_bookmark9) is called the g-closure of *A* and denoted by *Cl*\*(*A*), and the g-interior of *A* [[15]](#_bookmark10) is the union of all g-open sets contained in *A* and is denoted by *Int*\*(*A*).

We need the following notations:

* *αO*(*X*) (resp. *αC*(*X*)) denotes the family of all *α*-open sets (resp.

*α*-closed sets) in (*X*, *τ*).

* *GO*(*X*) (resp. *GC*(*X*)) denotes the family of all generalized open sets (resp. generalized closed sets) in (*X*, *τ*).

Definition 1.4. A topological space (*X*, *τ*) is said to be:

* 1. *α*-*T*1 [[9]](#_bookmark7) (resp. g-*T*1 [[22]](#_bookmark17)) if for any distinct pair of points *x* and *y* in *X*, there exist *α*-open (resp. g-open) set *U* in *X* containing *x* but not *y* and an *α*-open (resp. g-open) set *V* in *X* containing *y* but not *x*.
  2. *α*-*T*2 [[8]](#_bookmark6) (resp. g-*T*2 [[22]](#_bookmark17)) if for any distinct pair of points *x* and *y* in *X*, there exist *α*-open (resp. g-open) sets *U* and *V* in *X* containing *x* and *y*, respectively, such that *U* ∩ *V* = *ϕ*.

Lemma 1.5. Let *A* ⊆ *X*, then

1. *X* \ *Cl*\*(*A*)  *Int*\*(*X* \ *A*) .
2. *X* \ *Int*\*(*A*)  *Cl*\*(*X* \ *A*) .

Lemma 1.6. A function *f* : (*X*, ** )  (*Y*, ** ) has a closed graph [[19]](#_bookmark14) if for each (*x*, *y*) (*X*  *Y*) \ *G*( *f* ), there exist *U* ∈ *O*(*X*, *x*) and *V* ∈ *O*(*Y*, *y*) such that *f*(*U*) ∩ *V* = *ϕ*.

Lemma 1.7. The graph *G*(*f*) is strongly closed [[23]](#_bookmark18) if and only if for each point (*x*, *y*) *G*( *f* ), there exist open sets *U* ⊂ *X* and *V* ⊂ *Y* containing *x* and *y*, respectively, such that *f* (*U*)  *Cl*(*V*)  **.

# *Dα*-closed sets

In this section we introduce *Dα*-closed sets and investigate some of their basic properties.

Definition 2.1. A subset *A* of a space *X* is called *Dα*-closed if

*Cl*\*(*Int*(*Cl*\*(*A*)))  *A* .

The collection of all *Dα*-closed sets in *X* is denoted by *DαC*(*X*).

* + *O*(*X*, *x*)  {*U* | *x* *U* *O*(*X*, ** )} , *O*(*X*, *x*)  {*U* | *x* *U* ** }

*C*(*X*, *x*)  {*U* | *x* *U* *C*(*X*, ** )} .

and

Lemma 2.2. If there exists an g-closed set *F* such that

*Cl*\*(*Int*(*F*))  *A*  *F* , then *A* is *Dα*-closed.

Definition 1.2. A function *f* : *X* → *Y* is said to be:

1. *α*-continuous [[16]](#_bookmark11) (resp. g-continuous [[17]](#_bookmark12)) if the inverse image of each open set in *Y* is *α*-open (resp. g-open) in *X*.
2. *α*-open [[16]](#_bookmark11) (resp. *α*-closed [[16]](#_bookmark11)) if the image of each open (resp. closed ) set in *X* is *α*-open (resp. *α*-closed) in *Y*.
3. g-open [[18]](#_bookmark13) (resp. g-closed [[18]](#_bookmark13)) if the image of each open (resp. closed) set in *X* is g-open (resp. g-closed) in *Y*.

Definition 1.3. Let *f* : *X* → *Y* be a function:

1. The subset {(*x*, f(*x*)) | *x*  *X*} of the product space *X* × *Y* is called the graph of *f* [[19]](#_bookmark14) and is usually denoted by *G*(*f*).
2. a closed graph [[19]](#_bookmark14) if its graph *G*(*f*) is closed sets in the product space *X* × *Y*.
3. a strongly closed graph [[20]](#_bookmark15) if for each point (*x*, *y*) *G*( *f* ),

there exist open sets *U* ⊂ *X* and *V* ⊂ *Y* containing *x* and

*y*, respectively, such that (*U*  *Cl*(*V*))  *G*( *f* )  ** .

1. a strongly *α*-closed graph [[21]](#_bookmark16) if for each (*x*, *y*) (*X*  *Y*) \ *G*( *f* ) , there exist *U* *O*(*X*, *x*) and *V* ∈ *O*(*Y*, *y*) such that (*U*  *Cl*(*V*))  *G*( *f* )  ** .

Proof. Since *F* is g-closed, *Cl*\*(*F*) = *F*. Therefore, *Cl*\*(*Int*(*Cl*\*(*A*))) 

*Cl*\*(*Int*(*Cl*\*(*F*)))  *Cl*\*(*Int*(*F*))  *A* . Hence *A* is *Dα*-closed.

Remark 2.3. The converse of above lemma is not true as shown in the following example.

Example 2.4. Let (*X*, *τ*) be a topological space, where *X* = {*a*, *b*, *c*} and **  {**, {*a*}, {*a*, *b*}, *X*}. Then *FX*  {**, {*c*}, {*b*, *c*}, *X*}, *GC*(*X*) 

{**, {*c*}, {*a*, *c*}, {*b*, *c*}, *X*} , *GO*(*X*)  {**, {*a*}, {*b*}, {*a*, *b*}, *X*}, *DC*(*X*)  {**,

{*b*}, {*c*}, {*a*, *c*}, {*b*, *c*}, *X*} , *DO*(*X*)  {**, {*a*}, {*b*}, {*a*, *b*}, {*a*, *c*}, *X*}. There- fore {*c*}  *DC*(*X*) and {*a*, *c*} *GC*(*X*) but *Cl*\*(*Int*{*a*, *c*})  {*a*, *c*} 

{*c*}  {*a*, *c*} .

Theorem 2.5. Let (*X*, *τ*) be a topological space. Then

1. Every *α*-closed subset of (*X*, *τ*) is *Dα*-closed.
2. Every g-closed subset of (*X*, *τ*) is *Dα*-closed.

Proof. (i) Since closed set is g-closed, *Cl*\*(*A*)  *Cl*(*A*) [[14]](#_bookmark9). Now, suppose *A* is *α*-closed in *X*, then *Cl*(*Int*(*Cl*(*A*)))  *A*. There- fore, *Cl*\*(*Int*(*Cl*\*(*A*)))  *Cl*(*Int*(*Cl*(*A*)))  *A* . Hence *A* is *Dα*-closed in *X*.

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(ii) Suppose *A* is g-closed. Then *Cl*\*(*A*) = *A* [[14]](#_bookmark9). There- fore, *Int*(*Cl*\*(*A*))  *Cl*\*(*A*). Then *Cl*\*(*Int*(*Cl*\*(*A*)))  *Cl*\*(*Cl*\*(*A*))  *Cl*\*(*A*)  *A* [[14]](#_bookmark9). Hence *A* is *Dα*-closed.

Remark 2.6. The converse of above theorem is not true as

Theorem 2.14. Let *A* and *B* be subsets of *X*. Then the follow- ing results hold.

1. *A*  *ClD*(*A*)  *Cl* (*A*), *ClD*(*A*)  *Cl*\*(*A*) .

