Electronic Notes in Theoretical Computer Science 100 (2004) 95–109 

[www.elsevier.com/locate/entcs](http://www.elsevier.com/locate/entcs)

The Homotopy Branching Space of a Flow

# Philippe Gaucher

*Preuves Programmes et Syst`emes Universit´e Paris 7–Denis Diderot Case 7014*

*2 Place Jussieu*

*75251 PARIS Cedex 05*

*France*

[*http: // www. pps. jussieu. fr/ ~gaucher/*](http://www.pps.jussieu.fr/~gaucher/)

Abstract

In this talk, I will explain the importance of the homotopy branching space functor (and of the homotopy merging space functor) in dihomotopy theory.

*Keywords:* concurrency, homotopy, homotopy colimit, model category, simplicial model category, exact sequence, cone, homology, localization

# Introduction

In [[10](#_bookmark12)], the reader will be able to find a survey of the different geometric approaches of concurrency. The model category of flows was introduced in [[8](#_bookmark10)] to model higher dimensional automata (HDA). It allows the study of HDA up to homotopy (cf. also [[6](#_bookmark8),[7](#_bookmark9)]). A good notion of homotopy of flows must preserve the computer scientific properties of the HDA to be modeled like the initial and final states, the deadlocks and the unreachable states. In particular, it

must preserve the direction of time, hence the terminology *dihomotopy* for a

contraction of *directed homotopy*. This way, instead of working in the category of flows itself, one can work in the localization of the category of flows with respect to *dihomotopy equivalences*.

I will explain in this talk the powerfulness of the homotopy branching space functor in dihomotopy theory. The corresponding papers are “Homotopy

1571-0661 © 2004 Elsevier B.V. Open access under [CC BY-NC-ND license.](http://creativecommons.org/licenses/by-nc-nd/3.0/)

doi:10.1016/j.entcs.2004.08.015

branching space and weak dihomotopy” [[4](#_bookmark5)] and “A long exact sequence for the branching homology” [[3](#_bookmark6)].

# Model category

If C is a category, one denotes by *Map*(C) the category whose objects are the morphisms of C and whose morphisms are the commutative squares of C.

In a category C, an object *x* is *a retract* of an object *y* if there exists *f* : *x* −→ *y* and *g* : *y* −→ *x* of C such that *g* ◦ *f* = Id*x*. A *functorial factorization* (*α, β*) of C is a pair of functors from *Map*(C) to *Map*(C) such that for any *f* object of *Map*(C), *f* = *β*(*f* ) ◦ *α*(*f* ).

Definition 2.1 [[12](#_bookmark14),[11](#_bookmark13)] Let *i* : *A* −→ *B* and *p* : *X* −→ *Y* be maps in a category C. Then *i* has the left lifting property (LLP) with respect to *p* (or *p* has the right lifting property (RLP) with respect to *i*) if for any commutative square

*A*  *α*  ¸*X* ,

*i g* *p*

J *β*  J

*B Y*

there exists *g* making both triangles commutative.

There are several versions of the notion of *model category*. The following definitions give the one we are going to use.

Definition 2.2 [[12](#_bookmark14),[11](#_bookmark13)] A model structure on a category C is three subcate- gories of *Map*(C) called weak equivalences, cofibrations, and fibrations, and two functorial factorizations (*α, β*) and (*γ, δ*) satisfying the following proper- ties :

* 1. (2-out-of-3) If *f* and *g* are morphisms of C such that *g* ◦ *f* is defined and two of *f* , *g* and *g* ◦ *f* are weak equivalences, then so is the third.
  2. (Retracts) If *f* and *g* are morphisms of C such that *f* is a retract of *g* and

*g* is a weak equivalence, cofibration, or fibration, then so is *f* .

* 1. (Lifting) Define a map to be a trivial cofibration if it is both a cofibration and a weak equivalence. Similarly, define a map to be a trivial fibration if it is both a fibration and a weak equivalence. Then trivial cofibrations have the LLP with respect to fibrations, and cofibrations have the LLP with respect to trivial fibrations.
  2. (Factorization) For any morphism *f* , *α*(*f* ) is a cofibration, *β*(*f* ) a trivial fibration, *γ*(*f* ) is a trivial cofibration , and *δ*(*f* ) is a fibration.

Definition 2.3 [[12](#_bookmark14),[11](#_bookmark13)] A model category is a complete and cocomplete cat- egory C together with a model structure on C.

