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Topological Properties of Event Structures[1](#_bookmark0)

# Luigi Santocanale[2](#_bookmark1)

*Laboratoire d’Informatique Fondamentale, Universite´ de Provence Marseille, France*

Abstract

Motivated by the nice labelling problem for event structures, we study the topological properties of the associated graphs. For each *n* ≥ 0, we exhibit a graph *G*n that cannot occur on an antichain as a subgraph of the graph of an event structure of degree *n*. The clique complexes of the graphs *G*n are disks (*n* even) and spheres (*n* odd) in increasing dimensions. We strengthen the result for event structures of degree 3: cycles of length greater than 3 do not occur on antichains as subgraphs. This amounts to saying that the clique complex of the graph of an event structure of degree 3 is acyclic.

*Keywords:* Event structures, clique complexes, nice labelling problem

# Introduction

In this note we present some ideas on the use of algebraic topology finalized to the understanding of mathematical structures modeling concurrency, finitary coherent domains and event structures. These ideas are part of a larger investigation of the finite labelling problem for event structures [[2](#_bookmark12)]. The purpose of the note is to show how many natural geometrical questions arise from this working context.

Roughly speaking, the nice labelling problem consists in reconstructing a given finite coherent domain – i.e. a poset which represents the possible of executions of a concurrent system – from the standard ingredients of trace theory [[3](#_bookmark14)]. These are an alphabet, a local independence relation, and a prefix closed subset of the free monoid, see [[1](#_bookmark13),[7](#_bookmark15)]. The problem always has a solution, and we are asked to find a solution of minimal cardinality (of the alphabet). The problem is equivalent to a graph coloring problem in that we can associate to a finite coherent domain a graph, of which we are asked to compute its chromatic number. The main technical contribution in [[2](#_bookmark12)] is to show that some simple graph cannot occur as a subgraph

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2 Email:[luigi.santocanale@lif.univ-mrs.fr](mailto:luigi.santocanale@lif.univ-mrs.fr)

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of the restriction to an antichain of the graph of an event structure of degree 2. We develop this idea for event structures of higher degree an discover a family of graphs that are avoided on antichains. These graphs have a geometrical flavor as they are iteratively constructed by cones and suspensions. This is among the reasons to move from a graph theoretic perspective and to consider instead the clique complex of the graph of an event structure. For degree 3, we show that one dimensional spheres, that is cycles, cannot occur on antichains, unless they are boundaries. This lead to an explicit computation of the homology groups (of antichains of event structure of degree 3) that are shown to be trivial in all the dimension greater than zero.

We conjecture that similar results hold in higher dimensions and degrees. To- ward this goal we make explicit the sense for which the homology of antichains makes a functor from a poset of antichains – isomorphic to the poset of upper sets of the event structure – to the category of sequences of abelian groups.

The usual definition of event structures [[11](#_bookmark20)] already suggests these are sort of *ordered simplicial complexes*. The collection of results we present here might be thought as witnessing the value of such an elementary connection between topology and concurrency. Yet, in our knowledge, we lack a comprehensive clarification of how this approach, grounded on the notion of ordered simplicial complex, compares to other topological analysis of concurrent computation [[6](#_bookmark16),[5](#_bookmark17)]. This might be the object of future researches.

The note is structured as follows. We present in the first section the background for our remarks. This comprises domains and event structures, elements from trace theory, and the nice labelling problem. The reader shall find the definition of the graph of an event structure and the reasons that induce us to study these graphs with fixed clique number. In the second section we shall exhibit some graphs that are avoided on antichains of this class of graphs. In the third section we use the previous considerations to determine the homology groups of the clique complexes of graphs with clique number 3. In the final section we formally define the homology of an event structure, and sketch some conjectures and directions for future researches.

# Concurrency by Partial Orders and Graphs

* 1. *Finitary Coherent Domains*

Recall that an element *p* of a poset ⟨*P,* ≤⟩ is a *complete join prime* if whenever the least upper bound ⋁ *X* of a possibly infinite set *X* ⊆ *P* exists and *p* ≤ ⋁ *X*, then *p* ≤ *x* for some *x* ∈ *X*. For *x* ∈ *P* we let *P* (*x*) = { *p* ≤ *x* ∣ *p* is a complete join prime }.