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1. *ClD*(**)  ** and *ClD*(*X*)  *X* .

**

**

shown in the following example.

1. *Dα*-closed set need not be *α*-closed. (see Example 2.7 below)
2. *Dα*-closed set need not be g-closed. (see Example 2.7 below)

Example 2.7. Let (*X*, *τ*) be a topological space, where *X*  {*a*, *b*, *c*}

1. If *A* ⊆ *B*, Then *ClD*(*A*)  *ClD*(*B*).
2. *ClD*(*ClD*(*A*))  *ClD*(*A*).

** **

** ** **

1. *ClD*(*A*)  *ClD*(*B*)  *ClD*(*A*  *B*) .

** ** **

1. *ClD*(*A*  *B*)  *ClD*(*A*)  *ClD*(*B*) .

** ** **

Proof. (i) Follows From Theorem 2.5 (i) and (ii), respectively.

(ii) and (iii) are obvious.

1. If *A* ⊆ *F*, *F* ∈ *DαC*(*X*), then from (iii) and Theorem 2.13,

*ClD*(*A*)  *ClD*(*F*)  *F* . Again *ClD*(*ClD*(*A*))  *ClD*(*F*)  *F*. Therefore

and **  {**, {*a*, *b*}, *X*}. Then *FX*  *C*(*X*)  {**, {*c*}, *X*}, *O*(*X*)  {**, {*a*, ** **

** ** **

** ** **

*b*}, *X*} , *GC*(*X*)  {**, {*c*}, {*a*, *c*}, {*b*, *c*}, *X*}, *GO*(*X*)  {**, {*a*}, {*b*}, {*a*, *b*}, *X*},

*DC*(*X*)  {**, {*a*}, {*b*}, {*c*}, {*a*, *c*}, {*b*, *c*}, *X*} , *DO*(*X*)  {**, {*a*}, {*b*}, {*a*, *b*},

{*a*, *c*}, {*b*, *c*}, *X*} . Therefore {*a*}  *DC*(*X*), but {*a*} *C*(*X*) and

{*a*} *GC*(*X*) .

From the above discussions we have the following diagram

*ClD*(*ClD*(*A*))  {*F* : *A*  *F*, *F*  *DC*(*X*)}  *ClD*(*A*).

1. and (vi) follows from (iii).

Remark 2.15. The equality in the statements (v) of the above theorem need not be true as seen from Example 2.7, where *A* = {*a*}, *B* = {*b*}, and *A*  *B*  {*a*, *b*}. Then one can have that, *ClD*(*A*)  {*a*} ; *ClD*(*B*)  {*b*}; *ClD*(*A*  *B*)  *X*; *ClD*(*A*)  *ClD*(*B*)  {*a*, *b*}.

in which the converses of implications need not be true. ** ** ** ** **

**-closed set  *D*-closed set  *g*-closed set

Theorem 2.8. Arbitrary intersection of *Dα*-closed sets is

*Dα*-closed.

Proof. Let {*Fi* : *i* ∈ Λ} be a collection of *Dα*-closed sets in *X*. Then *Cl*\*(*Int*(*Cl*\*(*Fi* )))  *Fi* for each *i*. Since ∩ *Fi* ⊆ *Fi* for each *i*, *Cl*\*(*Fi* )  *Cl*\*(*Fi* ) for each *i*. Hence *Cl*\*(*Fi* )  *Cl*\*(*Fi* ), *i*  .

Further more the equality in the statements (iv) of the above theorem need not be true as shown in the following example.

Example 2.16. Let (*X*, *τ*) be a topological space, where *X* = {*a*,

*b*, *c*} and **  {**, {*b*}, {*c*}, {*b*, *c*}, *X*}. Then *FX*  *GC*(*X*)  *DC*(*X*) 

{**, {*a*}, {*a*, *b*}, {*a*, *c*}, *X*} , *GO*(*X*)  {**, {*b*}, {*c*}, {*b*, *c*}, *X*}. Let *A* = {*a*},

*B* = {*b*}, and *A* ∩ *B* = *ϕ*. Then one can have that, *ClD*(*A*)  {*a*};

**

*ClD*(*B*)  {*a*, *b*} ; *ClD*(*A*  *B*)  ** ; *ClD*(*A*)  *ClD*(*B*)  {*a*}.

**

**

** **

Therefore *Cl*\*(*Int*(*Cl*\*(*Fi* )))  *Cl*\*(*Int*(*Cl*\*(*Fi* )))  *Cl*\*(*Int*(*Cl*\*(*Fi* ))) 

*Cl*\*(*Int*(*Cl*\*(*Fi* )))  *Fi* . Hence ∩*Fi* is *Dα*-closed.

# *Dα*-open sets

Remark 2.9. The union of two *Dα*-closed sets need not to be *Dα*-closed as shown in Example 2.7, where both

{*a*} and {*b*} are *Dα*-closed sets but {*a*}  {*b*}  {*a*, *b*} is not

*Dα*-closed.

Corollary 2.10. If a subset *A* is *Dα*-closed and *B* is *α*-closed, then

*A* ∩ *B* is *Dα*-closed.

Proof. Follows from Theorem 2.5 (i) and Theorem 2.8. Corollary 2.11. If a subset *A* is *Dα*-closed and *F* is g-closed, then

*A* ∩ *F* is *Dα*-closed.

Proof. Follows from Theorem 2.5 (ii) and Theorem 2.8.

Definition 2.12. Let *A* be a subset of a space *X*. The *Dα*- closure of *A*, denoted by *ClD*(*A*), is the intersection of all *Dα*-closed sets in *X* containing *A*. That is *ClD*(*A*)  {*F* : *A*  *F* and *F*  *DC*(*X*)} .

**

**

Theorem 2.13. Let *A* be a subset of *X*. Then *A* is *Dα*-closed set in *X* if and only if *ClD*(*A*)  *A*.

**

Proof. Suppose *A* is *Dα*-closed set in *X*. By Definition 2.12,

*ClD*(*A*)  *A* . Conversely, suppose *ClD*(*A*)  *A*. By Theorem 2.8 *A*

In this section we introduce *Dα*-open sets and investigate some of their basic properties.

Definition 3.1. A subset *A* of a space *X* is called an *Dα*-open if *X* \ *A* is *Dα*-closed. Let *DαO*(*X*) denote the collection of all an *Dα*-open sets in *X*.

Lemma 3.2. Let *A* ⊆ *X*, then

1. *X* \ *Cl*\*(*X* \ *A*)  *Int*\*(*A*) .
2. *X* \ *Int*\*(*X* \ *A*)  *Cl*\*(*A*) .

Proof. Obvious.

Theorem 3.3. A subset *A* of a space *X* is *Dα*-open if and only if *A*  *Int*\*(*Cl*(*Int*\*(*A*))).

Proof. Let *A* be *Dα*-open set. Then *X* \ *A* is *Dα*-closed and *Cl*\*(*Int*(*Cl*\*(*X* \ *A*)))  *X* \ *A* . By Lemma 3.2 *A*  *Int*\*(*Cl*(*Int*\*(*A*))). Conversely, suppose *A*  *Int*\*(*Cl*(*Int*\*(*A*))). Then *X* \ *Int*\*(*Cl*(*Int*\* (*A*)))  *X* \ *A* . Hence (*Int*\*(*Cl*(*Int*\*(*X* \ *A*))))  *X* \ *A*. This shows that *X* \ *A* is *Dα*-closed. Thus *A* is *Dα*-open.