Proposition and Definition 2.4 *[*[*12*](#_bookmark14)*,*[*11*](#_bookmark13)*] A Quillen adjunction is a pair of adjoint functors F* : C « D : *G between the model categories* C *and* D *such that one of the following equivalent properties holds :*

1. *if f is a coﬁbration (resp. a trivial coﬁbration), then so does F* (*f* )
2. *if g is a ﬁbration (resp. a trivial ﬁbration), then so does G*(*g*)*.*

*One says that F is a* left Quillen functor*. One says that G is a* right Quillen functor*.*

fibrant) if and only if the canonical morphism ∅ −→ *X* from the initial object Definition 2.5 [[12](#_bookmark14),[11](#_bookmark13)] An object *X* of a model category C is cofibrant (resp. of C to *X* (resp. the canonical morphism *X* −→ 1 from *X* to the final object

1) is a cofibration (resp. a fibration).

For any object *X* of a model category, the canonical morphism ∅*X* : ∅ −→

*X* from the initial object to *X* can be factored as a composite

∅ *α*(∅X ) *Q* (*X*) *β*(∅X ) *X*

where, by definition, *Q*(*X*) is a cofibrant object which is weakly equivalent to

*X*. The functor *Q* : C −→ C is called the *coﬁbrant replacement functor*.

# Reminder about the category of flows

In the sequel, any topological space will be supposed to be compactly gener- ated (more details for this kind of topological spaces in [[1](#_bookmark3),[14](#_bookmark15)], the appendix of [[13](#_bookmark16)] and also the preliminaries of [[8](#_bookmark10)]).

Let *n* ≥ 1. Let D*n* be the closed *n*-dimensional disk. Let S*n*−1 = *∂*D*n* be the boundary of D*n* for *n* ≥ 1. Notice that S0 is the discrete two-point topological space {−1*,* +1}. Let D0 be the one-point topological space. Let

S−1 = ∅ be the empty set. The following theorem is well-known.

Theorem 3.1 *[*[*11*](#_bookmark13)*,*[*12*](#_bookmark14)*] The category of compactly generated topological spaces*

Top *can be given a model structure such that:*

* 1. *The weak equivalences are the weak homotopy equivalences.*
  2. *The ﬁbrations (sometime called Serre ﬁbrations) are the continuous maps satisfying the RLP (right lifting property) with respect to the continuous maps* D*n* −→ [0*,* 1] × D*n such that x* '→ (0*, x*) *and for n* ≥ 0*.*
  3. *The coﬁbrations are the continuous maps satisfying the LLP (left lifting property) with respect to any maps satisfying the RLP with respect to the inclusion maps* S*n*−1 −→ D*n.*
  4. *Any topological space is ﬁbrant.*
  5. *The homotopy equivalences arising from this model structure coincide with the usual one.*

Definition 3.2 [[8](#_bookmark10)] A flow *X* consists of a topological space P*X*, a discrete space *X*0, two continuous maps *s* and *t* from P*X* to *X*0 and a continuous and associative map ∗ : {(*x, y*) ∈ P*X* × P*X*; *t*(*x*) = *s*(*y*)} −→ P*X* such that

of a set map *f* 0 : *X*0 −→ *Y* 0 together with a continuous map P*f* : P*X* −→ P*Y s*(*x* ∗ *y*) = *s*(*x*) and *t*(*x* ∗ *y*) = *t*(*y*). A morphism of flows *f* : *X* −→ *Y* consists such that *f* (*s*(*x*)) = *s*(*f* (*x*)), *f* (*t*(*x*)) = *t*(*f* (*x*)) and *f* (*x* ∗ *y*) = *f* (*x*) ∗ *f* (*y*).

The corresponding category will be denoted by Flow.

space P*X* is called the *path space* and its elements the *non constant execution* The topological space *X*0 is called the 0*-skeleton* of *X*. The topological *paths* of *X*. The initial object ∅ of Flow is the empty set. The terminal object 1 is the flow defined by 10 = {0}, P1 = {*u*} and necessarily *u* ∗ *u* = *u*.

flow Glob(*Z*) defined as follows: Glob(*Z*)0 = {0*,* 1}, PGlob(*Z*) = *Z*, *s* = 0, Definition 3.3 [[8](#_bookmark10)] Let *Z* be a topological space. Then the globe of *Z* is the *t* = 1 and the composition law is trivial.