Definition 1.1 A *ﬁnitary domain* is a poset D = ⟨*D,* ≤⟩ such that, for each *d* ∈ *D*, the set *P* (*d*) is finite and *d* = ⋁ *P* (*d*).

In this paper we are shall be concerned with finite structures, hence the word “finitary” can be safely replaced by “finite”. Let us recall that *d*′ is an upper cover of *d* (denoted by *d* ≺ *d*′) if the open interval { *x* ∣ *d* < *x* < *d*′ } is empty. The degree of *d* ∈ D, deg(*d*), is the number of upper covers of *d*. The degree of a finitary domain

D is defined by

deg(D) = max deg(*d*) *.*

*d*∈D

We shall deal with special domains: a finitary domain is *coherent* if whenever

{ *di, dj* } are bounded for each *i, j* ∈ { 1*,* 2*,* 3 }, then the set { *d*1*, d*2*, d*3 } is also bounded.

* 1. *Local Independence Relations*

We see now how finitary domains arise from trace theory [[3](#_bookmark14)].

Definition 1.2 A *local independence relation* over an alphabet Σ is a relation *R* ⊆

Σ∗ × *P*2(Σ).

A relation of the form *wR*{ *a, b* } informally means that the events *a* and *b* are independent (i.e. commute) immediately after the sequence of events encoded in the word *w*. We denote by ∼*R* the least right congruence containing the pairs *wab* = *wba* whenever *wR*{ *a, b* }, so that [*w*]*R* denotes the equivalence class of *w* modulo ∼*R*. By definition the quotient Σ∗/∼*R* is a right module, the action on equivalence classes being defined in the natural way: [*w*]*Ra* = [*wa*]*R*.

We say that the local independence relation *R* is *stable* if and only if the cube axiom holds for each ∼*R*-equivalence class *q*:

, *qb*¸*c*  *a*  *qa* ¸*bc* , *qb*,¸*c* , *a*  *qa* *bc*

*b*,,

,*b*,, ,,

*b*,, ,,

*qc* ,, *a*  *qa* *c* , *c qc* ,, *c c*

,, ,,

*c c*  *qa*¸*b*

⇐⇒ ,,

*c* , *qb*¸

*a*  *qa* ¸*b*

*b*,,,

,*b*,,

*b*,,,

*q*  *a*  *qa* ,, *q* ,, *a*  *qa* ,,

This diagram is asserting two implications. The implication from left to right is read as follows: if *qac* = *qca* and *qabc* = *qacb* = *qcab* = *qcba*, then *qab* = *qba*, *qbc* = *qcb*, and *qbac* = *qbca*. We leave the reader to make explicit the implication from right to left. We say that *R* is *coherent* if and only if the implication

, *qb*,¸*c* ,

*qb*¸*c*  *a*  *qa* ¸*bc*

*b*,,

*b*,,, ,, ,*b*,, ,,

*qc* ,, *ac*

,,

, *qb*¸*c*

*a*

*a*  *qa* ,,

*b*,,,

*qa* ¸*b*

,,

*c*

*qa* *c*

*qc* ,,

=⇒

*ac* *qa* *c* ,

*qb*

*c*

,,

*a*

*qa* ,

,

*b* ,

,

,

*c*

*qa* ¸*b*

,*b*,,

,,

*q*

holds for every ∼*R*-equivalence class *q*. Explicitly, if *qab* = *qba*, *qac* = *qca*, and

*qbc* = *qcb*, then *qabc* = *qabc*, *qbac* = *qbca*, and *qcab* = *qcba*.

We can define on a right Σ∗-module a preorder by saying *q* ≤ *q*′ if and only if

*q*′ = *qw* for some *w* ∈ Σ∗. The following is the main result of [[7](#_bookmark15)].