Lemma 3.4. If there exists g-open set *V* such that *V*  *A*  *Int*\*

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is *Dα*-closed.

(*Cl*(*V*)) , then *A* is *Dα*-open.

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Proof. Since *V* is g-open, *X* \ *V* is g-closed and *X* \ *Int*\*(*Cl*(*V*))

 *X* \ *A*  *X* \ *V* . Therefore From Lemma 3.2 *Cl*\*(*Int*(*X* \ *V*))

 *X* \ *A*  *X* \ *V* . From Lemma 2.2 we have *X* \ *A* is *Dα*-closed. Hence *A* is *Dα*-open.

Theorem 3.15. Let *A* and *B* be subsets of *X*. Then the follow- ing results hold.

* 1. *Int* (*A*)  *IntD*(*A*)  *A* , *Int*\*(*A*)  *IntD*(*A*).

** **

* 1. *IntD*(**)  ** and *IntD*(*X*)  *X*.

** **

Remark 3.5. The converse of Lemma 3.4 need not to be true as seen from Example 2.4, where {*a*, *b*}  *DO*(*X*) and

{*b*} *GO*(*X*) but {*b*}  {*a*, *b*}  {*b*} .

Theorem 3.6. Let (*X*, *τ*) be a topological space. Then

1. Every *α*-open subset of (*X*, *τ*) is *Dα*-open.
2. Every g-open subset of (*X*, *τ*) is *Dα*-open.

Proof. From Theorem 2.5, the proof is obvious.

1. If *A* ⊆ *B*, then *IntD*(*A*)  *IntD*(*B*).
2. *IntD*(*IntD*(*A*))  *IntD*(*A*) .

** **

** ** **

1. *IntD*(*A*)  *IntD*(*B*)  *IntD*(*A*  *B*).

** ** **

1. *IntD*(*A*  *B*)  *IntD*(*A*)  *IntD*(*B*).

** ** **

Proof. Obvious.

Remark 3.16. The equality in the statements (v) of Theorem

3.15 need not be true as seen from Example 2.7, where *A* = {*b*, *c*}, *B* = {*a*, *c*}, and *A* ∪ *B* = *X*. Then one can have that, *IntD*(*A*)  {*b*, *c*}; *IntD*(*B*)  {*c*} ; *IntD*(*A*)  *IntD*(*B*)  {*b*, *c*}; *IntD*(*A*  *B*)  *X* . Further-

**

** ** ** **

Remark 3.7. The converse of the above theorem is not true as

seen from Example 2.7, where {*b*, *c*}  *DO*(*X*) but {*b*, *c*} *O*(*X*)

and {*b*, *c*} *GO*(*X*).

From the above discussions we have the following diagram in which the converses of implications need not be true.

**-open set  *D*-open set  *g*-open set

Theorem 3.8. Arbitrary union of *Dα*-open set is *Dα*-open. Proof. Follows from Theorem 2.8.

Remark 3.9. The intersection of two *Dα*-open sets need not be *Dα*-open as seen from Example 2.7, where both {*b*, *c*}

more the equality in the statements (iv) of the above theorem need not be true as seen from Example 2.7, where *A* = {*b*, *c*}, *B* = {*a*, *c*}, and *A* ∩ *B* = {*c*}. Then one can have that, *IntD*(*A*)  {*b*, *c*} ; *IntD*(*B*)  {*a*, *c*} ; *IntD*(*A* ∩ *B*)  ** ; *IntD*(*A*)  *IntD*

(*B*)  {*c*} .

** ** ** ** **

Theorem 3.17. Let *x* ∈ *X*. Then *x* *ClD*(*A*) if and only if *U* ∩ *A* ≠ *ϕ*

**

for every *Dα*-open set *U* containing *x*.

Proof. Let *x* *ClD*(*A*) and there exists *Dα*-open set *U* contain- ing *x* such that *U* ∩ *A* = *ϕ*. Then *A*  *X* \ *U* and *X*\*U* is *Dα*- closed. Therefore *ClD*(*A*)  *ClD*(*X* \ *U*)  *X* \ *U*. This implies *x* *ClD*(*A*), which is a contradiction. Conversely, assume that *U* ∩ *A* ≠ *ϕ* for every *Dα*-open set *U* containing *x* and *x* *ClD*(*A*). Then there exists *Dα*-closed subset *F* containing *A* such that

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and {*a*, *c*} are *Dα*-open sets but {*b*, *c*}  {*a*, *c*}  {*c*}

*Dα*-open.

is not

*x* *F* . Hence *x* ∈ *X* \ *F* and *X* \ *F* is *Dα*-open. Therefore *A* ⊆ *F*,

(*X* \ *F*)  *A*  ** This is a contradiction to our assumption.

Corollary 3.10. If a subset *A* is *Dα*-open and *B* is *α*-open, then

*A* ∪ *B* is *Dα*-open.

Proof. Follows from Theorem 3.6 (i) and Theorem 3.8. Corollary 3.11. If a subset *A* is *Dα*-open and *U* is g-open, then

*A* ∪ *U* is *Dα*-open.

Proof. Follows from Theorem 3.6 (ii) and Theorem 3.8.

Definition 3.12. Let *A* be a subset of a space *X*. The *Dα*- interior of *A* is denoted by *IntD*(*A*), is the union of all an *Dα*-open sets in *X* contained in *A*. That is *IntD*(*A*)  {*U* : *U*  *A*,

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Lemma 3.18. Let *A* be any subset of (*X*, *τ*). Then

1. *A*  *Int*\*(*Cl*(*Int*\*(*A*))) is *Dα*-open;
2. *A*  *Cl*\*(*Int*(*Cl*\*(*A*))) is *Dα*-closed.

Proof.

1. *Int*\*(*Cl*(*Int*\*(*A*  *Int*\*(*Cl*(*Int*\*(*A*))))))  *Int*\*(*Cl*(*Int*\*(*A*)  *Int*\*(*Cl*

(*Int*\*(*A*)))))  *Int*\*(*Cl*(*Int*\*(*A*))) . This implies that

*A*  *Int*\*(*Cl*(*Int*\*(*A*)))  *A*  *Int*\*(*Cl*(*Int*\*(*A*  *Int*\*(*Cl*(*Int*\*(*A*))))))

 *Int*\*(*Cl*(*Int*\*(*A*  *Int*\*(*Cl*(*Int*\*(*A*)))))) . Therefore *A*  *Int*\*(*Cl*

(*Int*\*(*A*))) is *Dα*-open.

1. From (i) we have *X* \ (*A*  *Cl*\*(*Int*(*Cl*\*(*A*)))  (*X* \ *A*)

*U*  *DO*(*X*)} .

 *Int*\*(*Cl*(*Int*\*(*X* \ *A*)))

is *Dα*-open that further implies

Lemma 3.13. If *A* is a subset of *X*, then

1. *X* \ *ClD*(*A*)  *IntD*(*X* \ *A*) .

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1. *X* \ *IntD*(*A*)  *ClD*(*X* \ *A*) .

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Proof. Obvious.

*A*  *Cl*\*(*Int*(*Cl*\*(*A*))) is *Dα*-closed.

Theorem 3.19. If *A* is a subset of a topological space *X*,

1. *IntD*(*A*)  *A*  *Int*\*(*Cl*(*Int*\*(*A*))) .

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1. *ClD*(*A*)  *A*  *Cl*\*(*Int*(*Cl*\*(*A*))) .