Theorem 3.4 *[*[*8*](#_bookmark10)*] The category of flows can be given a model structure such that:*

1. *The weak equivalences are the* weak S-homotopy equivalences*, that is*

*isomorphism of sets and f* : P*X* −→ P*Y a weak homotopy equivalence of a morphism of flows f* : *X* −→ *Y such that f* : *X*0 −→ *Y* 0 *is an topological spaces.*

1. *The ﬁbrations are the continuous maps satisfying the RLP with respect to the morphisms* Glob(D*n*) −→ Glob([0*,* 1] × D*n*) *for n* ≥ 0*. The ﬁbrations*

*are exactly the morphisms of flows f* : *X* −→ *Y such that* P*f* : P*X* −→

P*Y is a Serre ﬁbration of* Top*.*

1. *The coﬁbrations are the morphisms satisfying the LLP with respect to any*

*map satisfying the RLP with respect to the morphisms* Glob(S*n*−1) −→

Glob(D*n*) *for n* ≥ 0 *and with respect to the morphisms* ∅ −→ {0} *and*

{0*,* 1} −→ {0}*.*

1. *Any flow is ﬁbrant.*

Let *Igl* be the set of morphisms of flows Glob(S*n*−1) → Glob(D*n*) for

*n* ≥ 0. Denote by *Igl* be the union of *Igl* with the two morphisms of flows

+

*R* : {0*,* 1}→ {0} and *C* : ∅ ⊂ {0}.

morphism of flows ∅ −→ *X* from the initial object of Flow to *X* is a transfinite Definition 3.5 [[4](#_bookmark5)] An *Igl*-cell complex is a flow *X* such that the canonical composition of pushouts of elements of *Igl*. The full and faithful subcategory

+

+

of Flow whose objects are the *Igl*-cell complexes will be denoted by *Igl*cell.

+ +

The category *Igl*cell of *Igl*-cell complexes is a subcategory of the category

+ +

of flows which is sufficient to model higher dimensional automata (HDA), at

least those modeled by precubical sets [[9](#_bookmark11),[2](#_bookmark4)]. This geometric model of HDA is designed to define and study equivalence relations preserving the computer- scientific properties of the HDA to be modeled so that it then suffices to work in convenient localizations of *Igl*cell. The properties which are preserved are for instance the initial or final states, the presence or not of deadlocks and of unreachable states [[8](#_bookmark10)].

+

The cofibrant replacement functor is a functor *Q* : Flow −→ *Igl*cell. The

+

flows coming from concrete HDAs are all cofibrant.

# The homotopy branching space functor

paths beginning in the same way. The branching space functor P− from the The branching space of a flow is the space of germs of non-constant execution category of flows Flow to the category of compactly generated topological

spaces Top was also introduced in [[8](#_bookmark10)] to fit the definition of the branching semi- globular nerve of a strict globular *ω*-category modeling an HDA introduced in [[5](#_bookmark7)].

Proposition 4.1 *[*[*8*](#_bookmark10)*,*[*4*](#_bookmark5)*] Let X be a flow. There exists a topological space* P−*X unique up to homeomorphism and a continuous map h*− : P*X* −→ P−*X satis- fying the following universal property:*

* 1. *For any x and y in* P*X such that t*(*x*) = *s*(*y*)*, the equality h*−(*x*) =

*h*−(*x* ∗ *y*) *holds.*

* 1. *Let φ* : P*X* −→ *Y be a continuous map such that for any x and y of* P*X such that t*(*x*) = *s*(*y*)*, the equality φ*(*x*) = *φ*(*x* ∗ *y*) *holds. Then there exists a unique continuous map φ* : P−*X* −→ *Y such that φ* = *φ* ◦ *h*−*.*

*Moreover, one has the homeomorphism*

P−*X* ∼= .

*α*

*α*∈*X*0

P−*X*

*where* P−*X* := *h*− .

*α*

*β*∈*X*

0 P*α,βX* *. The mapping X* '→ P−*X yields a functor*

P− *from* Flow *to* Top*.*

Definition 4.2 [[8](#_bookmark10),[4](#_bookmark5)] Let *X* be a flow. The topological space P−*X* is called the branching space of the flow *X*.

*f* : *X* −→ *Y such that the topological spaces* P−*X and* P−*Y are not weakly* Proposition 4.3 *[*[*4*](#_bookmark5)*] There exists a weak S-homotopy equivalence of flows homotopy equivalent.*

*X*, by Proposition [4.1](#_bookmark0), the topological space P−*X* is the coequalizer of the The idea for the proof of Proposition [4.3](#_bookmark1) is as follows. For a given flow continuous map P*X* ×*X*0 P*X* −→ P*X* induced by the composition law of *X* and of the projection map P*X* ×*X*0 P*X* −→ P*X* on the first factor. And one cannot

expect a coequalizer to transform a objectwise weak homotopy equivalence into

a weak homotopy equivalence. One must use a kind of homotopy coequalizer instead.