Theorem 1.3 *If R is stable and coherent, then for any lower set L* ⊆ Σ∗/∼*R, the pair* ⟨*L,* ≤⟩ *is a ﬁnitary coherent domain.*

A lower set *L* ⊆ Σ∗/ ∼*R* can be identified with a prefix closed subset of Σ∗ which moreover is closed w.r.t. ∼*R*. We denote the domain arising from the data ⟨Σ*, R, L*⟩ by D(Σ*, R, L*). The reader will have no difficulties in verifying that deg(D(Σ*, R, L*)) ≤ card(Σ). We are ready to state the nice labelling problem: [3](#_bookmark4) *given a ﬁnite coherent domain* D *compute the least n* ≥ 0 *such that, for some data*

⟨Σ*, R, L*⟩ *with* card(Σ)= *n,* D *is order isomorphic to* D(Σ*, R, L*)*.* It is quite easy to find some data ⟨Σ*, R, L*⟩ giving rise to D. Indeed let *CV* (D) = { (*d, d*′)∣ *d* ≺ *d*′ } and say that (*x, x*′)*,* (*y, y*′) ∈ *CV* (D) are *perspective* if *x* = *x*′ ∧ *y* and *y*′ = *y* ∨ *x*′. Then we need to find Σ and *λ* ∶ *CV* (D) —→ Σ such that

1. *λ*(*x, x*′)= *λ*(*y, y*′) if (*x, x*′)*,* (*y, y*′) are perspective,
2. *λ*(*x, x*′)≠ *λ*(*x, x*′′) if *x* ≠ *x*′′.

It is not difficult to satisfy these constraints, provided that Σ is large. Since some data ⟨Σ*, R, L*⟩ giving rise to D always exists, we let nl(D) be such least *n*. More generally, for *d* ≥ 0, we define

nl(*d*)= max{ nl(D) ∣ deg(D) = *d* } *.*

It was shown in [[2](#_bookmark12)] that nl(*d*) = *d* if *d* ≤ 2 and that nl(*d*)> *d* otherwise.

* 1. *Event Structures*

Finitary domains are almost lattices, since the join of two elements might not exist. Nonetheless, they are distributive, meaning also that a Birkhoff-like representation theorem holds: finitary domains are dual to event structures.

Definition 1.4 An *event structure* [4](#_bookmark5) is a triple E = ⟨*P,* ≤*,* ⌣⟩ such that

* ⟨*P,* ≤⟩ is a poset, such that for each *p* ∈ *P* the lower set { *x* ∣ *x* ≤ *p* } is finite,
* ⌣⊆ *P* × *P* is a symmetric binary relation upper closed w.r.t. the order (i.e. *p* ⌣ *q*

and *p* ≤ *p*′ implies *p*′ ⌣ *q*). Moreover, if *p* ⌣ *q*, then { *p, q* } is an antichain.

The order ≤ is known as the *causality* of events, where the binary relation ⌣ is called *conflict*. Given an event structure ⟨*P,* ≤*,* ⌣⟩, we define the *concurrency* relation

= as the complement of ⌣, i.e. *p* = *q* if and only if it not the case that *p* ⌣ *q*. The concurrency relation = is closed under the order (*x*′ ≤ *x* = *y* implies *x*′ = *y*) and every comparable pair is concurrent. Event structures could have been defined by taking the concurrency relation as a primitive notion, and in the following we shall be oblivious of the conflict relation. Given an event structure E = ⟨*P,* ≤*,* ⌣⟩ a lower set of E is a subset *I* of *P* such that *x* ≤ *y* ∈ *I* implies *x* ∈ *I*. We let

*CL*(E) = { *I* ∣ *I* is a lower set and a clique w.r.t. =} *,* D(E) = ⟨*CL*(E)*,* ⊆⟩ *.*

Theorem 1.5 *The poset* D(E) *is a ﬁnitary coherent domain. Moreover, every ﬁnite coherent domain* D *is order isomorphic to some domain of the form* D(E) *for*

3 See Corollary [1.7](#_bookmark6) for the meaning of the word “labelling”.

4 We define here event structures with binary conflict.

*some event structure* E *.*

The statement is a well known result of concurrency theory, see [[11](#_bookmark20)]. We recall that given a domain D, we can define ED = ⟨*P,* ≤*,* ⌣⟩ with the property that D is order isomorphic to D(ED) as follows: we let *P* be the set of complete join prime elements of D, ≤ is the restriction of the order of D to *P* , and we let *p* = *p*′ if and only if the pair { *p, p*′ } is bounded in D.