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Theorem 3.14. Let *A* be a subset of *X*. Then *A* is *Dα*-open if and only if *IntD*(*A*)  *A* .

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Proof. Follows from Theorem 2.13 and Lemma 3.13.

Proof.

1. Let *B*  *IntD*(*A*). Clearly *B* is *Dα*-open and *B* ⊆ *A*. Since *B* is *Dα*-open, *B*  *Int*\*(*Cl*(*Int*\*(*B*)))  *Int*\*(*Cl*(*Int*\*(*A*))). This proves that *B*  *A*  *Int*\*(*Cl*(*Int*\*(*A*))). By Lemma 3.18,

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*A*  *Int*\*(*Cl*(*Int*\*(*A*))) is *Dα*-open. By the definition of *IntD*(*A*), *A*  *Int*\*(*Cl*(*Int*\*(*A*)))  *B* . Then it follows that *B*  *A*  *Int*\*(*Cl*(*Int*\*(*A*))) . Therefore *IntD*(*A*)  *A*  *Int*\*(*Cl*

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1. *f* (*ClD*(*A*))  *Cl*( *f* (*A*)) for every subset *A* of *X*.
2. *ClD*( *f* 1(*B*))  *f* 1(*Cl*(*B*)) for every subset *B* of *Y*.

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1. *f* 1(*Int*(*B*))  *IntD*( *f* 1(*B*)) for every subset *B* of *Y*.

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(*Int*\*(*A*))) .

1. By Lemma 3.13 we have *ClD*(*A*)  *X* \ *IntD*(*X* \ *A*),

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 *X* \ ((*X* \ *A*)  *Int*\*(*Cl*(*Int*\*(*X* \ *A*)))) , using (i)

 *X* \ (*X* \ *A*)  (*X* \ *Int*\*(*Cl*(*Int*\*(*X* \ *A*)))

 *A*  *Cl*\*(*Int*(*Cl*\*(*A*))).

Proof. (i)→(ii) Since *V* ⊂ *Y* containing *f*(*x*) is open, then *f* 1(*V*)  *DO*(*X*). Set *W*  *f* 1(*V*) which contains *x*, therefore *f*(*W*) ⊂ *V*.

(ii)→(i) Let *V* ⊂ *Y* be open, and let *x*  *f* 1(*V*), then *f*(*x*) ∈ *V* and

thus there exists *Wx*  *DO*(*X*) such that *x* ∈ *Wx* and *f*(*Wx*) ⊂ *V*.

Then *x*  *Wx*  *f* 1(*V*), and so *f* 1(*V*) 

∪ *Wx* but ∪ *Wx* 

*x* *f* 1 ( *V* ) *x* *f* 1 ( *V* )

# *Dα*-continuous functions

*DO*(*X*)

by Theorem 3.8. Hence *f* 1(*V*)  *DO*(*X*), and there-

In this section we introduce *Dα*-continuous functions and in- vestigate some of their basic properties.

Definition 4.1. A function *f* : *X* → *Y* is called *Dα*-continuous if the inverse image of each open set in *Y* is *Dα*-open in *X*.

Theorem 4.2.

1. Every *α*-continuous function is *Dα*-continuous.
2. Every g-continuous function is *Dα*-continuous.

Proof. It is obvious from Theorem 3.6. Remark 4.3.

1. *Dα*-continuous function need not be *α*-continuous.

fore *f* is *Dα*-continuous.

1. →(iii) Let *F* ⊂ *Y* be closed. Then *Y*\*F* is open and *f* 1(*Y* \ *F*)  *DO*(*X*) , i.e. *X*  *f* 1(*F*)  *DO*(*X*). Then *f* 1(*F*) is *Dα*- closed of *X*.
2. →(iv) Let *A*  *X* and *F* be a closed set in *Y* containing *f*(*A*). Then by (iii), *f* 1(*F*) is *Dα*-closed set containing *A*. It follows that

*ClD*(*A*)  *ClD*( *f* 1(*F*))  *f* 1(*F*) and hence *f* (*ClD*(*A*))  *F* . Therefore

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*f* (*ClD*(*A*))  *Cl*( *f* (*A*)) .

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1. →(v) Let *B*  *Y* and *A*  *f* 1(*B*). Then by assumption, *f* (*ClD*(*A*))  *Cl*( *f* (*A*))  *Cl*(*B*) . This implies that *ClD*(*A*)  *f* 1 (*Cl*(*B*)) . Hence *ClD*( *f* 1(*B*))  *f* 1(*Cl*(*B*)).

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1. →(vi) Let *B*  *Y*. By assumption, *ClD*( *f* 1(*Y* \ *B*))  *f* 1

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(*Cl*(*Y* \ *B*)) . This implies that, *ClD*(*X* \ *f* 1(*B*))  *f* 1(*Y* \ *Int*(*B*)) and hence *X* \ *IntD*( *f* 1(*B*))  *X* \ *f* 1(*Int*(*B*)). By taking complement on both sides we get *f* 1(*Int*(*B*))  *IntD*( *f* 1(*B*)).

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1. →(i) Let *U* be any open set in *Y*. Then *Int*(*U*) = *U*. By as-

sumption, *f* 1(*Int*(*U*))  *IntD*( *f* 1(*U*)) and hence *f* 1(*U*)  *IntD*( *f* 1

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(see Example 4.4 (i) below)

1. *Dα*-continuous function need not be g-continuous. (see Example 4.4 (ii) below)

Example 4.4. (i) Let *X* = {*a*, *b*, *c*} associated with the topology

(*U*)) . Then *f* 1(*U*)  *IntD*( *f* 1(*U*)). Therefore by Theorem 3.14,

*f* 1(*U*) is *Dα*-open in *X*. Thus *f* is *Dα*-continuous.

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Theorem 4.6. Let *f* : *X* → *Y* be *Dα*-continuous and let *g* : *Y* → *Z*

be continuous. Then *gof* : *X* → *Z* is *Dα*-continuous.

**  {**, {*a*}, *X*} and *Y* = {*x*, *y*, *z*} associated with the topology

**  {**, {*x*, *y*}, {*z*}, *Y*} . Let *f* : *X* → *Y* be a function defined by *f*(*a*) = *f*(*b*) = *x*, *f*(*c*) = *z*. One can have that *FX*  {**, {*b*, *c*}, *X*}, *GC*(*X*)  {**, {*b*}, {*c*}, {*a*, *b*}, {*a*, *c*}, {*b*, *c*}, *X*} , *GO*(*X*)  {**, {*a*}, {*b*}, {*c*}, {*a*,

*b*}, {*a*, *c*}, *X*} , *O*(*X*)  {**, {*a*}, {*a*, *b*}, {*a*, *c*}, *X*}, *DαC*(*X*) = *DαO*(*X*) =

*P*(*X*). Since {*z*} is open in *Y*, *f* 1({*z*})  {*c*}  *DO*(*X*), but {*c*} *O*(*X*). Therefore *f* is *Dα*-continuous but not *α*-continuous.

(ii) Let (*X*,*τ*) and (*Y*,*σ*) be the topological spaces in (i) and *f* : *X* → *Y* be a function defined by *f*(*a*) = *x*, *f*(*b*) = *f*(*c*) = *z*. Since {*z*} is open in *Y*, *f* 1({*z*})  {*b*, *c*}  *DO*(*X*), but {*b*, *c*} *GO*(*X*). There-

fore *f* is *Dα*-continuous but not g-continuous.