If two flows are weakly S-homotopy equivalent, then they are supposed to satisfy the same computer-scientific properties. With the example above, one obtains two such flows but with very different branching spaces. But

*between coﬁbrant flows, then the topological spaces* P−*X and* P−*Y are homo-* Theorem 4.4 *[*[*4*](#_bookmark5)*] If f* : *X* −→ *Y is a weak S-homotopy equivalence of flows topy equivalent.*

This suggests that the definition of the branching space is the good one up to homotopy for cofibrant flows. Indeed, we have the theorems:

*pair of functors* P− : Flow « Top : *C*− *is a Quillen adjunction. In particular,* Theorem 4.5 *[*[*4*](#_bookmark5)*] There exists a functor C*− : Top −→ Flow *such that the there is an homeomorphism* P−(l−i→m *Xi*) ∼= l−i→m P−*Xi.*

Definition 4.6 The homotopy branching space hoP− *X* of a flow *X* is by definition the topological space P−*Q*(*X*).

Theorem 4.7 *[*[*4*](#_bookmark5)*] The functor* hoP− : Flow −→ Top −→ Ho(Top) *satisﬁes the following universal property: if F* : Flow −→ Ho(Top) *is another functor*

*natural transformation F* ⇒ P−*, then the latter natural transformation factors sending weak S-homotopy equivalences to isomorphisms and if there exists a uniquely as a composite F* ⇒ hoP− ⇒ P−*.*

Up to homotopy, the homotopy branching space hoP−(*X*) is well-defined and coincides with P−*X* for any cofibrant flow, so in particular for any flow coming from a HDA. The behavior of the branching space functor and the

homotopy branching space functor are the same up to homotopy for flows modeling HDAs and may differ for other flows.

# The homotopy merging space functor

This is the dual version of the preceding functor. Some results are collected in this section about it.

Proposition 5.1 *[*[*4*](#_bookmark5)*] Let X be a flow. There exists a topological space* P+*X unique up to homeomorphism and a continuous map h*+ : P*X* −→ P+*X satis- fying the following universal property :*

* 1. *For any x and y in* P*X such that t*(*x*) = *s*(*y*)*, the equality h*+(*y*) =

*h*+(*x* ∗ *y*) *holds.*

* 1. *Let φ* : P*X* −→ *Y be a continuous map such that for any x and y of* P*X such that t*(*x*) = *s*(*y*)*, the equality φ*(*y*) = *φ*(*x* ∗ *y*) *holds. Then there exists a unique continuous map φ* : P+*X* −→ *Y such that φ* = *φ* ◦ *h*+*.*

*Moreover, one has the homeomorphism*

P+*X* ∼= .

*α*

*α*∈*X*0

P+*X*

*where* P+*X* := *h*+ .

*α*

P+ : Flow −→ Top*.*

*β*∈*X*

0 P*β,αX* *. The mapping X* '→ P+*X yields a functor*

Definition 5.2 [[4](#_bookmark5)] Let *X* be a flow. The topological space P+*X* is called the

merging space of the flow *X*.

*pair of functors* P+ : Flow « Top : *C*+ *is a Quillen adjunction. In particular,* Theorem 5.3 *[*[*4*](#_bookmark5)*] There exists a functor C*+ : Top −→ Flow *such that the there is an homeomorphism* P+(l−i→m *Xi*) ∼= l−i→m P+*Xi.*

Definition 5.4 [[4](#_bookmark5)] The homotopy merging space hoP+ *X* of a flow *X* is by definition the topological space P+*Q*(*X*).

Theorem 5.5 *[*[*4*](#_bookmark5)*] The functor* hoP+ : Flow −→ Top −→ Ho(Top) *satisﬁes the following universal property : if F* : Flow −→ Ho(Top) *is another functor*

*natural transformation F* ⇒ P+*, then the latter natural transformation factors sending weak S-homotopy equivalences to isomorphisms and if there exists a uniquely as a composite F* ⇒ hoP+ ⇒ P+*.*

# First application: studying weak dihomotopy

The class S of weak S-homotopy equivalences is an example of class of mor- phisms of flows which is supposed to preserve various computer-scientific prop- erties. This class of morphisms of flows satisfies the following properties:

* 1. The two-out-of-three axiom, that is if two of the three morphisms *f* , *g* and *g* ◦ *f* belong to S, then so does the third one: this condition means that the class S defines an equivalence relation.
  2. The embedding functor *I* : *Igl*cell −→ Flow induces a functor *I* : *Igl*cell[S−1] −→ Flow[S−1] between the localization of respectively the category of *Igl*-cell complexes and the category of flows with respect to weak S-homotopy equivalences which is an equivalence of categories.