We shall need one more relation:

*p* = *q* if and only if { *p, q* } is an antichain,

*p*′ = *q* for all *p*′ < *p,* and *p* = *q*′ for all *q*′ < *q.*

Given an event structure E = ⟨*P,* ≤*,* ⌣⟩ we define the undirected graph G(E) as the pair ⟨*P,* =⟩.

Lemma 1.6 *A set* { *x*1*,... , xn* } *is a clique in the graph* G(E) *iff there exists an ideal I* ∈ *CL*(E) *such that the* { *xi* }∪ *I, i* = 1*,... , n, are distinct upper covers of I in* D(E)*.*

Proof. Suppose that { *xi* }∪ *I* and { *xj* }∪ *I* are distinct upper covers of some *I* in

D(E). Then {*xi, xj*} is an antichain since *xi* ≤ *xj* implies that { *xi* }∪ *I* ≤ { *xj* }∪ *I*. If *x*′ < *xi* then *I* ⊆ { *x*′ }∪ *I* ⊆ { *xj* }∪ *I* hence *I* = { *x*′ }∪ *I* and *x*′ ∈ *I* ⊆ { *xj* }∪ *I*. Since { *xj* }∪ *I* is a =-clique, then *x*′ = *xj*. Similarly *y*′ < *xj* implies *xi* = *y*′. Thus,

{ *x*1*,... , xn* } is a clique.

Conversely, let us suppose that *xi* = *xj* whenever *i* ≠ *j*. Observe that *xi* = *xj*

implies *x*′ = *y*′ for *x*′ < *x* and *y*′ < *y*. Thus the ideal *I* = ⋃*n* { *x*′ ∣ *x*′ < *xi* } belongs

*i*=1

to *CL*(E) and { *xi* } ∪ *I* belongs to *CL*(E) for *i* = 1*,... , n*. Since { *xi, xj* } is an

antichain, then { *xi* }∪ *I* and { *xj* }∪ *I* are distinct upper covers of *I*. ◻

For a graph G, let *χ*(G) be the chromatic number of G and let *ω*(G) be its clique number, that, is the cardinality of the greatest clique.

Corollary 1.7 *The following relations hold:*

deg(D(E )) = *ω*(G(E )) *,* nl(D(E )) = *χ*(G(E ))*.*

Proof. The first statement clearly follows from the previous Lemma.

A labelling of D(E) satisfies condition ([i](#_bookmark2)) if and only if it is uniformly defined on prime elements of D(E), that is if it is defined on on elements of *P* . By the previous Lemma, it satisfies ([ii](#_bookmark3)) of and only if it *λ*(*p*) ≠ *λ*(*q*) whenever *p* = *q*. Hence a nice labelling of D(E) corresponds to a coloring of the graph G(E). ◻

Thus, for an event structure, we define the degree of E as the degree of D(E) or, in an equivalent way, as the clique number of G(E). The nice labelling problem for D amounts to find a coloring of G(ED) with the smallest number of colors.

The reader who’s not motivated by the finite labelling problem might object that studying the concurrency graph of an event structure – i.e. vertexes are elements of *P* and two elements are related if they are uncomparable and concurrent – might

be more interesting. However, it is a trivial observation that every graph can be realized as the concurrency graph of some event structure. On the other hand, not every graph is of the form G(E): many are the constraints on =, for example minimal elements form a clique w.r.t. =. Again, motived by nice labelling problem, it can be shown that some well known graphs with clique number 3 and increasing coloring number do not occur as subgraphs of some graph of the form G(E).

* + 1. *Characterization of data of the form* ⟨*P,* ≤*,* =⟩*.*

The following observations are trivial but worth to remark.