From the above discussions we have the following diagram in which the converses of implications need not be true.

**-continuity  *D*-continuity  *g*-continuity

Theorem 4.5. Let *f* : *X* → *Y* be a function. Then the following are equivalent:

1. *f* is *Dα*-continuous.
2. For each *x* ∈ *X* and each open set *V* ⊂ *Y* containing *f*(*x*), there exists *Dα*-open set *W* ⊂ *X* containing *x* such that *f*(*W*) ⊂ *V*.
3. The inverse image of each closed set in *Y* is *Dα*-closed in *X*.

Proof. Obvious.

Remark 4.7. Composition of two *Dα*-continuous functions need not be *Dα*-continuous as seen from the following example.

Example 4.8. Let *X*  {*a*, *b*, *c*} associated with the topology ** 

{**, {*b*}, {*a*, *b*}, *X*} , *Y*  {*x*, *y*, *z*} associated with the topology

**  {**, {*x*}, *Y*} and *Z*  {*p*, *q*, *r*} associated with the topology

**  {**, {*r*}, *Z*} and *f* : (*X*, ** )  (*Y*, ** ) by *f*(*a*) = *y*, *f*(*b*) = *x*, *f*(*c*) = *z*. Define *g* : (*Y*, ** )  (*Z*, **) by *g*(*x*)  *g*(*y*)  *p*, *g*(*z*) = *r*. One can have that *FX*  {**, {*c*}, {*a*, *c*}, *X*}, *GC*(*X*)  {**, {*c*}, {*a*, *c*}, {*b*, *c*}, *X*}, *GO*(*X*)

 {**, {*a*}, {*b*}, {*a*, *b*}, *X*} , *DC*(*X*)  {**, {*a*}, {*c*}, {*a*, *c*}, {*b*, *c*}, *X*}, *DO*(*X*)

 {**, {*a*}, {*b*}, {*a*, *b*}, {*b*, *c*}, *X*} , and *FY*  {**, {*y*, *z*}, *Y*}, *GC*(*Y*)  {**, {*y*},

{*z*}, {*x*, *y*}, {*x*, *z*}, {*y*, *z*}, *Y*} , *GO*(*Y*)  {**, {*x*}, {*y*}, {*z*}, {*x*, *y*}, {*x*, *z*}, *Y*},

*DC*(*Y*)  *DO*(*Y*)  *P*(*X*). Clearly, *f* and *g* are *Dα*-continuous. {*r*} is open in *Z*. But (*gof* )1({*r*})  *f* 1(*g*1({*r*}))  *f* 1({*z*})  {*c*}, which is not *Dα*-open in *X*. Therefore *gof* is not *Dα*-continuous.

# *Dα*-open functions and *Dα*-closed functions

In this section we introduce *Dα*-open functions and *Dα*-closed functions and investigate some of their basic properties.

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Definition 5.1. A function *f* : *X* → *Y* is said to be *Dα*-open (resp. Corollary 5.8. If *f* : *X* → *Y* is *Dα*-open, then *f* 1(*ClD*(*B*))  *Cl*( *f* 1(*B*))

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*Dα*-closed) if the image of each open (resp. closed) set in *X* is

*Dα*-open (resp. *Dα*-closed) in *Y*. Theorem 5.2.

1. Every *α*-open function is *Dα*-open.
2. Every g-open function is *Dα*-open.

Proof. It is obvious from Theorem 3.6. Remark 5.3.

1. *Dα*-open function need not be *α*-open. (see Example 5.4 below)
2. *Dα*-open function set need not be g-open. (see Example 5.5 below)

Example 5.4. (i) Let *X* = {*x*,*y*,*z*} associated with the topology

**  {**, {*x*}, *X*} and *Y* = {*a*,*b*,*c*} associated with the topology

**  {**, {*a*, *b*}, {*c*}, *Y*} . Let *f* : (*X*, ** )  (*Y*, ** ) be a function defined by *f*(*x*) = *a*, *f*(*y*) = *b* and *f*(*z*) = *c*. One can have that *FY*  *O*(*Y*) 

{**, {*a*, *b*}, {*c*}, *Y*} , *GC*(*Y*) = *GO*(*Y*) = *DαC*(*Y*) = *DαO*(*Y*) = *P*(*X*). Since

{*x*} is open in *X*, *f* ({*x*})  {*a*}  *DO*(*Y*), but {*a*} *O*(*Y*). There- fore *f* is *Dα*-open function but not *α*-open.

Example 5.5. (ii) Let *X* = {*x*,*y*,*z*} associated with the topology

**  {**, {*y*}, {*x*, *y*}, *X*} and *Y* = {*a*, *b*, *c*} associated with the topol- ogy **  {**, {*a*}, *Y*}. Let *f* : (*X*, ** )  (*Y*, ** ) be a function defined by

*f*(*x*) = *b*, *f*(*y*) = *c* and *f*(*z*) = *a*. One can have that *FY*  {**, {*b*, *c*}, *Y*},

*GC*(*Y*)  {**, {*b*}, {*c*}, {*a*, *b*}, {*a*, *c*}, {*b*, *c*}, *Y*} , *GO*(*Y*)  {**, {*a*}, {*b*}, {*c*}, {*a*,

*b*}, {*a*, *c*}, *Y*} , *DC*(*Y*)  *DO*(*Y*)  *P*(*X*). Since {*x*,*y*} is open in *X*,

*f* ({*x*, *y*})  {*b*, *c*}  *DO*(*Y*), but {*b*, *c*} *GO*(*Y*). Therefore *f* is *Dα*- open function but not g-open.

From the above discussions we have the following diagram in which the converses of implications need not be true.

**-open function  *D*-open function  *g*-open function

Theorem 5.6. Let *f* : *X* → *Y* be a function. The following state- ments are equivalent.

1. *f* is *Dα*-open.
2. For each *x* ∈ *X* and each neighborhood *U* of *x*, there exists

*Dα*-open set *W* ⊆ *Y* containing *f*(*x*) such that *W* ⊆ *f*(*U*).

Proof. (i)→(ii) Let *x* ∈ *X* and *U* is a neighborhood of *x*, then there exists an open set *V* ⊆ *X* such that *x* ∈ *V* ⊆ *U*. Set *W* = *f*(*V*). Since *f* is *Dα*-open, *f* (*V*)  *W*  *DO*(*Y*) and so *f* (*x*)  *W*  *f* (*U*).

(ii)→(i) Obvious.

Theorem 5.7. Let *f* : *X* → *Y* be *Dα*-open (resp. *Dα*-closed) func- tion and *W* ⊆ *Y*. If *A* ⊆ *X* is a closed (resp. open) set containing *f* 1(*W*) , then there exists *Dα*-closed (resp. *Dα*-open) set *H* ⊆ *Y* containing *W* such that *f* 1(*H*)  *A*.

Proof. Let *H*  *Y* \ *f* (*X* \ *A*). Since *f* 1(*W*)  *A*, we have *f* (*X* \ *A*)  *Y* \ *W* . Since *f* is *Dα*-open (resp. *Dα*-closed), then *H* is *Dα*-closed (resp. *Dα*-open) set and *f* 1(*H*)  *X* \ *f* 1( *f* (*X* \ *A*))  *X* \ (*X* \ *A*)  *A* .

for each set *B* ⊂ *Y*.