+

+

+

In particular, it reflects isomorphisms, that is *X* =∼ *Y* if and only if

*I*(*X*) ∼= *I*(*Y* ). In this case, one can use the whole category of flows which is a richer mathematical framework.

The class of T-homotopy equivalences was introduced in [[8](#_bookmark10)] to identify *Igl*- cell complexes equivalent from a computer-scientific viewpoint and which are

+

not identified in *Igl*cell[S−1]. Indeed, if two objects *X* and *Y* of *Igl*cell[S−1]

+ +

are isomorphic, then the 0-skeletons *X*0 and *Y* 0 are isomorphic. The merging

of the notions of weak S-homotopy equivalence and T-homotopy equivalence yields the class ST 0 of 0-dihomotopy equivalences.

Definition 6.1 [[8](#_bookmark10)] Let *X* be a flow. Let *A* and *B* be two subsets of *X*0. One

exists execution paths *γ*1 and *γ*2 of P*X* such that *s*(*γ*1) ∈ *B*, *t*(*γ*1) = *s*(*γ*2) = *α* says that *A* is surrounded by *B* (in *X*) if for any *α* ∈ *A*, either *α* ∈ *B* or there and *t*(*γ*2) ∈ *B*. We denote this situation by *A* ≪ *B*.

Definition 6.2 [[8](#_bookmark10)] Let *X* be a flow. Let *A* be a subset of *X*0. Then the restriction *X* T*A* of *X* over *A* is the unique flow such that (*X* T*A*)0 = *A* and

P (*X* T*A*) = .

(*α,β*)∈*A*×*A*

P*α,β X*

equipped with the topology induced by the one of P*X*.

Definition 6.3 [[8](#_bookmark10)] A morphism of flows *f* : *X* −→ *Y* is a 0-dihomotopy

equivalence if and only if the following conditions are satisfied :

1. The morphism of flows *f* : *X* −→ *Y* T*f*(*X*0 ) is a weak S-homotopy equiva- lence of flows. In particular, the set map *f* 0 : *X*0 −→ *Y* 0 is one-to-one.
2. For *α* ∈ *Y* 0\*f* (*X*0), the topological spaces P−*Y* and P+*Y* are singletons.

*α α*

1. *Y* 0 ≪ *f* (*X*0).

The class of 0-dihomotopy equivalences is denoted by ST 0.

But it turns out that

Theorem 6.4 *[*[*4*](#_bookmark5)*] The functor Igl*cell[ST −1] −→ Flow[ST −1] *does not re-*

+ 0 0

*flect isomorphisms. More precisely, there exists an Igl-cell complex C*3 *cor-*

+

*responding to the concurrent execution of three calculations which is not iso- morphic in Igl*cell[ST −1] *to the directed segment I, although the same flow*

+ 0

*C*3 *is isomorphic to I in* Flow[ST −1]*.*

0

The correct behavior is the one of ST 0 in Flow[ST −1]. Indeed, an HDA representing the concurrent execution of *n* processes must be equivalent to the directed segment in a good homotopical approach of concurrency. The interpretation of this fact is therefore that the class ST 0 of 0-dihomotopy equivalences is not big enough.

0

Definition 6.5 [[4](#_bookmark5)] A morphism of flows *f* : *X* −→ *Y* is a 1-dihomotopy equivalence if and only if the following conditions are satisfied :

1. The morphism of flows *f* : *X* −→ *Y* T*f*(*X*0 ) is a weak S-homotopy equiva- lence of flows. In particular, the set map *f* 0 : *X*0 −→ *Y* 0 is one-to-one.
2. For *α* ∈ *Y* 0\*f* (*X*0), the topological spaces P−*Y* and P+*Y* are weakly

*α α*

contractible.

1. *Y* 0 ≪ *f* (*X*0).

The class of 1-dihomotopy equivalences is denoted by ST 1.

Any 0-dihomotopy equivalence is of course a 1-dihomotopy equivalence. Moreover, the composite of a weak S-homotopy equivalence with a T-homoto- py equivalence can already give an element of ST 1\ST 0 ! And

Theorem 6.6 *[*[*4*](#_bookmark5)*] By slightly weakening the notion of T-homotopy as above, one obtains a class of morphisms* ST 1 *with* ST 0 ⊂ ST 1 *and such that the*

*flows C*3 *and I become isomorphic in the localization Igl*cell[ST −1]*.*

+ 1

There are actually two natural ways of weakening the definition of ST 0. One can replace in the statement the word *singleton* either by the word *weakly contractible*, or by the word *contractible*. This way, one obtains another class of morphisms ST '1 with ST '1 ⊂ ST 1 and one has:

Theorem 6.7 *[*[*4*](#_bookmark5)*] The localizations Igl*cell[ST '−1] *and Igl*cell[ST −1] *are equi-*

+ 1 + 1

*valent.*

Unfortunately, one has

Proposition 6.8 *[*[*4*](#_bookmark5)*] The composite of two morphisms of* ST 1 *does not nec- essarily belong to* ST 1*.*

Using the homotopy branching space functor, a new class ST 2 of mor- phisms of flows is introduced.