Let us consider a triple ⟨*P,* ≤*,E*⟩, where ⟨*P, E*⟩ is an undirected graph and ⟨*P,* ≤⟩ is a partially ordered set. Let us say that this triple is *order consistent* if *xEy* implies that (i) *x, y* are uncomparable and (ii) *x*′ < *x* implies *x*′*Ey* or *x*′ ≤ *y*. If the triple is order consistent, then we say that a pair (*x, y*) is *left induced* if either *x* ≤ *y*, or *x*′*Ey* for some *x*′ > *x*; it is *right induced* if either *x* ≥ *y*, or *xEy*′ for some *y*′ > *y*. A pair (*x, y*) is *E-induced* iff it is left or right induced. We define *E*○ to be the set of *E*-induced pairs.Observe that if *Comp* =≤ ∪ ≥ is the comparability relation, then *Comp* ⊆ *E*○ ⊆ *E* ∪ *Comp*. We say that an order consistent *E* is *closed* if whenever *x* and *y* are such that *x*′*E*○*y* for every *x*′ < *x* and *xE*○*y*′ for every *y*′ < *y*, then *xEy*.

Proposition 1.8 *The triple* ⟨*P,* ≤*,* =⟩ *is closed order consistent. Conversely, if a triple* ⟨*P,* ≤*,E*⟩ *is closed order consistent then E is of the form* = *for the concurrecny relation E*○*; the latter is the least concurrency relation giving rise to E.*

Proof. It is straightforward to observe that = is closed and consistent, since if

*x* =○ *y*, then *x* = *y*.

Let us consider *E* which is closed consistent. Let us define = as *E*○, then closed- ness of *E* means that *x* = *y* implies *xEy*. Conversely, let us suppose that *xEy*. Then *x, y* is an antichain and if *x*′ < *y*, then *x*′*E*○*y*, and similarly for *y*: thus *x* = *y*.

Finally, if = gives rise to *E*, and *xE*○*y*, then if *x* and *y* are comparable, then *x* = *y*, otherwise *x*′*Ey* for some *x*′ > *x* (or the symmetric case) then *x* = *yE*, showing that *E* ⊆=. ◻

The following Lemma is an example of how to use the abstract characterization of triples ⟨*P,* ≤*,* =⟩. It has the consequence that for any E = ⟨*P,* ≤*,* ⌣⟩ with deg(E) = *n* and any *p* ∈ *P* , the collection of =-neighbors of *p* is an event structure E*p* with deg(E*p*)≤ *n* − 1.

Proposition 1.9 *Consider a closed order consistent* ⟨*P,* ≤*,* =⟩ *triple and let p* ∈ *P. Let Pp* = { *p*′ ∣ *pEp*′ }*. Then* ⟨*Pp,* ≤∣*P , E*∣*P* ⟩ *id closed order consistent.*

p p

Proof. Clearly, the triple is order consistent. Thus suppose that for each *x, y* ∈ *Pp*, and suppose that *x*′ ∈ *Pp* and *x*′ < *x* implies *x*′*E*○*y* and the similar relation holds for

*y*. Let *x*′ < *x* with *x*′ ∈/ *Pp*. Since *E* is closed, *x*′ < *p* and thus *x*′*E*○*y*. Therefore *x, y*

is a critical pair and therefore *xEy*, since *E* is closed. ◻

# Avoided Graphs on Antichains

We are going to study the structure of a graph G(E) restricted to antichains. An antichain is meant to represent a collection of global states of a system, incompatible among them, that may share local similarities. A global state is characterized by the clique of its enabled unnamed events. Recall that the clique complex CL(*G*) of a graph *G* has as simplices the cliques of a graph. Therefore we shall emphasize that a global state is a simplex in the clique complex of G(E). Two global states may be similar in that they share some face. An antichain *B* might be dependent on an antichain *A* if each event in *B* depends on an event in *A*. Our goal is to analyze to what extent the topological properties of antichains are invariant under the dependency relation. These ideas are similar to (and indeed have been suggested by) those in the work [[6](#_bookmark16)] on asynchronous computability. An attempt to formalize this work in the present context of event structures has failed until now and suggested possible divergences. Thus we let A(E) the set of antichains of

⟨*P*E *,* ≤E ⟩. For *A, B* ∈ A(E), we say that *B* depends on *A*, written *B* >> *A*, if for all