Proof. Since *Cl*( *f* 1(*B*)) is closed in *X* containing *f* 1(*B*) for a set *B* ⊆ *Y*. By Theorem 5.7, there exists *Dα*-closed set *H* ⊆ *Y*, *B* ⊆ *H* such that *f* 1(*H*)  *Cl*( *f* 1(*B*)). Thus, *f* 1(*ClD*(*B*))  *f* 1(*ClD*(*H*))  *f* 1(*H*)  *Cl*( *f* 1(*B*)) .

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Theorem 5.9. A function *f* : *X* → *Y* is *Dα*-open if and only if

*f* (*Int*(*A*))  *IntD*( *f* (*A*)) for every subset *A* of *X*.

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Proof. Suppose *f* : *X* → *Y* is *Dα*-open function and *A* ⊆ *X*. Then *Int*(*A*) is open set in *X* and *f* (*Int*(*A*)) is *Dα*-open set contained in *f*(*A*). Therefore *f* (*Int*(*A*))  *IntD*( *f* (*A*)). Conversely, let be *f* (*Int*(*A*))  *IntD*( *f* (*A*)) for every subset *A* of *X* and *U* is open set in *X*. Then *Int*(*U*) = *U*, *f* (*U*)  *IntD*( *f* (*U*)). Hence *f* (*U*)  *IntD*( *f* (*U*)). By Theorem 3.14 *f*(*U*) is *Dα*-open.

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Theorem 5.10. For any bijective function *f* : (*X*, ** )  (*Y*, ** ) the following statements are equivalent.

1. *f* 1 is *Dα*-continuous function.
2. *f* is *Dα*-open function.
3. *f* is *Dα*-closed function.

Proof. (i)→(ii) Let *U* be an open set in *X*. Then *X* \ *U* is closed in *X*. Since *f* 1 is *Dα*-continuous, ( *f* 1 )1(*X* \ *U*) is *Dα*-closed in

*Y*. That is *f* (*X* \ *U*)  *Y* \ *f* (*U*) is *Dα*-closed in *Y*. This implies *f*(*U*) is *Dα*-open in *Y*. Hence *f* is *Dα*-open function.

(ii)→ (iii) Let *F* be a closed set in *X*. Then *X* \ *F* is open in *X*.

Since *f* is *Dα*-open, *f*(*X* \ *F*) is *Dα*-open in *Y*. That is *f* (*X* \ *F*)  *Y* \ *f* (*F*) is *Dα*-open in *Y*. This implies *f*(*F*) is *Dα*- closed in *Y*. Hence *f* is *Dα*-closed function.

(iii)→ (i) Let *F* be closed set in *X*. Since *f* is *Dα*-closed func-

tion, *f*(*F*) is *Dα*-closed in *Y*. That is ( *f* 1 )1(*F*) is *Dα*-closed in *Y*. Hence *f* 1 is *Dα*-continuous function.

Remark 5.11. Composition of two *Dα*-open functions need not be *Dα*-open as seen from the following example.

Example 5.12. Let *X* = {*x*, *y*, *z*} associated with the topology

**  {**, {*x*, *y*}, {*z*}, *X*} , *Y* = {*p*, *q*, *r*} associated with the topology

**  {**, {*p*}, *Y*} and *Z* = {*a*,*b*,*c*} associated with the topology, ** 

{**, {*b*}, {*a*, *b*}, *Z*} . Define *f* : (*X*, ** )  (*Y*, ** ) by *f*(*x*) = *p*, *f*(*y*) = *q*,

*f*(*z*) = *r* and *g* : (*Y*, ** )  (*Z*, **) by *g*(*p*) = *b*, *g*(*q*) = *a*, *g*(*r*) = *c*. One can have that; *FY*  {**, {*q*, *r*}, *Y*} , *GC*(*Y*)  {**, {*q*}, {*r*}, {*p*, *q*}, {*p*, *r*},

{*q*, *r*}, *Y*} , *GO*(*Y*)  {**, {*p*}, {*q*}, {*r*}, {*p*, *q*}, {*p*, *r*}, *Y*}, *DC*(*Y*)  *DO*(*Y*)

 *P*(*X*) and *FZ*  {**, {*c*}, {*a*, *c*}, *Z*}, *GC*(*Z*)  {**, {*c*}, {*a*, *c*}, {*b*, *c*}, *Z*},

*GO*(*Z*)  {**, {*a*}, {*b*}, {*a*, *b*}, *Z*} , *DC*(*Z*)  {**, {*a*}, {*c*}, {*a*, *c*}, {*b*, *c*}, *Z*},

*DO*(*Z*)  {**, {*a*}, {*b*}, {*a*, *b*}, {*b*, *c*}, *Z*} . Clearly, *f* and *g* are *Dα*-open

function. {*z*} is open in *X*. But *gof* ({*z*})  *g*( *f* ({*z*}))  *g*({*r*})  {*c*} which is not *Dα*-open in *Z*. Therefore *g*o*f* is not *Dα*-open function.

# *Dα*-closed graph and strongly *Dα*-closed

In this section we introduce *Dα*-closed graph and strongly

*Dα*-closed and investigate some of their basic properties.

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Definition 6.1. A function *f* : *X* → *Y* has *Dα*-closed graph if for each (*x*, *y*) (*X*  *Y*) \ *G*( *f* ), there exist *U*  *DO*(*X*, *x*) and *V* *GO*(*Y*, *y*) such that (*U*  *Cl*\*(*V*))  *G*( *f* )  ** .

Remark 6.2. Evidently every closed graph is *Dα*-closed. That the converse is not true is seen from the following example.

Example 6.3. Let *X* = {*a*, *b*, *c*} associated with the topology

Remark 6.7. The converse of the above theorem is not true as seen from Example 2.7.

Theorem 6.8. Let *f* : *X* → *Y* be any surjection with *G*(*f*) *Dα*-closed. Then *Y* is g-*T*1.

Proof. Let *y*1, *y*2(*y*1  *y*2 )  *Y* . The subjectivity of *f* gives the ex- istence of an element *xo* ∈ *X* such that *f*(*xo*) = *y*2. Now (*xo*, *y*1 ) (*X*  *Y*) \ *G*( *f* ) . The *Dα*-closeness of *G*(*f*) provides

**  {**, {*a*, *b*}, *X*}

and *Y* = {*x*, *y*, *z*} associated with the topology

*U*1  *DO*(*X*, *xo* ) , *V*1 *GO*(*Y*, *y*1 ) such that *f* (*U*1 )  *Cl*\*(*V*1 )  **.

**  {**, {*x*}, {*x*, *y*}, *Y*} . Let *f* : (*X*, ** )  (*Y*, ** ) be a function defined

by *f*(*a*) = *f*(*c*) = *x*, *f*(*b*) = *y*. One can have that *FX*  {**, {*c*}, *X*} ,

*GC*(*X*)  {**, {*c*}, {*a*, *c*}, {*b*, *c*}, *X*} , *GO*(*X*)  {**, {*a*}, {*b*}, {*a*, *b*}, *X*}, *DO*

Now *xo* *U*1  *f* (*xo* )  *y*2  *f* (*U*1 ). This and the fact that *f* (*U*1 )  *Cl*\*(*V*1 )  ** guarantee that *y*2 *V*1. Again from the sub- jectivity of *f* gives a *x*1 ∈ *X* such that *f*(*x*1) = *y*1. Now

(*X*)  {**, {*a*}, {*b*}, {*a*, *b*}, {*a*, *c*}, {*b*, *c*}, *X*}

and *FY*  {**, {*z*}, {*y*, *z*}, *Y*} ,

(*x*1, *y*2 ) (*X*  *Y*) \ *G*( *f* ) and the *Dα*-closedness of *G*(*f*) provides

*GC*(*Y*)  {**, {*z*}, {*x*, *z*}, {*y*, *z*}, *Y*} , *GO*(*Y*)  {**, {*x*}, {*y*}, {*x*, *y*}, *Y*}. Since *U*2  *DO*(*X*, *x*1 ) , *V*2 *GO*(*Y*, *y*2 ) such that *f* (*U*2 )  *Cl*\*(*V*2 )  ** .