Definition 6.9 [[4](#_bookmark5)] A morphism of flows *f* : *X* −→ *Y* is a 2-dihomotopy equivalence if and only if the following conditions are satisfied :

1. The morphism of flows *f* : *X* −→ *Y* T*f*(*X*0 ) is a weak S-homotopy equiva- lence of flows. In particular, the set map *f* 0 : *X*0 −→ *Y* 0 is one-to-one.
2. For *α* ∈ *Y* 0\*f* (*X*0), the topological spaces hoP− *Y* and hoP+ *Y* are weakly

*α α*

contractible.

1. *Y* 0 ≪ *f* (*X*0).

The class of 2-dihomotopy equivalences is denoted by ST 2.

And:

Theorem 6.10 *[*[*4*](#_bookmark5)*] One has the equivalence of categories*

*Igl*cell[ST −1] *Ig* *l*cell[ST −1]

+ 1 + 2

*where Igl*cell[ST −1] *(resp. Igl*cell[ST −1]*) is the localization of the category of*

+ 1 + 2

*Igl-cell complexes with respect to* 1*-dihomotopy equivalences (resp.* 2*-dihomo- topy equivalences).* ST 2 *is closed under composition. Moreover the embedding functor I* : *Igl*cell −→ Flow *induces an equivalence of categories*

+

+

*I* : *Igl*cell[ST −1] F low[ST −1] *.*

+ 2 2

*In particular, the functor Igl*cell[ST −1] −→ Flow[ST −1] *reflects isomor-*

*phisms.*

+ 2 2

The property *f* ∈ ST 2 and *g* ◦ *f* ∈ ST 2 =⇒ *g* ∈ ST 2 has no reasons to be satisfied by 2-dihomotopy equivalences. Indeed, if both *g* ◦ *f* and *f* are two one-to-one set maps, then *g* has no reasons to be one-to-one as well. Therefore in order to understand the isomorphisms of Flow[ST −1], we may introduce

2

another construction.

Definition 6.11 [[4](#_bookmark5)] Let *X* be a flow. Then a subset *A* of *X*0 is essential if

*X*0 ≪ *A* and if for any *α* ∈*/ A*, both topological spaces hoP− *X* and hoP+ *X*

*α α*

are weakly contractible.

Definition 6.12 [[4](#_bookmark5)] A morphism of flows *f* : *X* −→ *Y* is a 3-dihomotopy equivalence if the following conditions are satisfied :

1. *A* ⊂ *X*0 is essential if and only if *f* (*A*) ⊂ *Y* 0 is essential
2. for any essential *A* ⊂ *X*0 there exists an essential subset *B* ⊂ *A* such that the restriction *f* : *X* T*B*−→ *Y* T*f*(*B*) is a weak S-homotopy equivalence.

The class of 3-dihomotopy equivalences is denoted by ST 3.

Theorem 6.13 *[*[*4*](#_bookmark5)*] The localizations Igl*cell[ST −1] *and Igl*cell[ST −1] *are equ-*

+ 2 + 3

*ivalent and the class of morphisms* ST 3 *satisﬁes the two-out-of-three axiom.*

*Moreover the embedding functor I* : *Igl*cell −→ Flow *induces an equivalence*

+

*of categories*

*I* : *Igl*cell[ST −1] F low[ST −1] *.*

+ 3 3

*In particular, the functor Igl*cell[ST −1] −→ Flow[ST −1] *reflects isomor-*

*phisms.*

+ 3 3

The class ST 2 does not satisfy the two-out-of-three axiom but is invariant by retract. The class ST 3 does satisfy the two-out-of-three axiom but is probably not invariant by retract. So none of the definitions above allows to

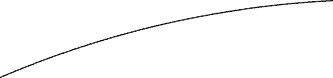
describe the isomorphisms of *Igl*cell[ST −1]. The situation can be summarized

+ 2

with the following diagram:

*gl*  /w *g* *l*

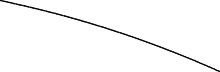
,,,,,

,,,,,

, , /w

*g* *l*

*Igl*cell

J

+

w *g* *l*

*gl* 1

*I* cell[S—1]

z˛

*I* cell[ST —1]

*I* cell[ST —1]

*I* cell[ST —1] *I*

cell[ST ]

+ + 0 + 1 + 2 + 3

w /w

J —1 /w

]

w??