{*a*, *c*}  *DO*(*X*, *c*)

and {*y*} *GO*(*Y*, *y*)

but {*a*, *c*} *O*(*X*)

and

Now *x*1 *U*2  *f* (*x*1 )  *y*1  *f* (*U*2 ) so that *y*1 *V*2. Thus we obtain

{*y*} *O*(*Y*) . Therefore *G*(*f*) is *Dα*-closed but not closed.

Theorem 6.4. Let *f* : (*X*, ** )  (*Y*, ** ) be a function and

1. *f* is *Dα*-closed graph;
2. For each (*x*, *y*) (*X*  *Y*) \ *G*( *f* ), there exist *U*  *DO*(*X*, *x*)

and *V* *GO*(*Y*, *y*) such that *f* (*U*)  *Cl*\*(*V*)  **.

1. For each (*x*, *y*) (*X*  *Y*) \ *G*( *f* ), there exist *U*  *DO*(*X*, *x*)

and *V*  *DO*(*Y*, *y*) such that (*U*  *ClD*(*V*))  *G*( *f* )  **.

**

1. For each (*x*, *y*) (*X*  *Y*) \ *G*( *f* ), there exist *U*  *DO*(*X*, *x*)

and *V*  *DO*(*Y*, *y*) such that *f* (*U*)  *ClD*(*V*)  ** . Then

**

* 1. (i)⇔ (ii)

sets *V*1, *V*2 *GO*(*Y*) such that *y*1 ∈ *V*1 but *y*2 *V*1 while *y*2 ∈ *V*2 but

*y*1 *V*2 . Hence *Y* is g-*T*1.

Corollary 6.9. Let *f* : *X* → *Y* be any surjection with *G*(*f*) *Dα*-closed. Then *Y* is *Dα*-*T*1.

Proof. Follows From Theorems 6.6 (i) and 6.8.

Theorem 6.10. Let *f* : *X* → *Y* be any injective with *G*(*f*) *Dα*-closed. Then *X* is *Dα*-*T*1.

Proof. Let *x*1, *x*2(*x*1  *x*2 )  *X*. The injectivity of *f* implies

* 1. (i)→ (iii)

*f* (*x*1 )  *f* (*x*2 )

whence one obtains that (*x*1, *f* (*x*2 )) (*X*  *Y*) \

* 1. (ii)→ (iv)
  2. (i)→ (iv)

*G*( *f* ) . The *Dα*-closedness of *G*(*f*) provides *U*1  *DO*(*X*, *x*1 ), *V*1 *GO*(*Y*, *f* (*x*2 )) such that *f* (*U*1 )  *Cl*\*(*V*1 )  **. Therefore *f* (*x*2 )  *f* (*U*1 ) and a fortiori *x*2 *U*1. Again (*x*2, *f* (*x*1 )) (*X*  *Y*) \

Proof. (i)→(ii) Suppose *f* is *Dα*-closed graph. Then for each

*G*( *f* )

and *Dα*-closedness of *G*(*f*) as before gives *U*2  *DO*(*X*,

(*x*, *y*) (*X*  *Y*) \ *G*( *f* ) , there exists *U*  *DO*(*X*, *x*) and *V* *GO*(*Y*, *y*) such that (*U*  *Cl*\*(*V*))  *G*( *f* )  ** . This implies that for each *f* (*x*)  *f* (*U*) and *y* ∈ *Cl*\*(*V*). Since *y* ≠ *f*(*x*), *f* (*U*)  *Cl*\*(*V*)  **.

(ii)→(i) Let (*x*, *y*) (*X*  *Y*) \ *G*( *f* ). Then there exists

*U*  *DO*(*X*, *x*) and *V* *GO*(*Y*, *y*) such that *f* (*U*)  *Cl*  \*(*V*)  **. This implies that *f*(*x*) ≠ *y* for each *x* ∈ *U* and *y* ∈ *Cl*\*(*V*). There- fore (*U*  *Cl*\*(*V*))  *G*( *f* )  ** .

1. →(iii) Suppose *f* is *Dα*-closed graph. Then for each (*x*, *y*) (*X*  *Y*) \ *G*( *f* ) , there exists *U*  *DO*(*X*, *x*) and *V* *GO*(*Y*, *y*) such that (*U*  *Cl*\*(*V*))  *G*( *f* )  ** . Since g-open set is *Dα*- open, *ClD*(*V*)  *Cl*\*(*V*). Therefore (*U*  *ClD*(*V*))  *G*( *f* )  **.

*x*2 ) , *V*2 *GO*(*Y*, *f* (*x*1 )) with *f* (*U*2 )  *Cl*\*(*V*2 )  ** , which guaran- tees that *f* (*x*1 )  *f* (*U*2 ) and so *x*1 *U*2. Therefore, we obtain sets *U*1 and *U*2  *DO*(*X*) such that *x*1 ∈ *U*1 but *x*2 *U*1 while *x*2 ∈ *U*2

but *x*1 *U*2. Hence *X* is *Dα*-*T*1.

Corollary 6.11. Let *f* : *X* → *Y* be any bijection with *G*(*f*) *Dα*-closed. Then both *X* and *Y* are *Dα*-*T*1.

Proof. It readily follows from Corollary 6.9 and Theorem 6.10. Definition 6.12. A topological space (*X*, *τ*) is said to be *Dα*-*T*2 if

** **

1. →(iv) Let (*x*, *y*) (*X*  *Y*) \ *G*( *f* ). Then there exists

*U*  *DO*(*X*, *x*) and *V* *GO*(*Y*, *y*) such that *f* (*U*)  *Cl*\*(*V*)  ** . Since *ClD*(*V*)  *Cl*\*(*V*), *f* (*U*)  *ClD*(*V*)  *f* (*U*)  *Cl*\*(*V*)  ** .

** **

(i)→(iv) From (ii).

Definition 6.5. A topological space (*X*,*τ*) is said to be *Dα*-*T*1 if for any distinct pair of points *x* and *y* in *X*, there exist *Dα*- open *U* in *X* containing *x* but not *y* and an *Dα*-open *V* in *X* containing *y* but not *x*.

Theorem 6.6.

1. Every *α*-*T*1 space is *Dα*-*T*1.
2. Every g-*T*1 space is *Dα*-*T*1.

Proof. It is obvious from Theorem 3.6.

for any distinct pair of points *x* and *y* in *X*, there exist *Dα*-

open sets *U* and *V* in *X* containing *x* and *y*, respectively, such that *U* ∩ *V* = *ϕ*.

Theorem 6.13.

1. Every *α*-*T*2 space is *Dα*-*T*2.
2. Every g-*T*2 space is *Dα*-*T*2.

Proof. Obvious.

Remark 6.14. The converse of the above theorem is not true as seen from Example 2.7.

Theorem 6.15. Let *f* : *X* → *Y* be any surjection with *G*(*f*) *Dα*-closed. Then *Y* is g-*T*2.