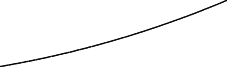
J —1 w??

ST 0

J —1 ¸w,??

w

—1 —1



,,

¸

J

Flow[S

¸ ¸

Flow[ ]

¸¸¸¸¸¸¸¸



¸¸¸¸¸¸

¸

Flow[ST 1 ] Flow[ST 2 ] Flow[ST 3 ]

Flow

The symbol ?? means that we do not know whether the functor is an equivalence of categories or not. The symbol / means that the corresponding functor is not an equivalence.

# Second application: a long exact sequence for the branching homology

The category of flows is a simplicial model category [[3](#_bookmark6)] in the following sense:

Definition 7.1 [[15](#_bookmark17),[12](#_bookmark14),[11](#_bookmark13)] x A simplicial model category is a model category C together with a simplicial set Map(*X, Y* ) for any object *X* and *Y* of C satisfying the following axioms:

* 1. the set Map(*X, Y* )0 is canonically isomorphic to C(*X, Y* )
  2. for any object *X*, *Y* and *Z*, there is a morphism of simplicial sets Map(*Y, Z*) × Map(*X, Y* ) −→ Map(*X, Z*)

which is associative

* 1. for any object *X* of C and any simplicial set *K*, there exists an object *X* ⊗ *K* of C such that there exists a natural isomorphism of simplicial sets

Map(*X* ⊗ *K, Y* ) ∼= Map(*K,* Map(*X, Y* ))

* 1. for any object *X* of C and any simplicial set *K*, there exists an object

*XK* such that there exists a natural isomorphism of simplicial sets Map(*X, Y K*) ∼= Map(*K,* Map(*X, Y* ))

* 1. for any cofibration *i* : *A* −→ *B* and any fibration *p* : *X* −→ *Y* of C, the morphism of simplicial sets

*Q*(*i, p*) : Map(*B, X*) −→ Map(*A, X*) ×Map(*A,Y* ) Map(*B, Y* )

is a fibration of simplicial sets. Moreover if either *i* or *p* is trivial, then the fibration *Q*(*i, p*) is trivial as well.

Recall that there exists a pair of adjoint functors |−| : SSet « Top : *S*∗ where | − | is the geometric realization functor and *S*∗ the singular nerve functor. The *n*-simplex of SSet is denoted by ∆[*n*]. Its boundary is denoted by *∂*∆[*n* − 1]. Let ∆*n* be the *n*-dimensional simplex.

The category of compactly generated topological spaces Top is a simplicial model category by setting Map(*X, Y* )*n* := Top(*X* ×∆*n,Y* ), *X* ⊗*K* := *X* ×|*K*| and *XK* := TOP(|*K*|*,X*). The category of simplicial sets SSet is a simplicial model category as well by setting Map(*X, Y* )*n* := Top(*X* ×∆[*n*]*,Y* ), *X* ⊗*K* := *X* × *K* and *XK* := Map(*K, X*) [[15](#_bookmark17)].

This means that the model category of flows can be enriched over the category of simplicial sets and that the enrichment is compatible with the model structure in the sense of Definition [7.1](#_bookmark2). The symbol ∆*n* is the simplicial set corresponding to the *n*-dimensional simplex.

Because of the existence of this enrichment, there exist explicit formulae for homotopy colimits [[11](#_bookmark13)]. In particular, the homotopy pushout of a diagram of flows looks as follows:

Definition 7.2 [[11](#_bookmark13)] The homotopy pushout of the diagram of flows

*A*  *B*

J

*C*

is the colimit of the diagram of flows

*A* ⊗ ∆0 *B*

*A* ⊗ ∆0 *A* ⊗ ∆1

J

J

*C*

It is then very easy to prove the:

Theorem 7.3 *[*[*3*](#_bookmark6)*] Let X be a diagram of flows. Then the topological spaces*

holim hoP−(*X*) *and* hoP−(holim *X*) *are homotopy equivalent (they are both*

−−−→ −−−→

*coﬁbrant indeed). So in particular, the homotopy branching space functor*

*commutes with homotopy pushouts.*

Definition 7.4 [[3](#_bookmark6)] Let *f* : *X* −→ *Y* be a morphism of flows. The cone *Cf* of

*f* is the homotopy pushout in the category of flows

*X f*  *Y*

J J

1 *C* *f*

where 1 is the terminal flow.