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Proof. Let *y*1, *y*2(*y*1  *y*2 )  *Y* . The subjectivity of *f* gives a *x*1 ∈ *X* such that *f*(*x*1) = *y*1. Now (*x*1, *y*2 ) (*X*  *Y*) \ *G*( *f* ). The *Dα*- closedness of *G*(*f*) provides *U*  *DO*(*X*, *x*1 ), *V* *GO*(*Y*, *y*2 ) such that *f* (*U*)  *Cl*\*(*V*)  ** . Now *x*1 *U*  *f* (*x*1 )  *y*1  *f* (*U*). This and the fact that *f* (*U*)  *Cl*\*(*V*)  ** guarantee that *y*1 *Cl*\*(*V*). This mean that there exists *W* *GO*(*Y*, *y*1 ) such that *W* ∩ *V* = *ϕ*. Hence

*Y* is g-*T*2.

Corollary 6.16. Let *f* : *X* → *Y* be any surjection with *G*(*f*) *Dα*-closed. Then *Y* is *Dα*-*T*2.

Proof. Follows from Theorems 6.13 (ii) and 6.15.

Definition 6.17. A function *f* : *X* → *Y* has a strongly *Dα*-closed graph if for each (*x*, *y*) (*X*  *Y*) \ *G*( *f* ), there exist *U*  *DO*(*X*, *x*) and *V* ∈ *O*(*Y*, *y*) such that (*U*  *Cl*(*V*))  *G*( *f* )  ** .

Corollary 6.18. A strongly *Dα*-closed graph is *Dα*-closed. That the converse is not true is seen from Example 6.3, where

{*y*} *GO*(*Y*, *y*) but {*y*} *O*(*Y*). Therefore *G*(*f*) is *Dα*-closed but

not strongly *Dα*-closed.

Remark 6.19. Evidently every strongly *α*-closed graph (resp. strongly closed graph) is strongly *Dα*-closed graph. That the converse is not true is seen from the following example.

Example 6.20. Let *X* = {*a*, *b*, *c*} associated with the topology

**  {**, {*a*, *b*}, *X*} and *Y* = {*x*, *y*, *z*} associated with the topology

**  {**, {*x*, *y*}, {*z*}, *Y*} . Let *f* :(*X*, ** )  (*Y*, ** ) be a function defined

by *f*(*a*) = *f*(*c*) = *x*, *f*(*b*) = *y*. One can have that *FX*  {**, {*c*}, *X*},

*GC* *X*  {**, {*c*}, {*a*, *c*}, {*b*, *c*}, *X*} , *GO* *X*  {**, {*a*}, {*b*}, {*a*, *b*}, *X*}, *O* *X*

 {**, {*a*, *b*}, *X*} , *DO* *X*  {**, {*a*}, {*b*}, {*a*, *b*}, {*a*, *c*}, {*b*, *c*}, *X*}. Since

Theorem 6.23. If *f* : *X* → *Y* is *Dα*-continuous function and *Y* is

*T*2. Then *G*(*f*) is strongly *Dα*-closed.

Proof. Let (*x*, *y*) (*X*  *Y*) \ *G*( *f* ). Since *Y* is *T*2, there exists a set *V* ∈ *O*(*Y*, *y*) such that *f* (*x*) *Cl*(*V*). But *Cl*(*V*) is closed. Now *Y* \ *Cl*(*V*) *O*(*Y*, *f* (*x*)). By Theorem 4.5 there exists

*U*  *DO*(*X*, *x*) such that *f* (*U*)  *Y* \ *Cl*(*V*). Consequently,

*f* (*U*)  *Cl*(*V*)  ** and therefore *G*(*f*) is strongly *Dα*-closed.

Theorem 6.24. Let *f* : *X* → *Y* be any surjection with *G*(*f*) strongly

*Dα*-closed. Then *Y* is *T*1 and *α*-*T*1.

Proof. Similar to the proof of Theorem 6.8 and *T*1-ness always guarantees *α*-*T*1-ness. Hence *Y* is *α*-*T*1.

Corollary 6.25. Let *f* : *X* → *Y* be any surjection with *G*(*f*) strongly

*Dα*-closed. Then *Y* is *Dα*-*T*1.

Proof. Follows From Theorems 6.6 (i) and 6.24.

Theorem 6.26. Let *f* : *X* → *Y* be any injective with *G*(*f*) strongly

*Dα*-closed. Then *X* is *Dα*-*T*1.

Proof. Similar to the proof of Theorem 6.10.

Corollary 6.27. Let *f* : *X* → *Y* be any bijection with *G*(*f*) strongly

*Dα*-closed. Then both *X* and *Y* are *Dα*-*T*1.

Proof. It readily follows from Corollary 6.25 and Theorem 6.26. Theorem 6.28. Let *f* : *X* → *Y* be any surjection with *G*(*f*) strongly

*Dα*-closed. Then *Y* is *T*2 and *α*-*T*2.

{*a*, *c*}  *DO*(*X*, *c*)

and {*z*} *O*(*Y*, *z*)

but {*a*, *c*} *O*(*X*)

(resp.

Proof. Similar to the proof Theorem 6.15 and *T*2-ness always

{*a*, *c*} *O*(*X*) ). Therefore *G*(*f*) is strongly *Dα*-closed but not strongly *α*-closed (resp. strongly closed).

Theorem 6.21. For a function *f* :(*X*, ** )  (*Y*, ** ), the following properties are equivalent:

1. *f* has strongly *Dα*-closed graph.
2. For each (*x*, *y*) (*X*  *Y*) \ *G*( *f* ), there exist *U*  *DO*(*X*, *x*)

and *V* ∈ *O*(*Y*, *y*) such that *f* (*U*)  *Cl*(*V*)  ** .

guarantees *α*-*T*2-ness. Hence *Y* is *α*-*T*2.

Corollary 6.29. Let *f* : *X* → *Y* be any surjection with *G*(*f*) strongly

*Dα*-closed. Then *Y* is *Dα*-*T*2.

Proof. Follows From Theorems 6.13 (i) and 6.28.

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1. For each (*x*, *y*) (*X*  *Y*) \ *G*( *f* ), there exist *U*  *DO*(*X*, *x*)

and *V* *O*(*Y*, *y*) such that (*U*  *Cl* (*V*))  *G*( *f* )  ** .

1. For each (*x*, *y*) (*X*  *Y*) \ *G*( *f* ), there exist *U*  *DO*(*X*, *x*)

and *V* *O*(*Y*, *y*) such that *f* (*U*)  *Cl* (*V*)  ** .

Proof. Similar to the proof of Theorem 6.4.

Theorem 6.22. If *f* : *X* → *Y* is a function with a strongly *Dα*- closed graph, then for each *x* ∈ *X*, *f* (*x*)  {*Cl* ( *f* (*U*)) : *U*  *DO*(*X*, *x*)} .

Proof. Suppose the theorem is false. Then there exists a *y* ≠ *f*(*x*) such that *y* {*Cl* ( *f* (*U*)) : *U*  *DO*(*X*, *x*)}. This implies that *y* *Cl* ( *f* (*U*)) for every *U*  *DO*(*X*, *x*). So *V* ∩ *f*(*U*) ≠ *ϕ* for every *V* *O*(*Y*, *y*) . This, in its turn, indicates that *Cl* (*V*)  *f* (*U*)  *V*  *f* (*U*)  ** , which contradicts the hypothesis that f is a func- tion with *Dα*-closed graph. Hence the theorem holds.

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