From the theorem

Theorem 7.5 *[*[*3*](#_bookmark6)*] The homotopy branching space of the terminal flow is con- tractible.*

one can easily deduce a long exact sequence for the branching homology.

Definition 7.6 [[3](#_bookmark6)] Let *X* be a flow. Then the (*n* + 1)st branching homol-

ogy group *H*−

*n*+1

1. is defined as the *n*st homology group of the augmented

simplicial set N −(*X*) defined as follows:

* 1. N −(*X*) = *Sn*(hoP− *X*) for *n* ≥ 0

∗

*n*

* 1. N − (*X*) = *X*0
  2. the augmentation map *ϵ* : *S*0(hoP− *X*) −→ *X*0 is induced by the mapping

−1

*γ* '→ *s*(*γ*) from hoP− *X* = *S*0(hoP− *X*) to *X*0.

Theorem 7.7 *[*[*3*](#_bookmark6)*] For any flow X, one has*

1. *H*−(*X*) = Z*X*0*/Im*(*s*)

0

1. *the short exact sequence*

0 → *H*−(*X*) → *H*0(hoP− *X*) → Z hoP− *X/Ker*(*s*) → 0

1

1. *H*− (*X*) = *Hn*(hoP− *X*) *for n* ≥ 1*.*

*n*+1

Theorem 7.8 *[*[*3*](#_bookmark6)*] For any morphism of flows f* : *X* −→ *Y , one has the long*

*exact sequence*

··· → *H*−(*X*) → *H*−(*Y* ) → *H*−(*Cf* ) → *...*

*n n n*

··· → *H*−(*X*) → *H*−(*Y* ) → *H*−(*Cf* ) →

3 3 3

*H*−(*X*) → *H*−(*Y* ) → *H*−(*Cf* ) →

2 2 2

*H*0(hoP− *X*) → *H*0(hoP− *Y* ) → *H*0(hoP− *Cf* ) → 0*.*

equivalence. The functor *X* '→ *H*0(hoP− *X*) is only invariant up to weak S- The functors *X* '→ *H*−(*X*) for *n* ≥ 0 are invariant up to 2-dihomotopy homotopy equivalence. So the long exact sequence above is not satisfactory.

*n*

It still remains to find an exact sequence whose each term would be a functor invariant up to 2-dihomotopy equivalence.

# References

1. Brown, R., “Topology,” Ellis Horwood Ltd., Chichester, 1988, second edition, xviii + 460 pp., a geometric account of general topology, homotopy types and the fundamental groupoid.
2. Cridlig, R., *Implementing a static analyzer of concurrent programs: Problems and perspectives*, in: *Logical and Operational Methods in the Analysis of Programs and Systems*, 1996, pp. 244– 259, <http://citeseer.nj.nec.com/11028.html>.
3. Gaucher, P., *A long exact sequence for the branching homology*, arXiv:math.AT/0305169.
4. Gaucher, P., *Homotopy branching space and weak dihomotopy*, arXiv:math.AT/0304112.
5. Gaucher, P., *The branching nerve of HDA and the Kan condition*, Theory and Applications of Categories 11 (2003), pp. pp.75–106.
6. Gaucher, P., *Concurrent process up to homotopy (I)*, C. R. Acad. Sci. Paris Ser. I Math. 336

(2003), pp. 593–596, french.

1. Gaucher, P., *Concurrent process up to homotopy (II)*, C. R. Acad. Sci. Paris Ser.I Math. 336

(2003), pp. 647–650, french.

1. Gaucher, P., *A model category for the homotopy theory of concurrency*, Homology, Homotopy and Applications 5 (2003), pp. p.549–599.
2. Gaucher, P. and E. Goubault, *Topological deformation of higher dimensional automata*, Homology, Homotopy and Applications 5 (2003), pp. p.39–82.
3. Goubault, E., *Some geometric perspectives in concurrency theory*, Homology, Homotopy and Applications 5 (2003), pp. p.95–136.
4. Hirschhorn, P. S., “Model categories and their localizations,” Mathematical Surveys and Monographs 99, American Mathematical Society, Providence, RI, 2003, xvi+457 pp.
5. Hovey, M., “Model categories,” American Mathematical Society, Providence, RI, 1999, xii+209 pp.
6. Lewis, L. G., “The stable category and generalized Thom spectra,” Ph.D. thesis, University of Chicago (1978).
7. May, J. P., “A concise course in algebraic topology,” University of Chicago Press, Chicago, IL, 1999, x+243 pp.
8. Quillen, D. G., “Homotopical algebra,” Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967, iv+156 pp.