*b* ∈ *B* there exists an *a* ∈ *A* such that *a* ≤ *b*. We shall come back to this order on

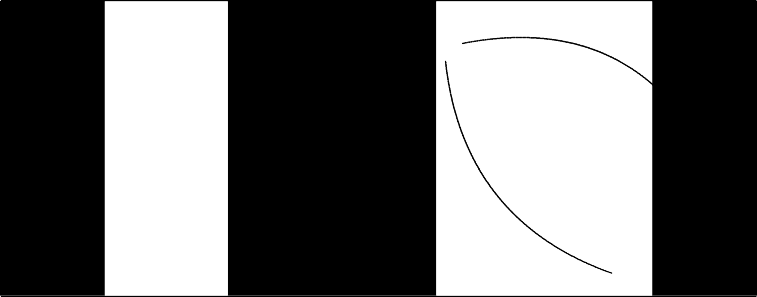
A(E) in the last section.

* 1. *Disks and spheres . . .*

We define a sequence of graphs *Gn*, for *n* ≥ −1. To this goal, for a graph *G* = ⟨*V, E*⟩, let *v* ∗ *G* = ⟨*V* ⊎{ *v* }*,E* ∪ *E*′) where *E*′ = { { *v, v*′ }∣ *v*′ ∈ *V* }. Observe that a clique in *v* ∗ *G* is either *v*, or a clique in *G*, or a clique of the form { *v* }∪ *S*, with *S* a clique of *G*. That is, this operation amounts to *adding a cone* to *G* in the clique complex of *G*, CL(*v* ∗ *G*) = *v* ∗ CL(*G*). The *suspension* of *G*, i.e. adding two cones to *G*, is defined similarly and it is well defined w.r.t. the clique complex. It will be denoted by (*v, v*˜)∗ *G*.

The graph *G*−1 is the empty graph. If *n* is even, then *Gn* = *vn* ∗ *Gn*−1, and, if *n* is odd, *Gn* = (*vn, v*˜*n*)∗*Gn*−2. We sketch the structure of the graphs *Gn* for *n* = 0*,... ,* 4:

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| ∶ | *v*0 | ∶ | *v*1 | *v*˜1 | ∶ | *v*1 | *v*2,,  ,,,  ,,  *v*˜1 |



,

*v*˜3

*v*˜1

,,,,

*v*1,,

,,,,

3

,,,

,,,,

¸¸¸¸¸ *v*

,,

,,,,

*v*4,¸¸¸¸¸¸¸

1

,

*v*˜3

,,,

*v*˜

1,,,

,,,,

,,

*v*

*v*3,

Lemma 2.1 *For each n* ≥ 1*, Gn is a suspension of Gn*−2*.*

Proof. The property holds by definition if *n* is odd. Thus we shall prove that the property holds for *G*2*n* with *n* ≥ 1. The following diagrams should be self- explanatory.

*v*2*n*¸¸

¸¸¸

*v*2*n*−1

¸¸¸

*G*2*n*−3 *v*˜2*n*−1

*G*2*n* = *v*2*n*

*G*2*n*−1

= =

*v v*˜

= *v v*˜ ◻

◻

*v*2*n*−,2

,,,,

,,,

*G*2*n*−3

*G*2*n*−2

The following proposition immediately follows by the definitions and from the above lemma.

Proposition 2.2 *For each n* ≥ 0*,* CL(*G*2*n*) *is a disk in dimension n and* CL(*G*2*n*+1)

*is a sphere in dimension n.*

Proof. The property holds for *n* = 0. The previous lemma allows us to induce the property from *n* to *n* + 1. ◻

We now give a graph-theoretic characterization of the graphs *Gn*. To this goal, let *P* be the following property: *if x, y are distinct nodes of G* = ⟨*V, E*⟩ *such that*

{ *x, y* } ∈/ *E, then they both form a cone over G* ∖{ *x, y* }.

Lemma 2.3 *The graphs Gn have property P. Moreover, if a graph G has property*

*P, then it contains a copy of Gn as a subgraph, where n* = card(*A*)− 1*.*

Proof. Property *P* clearly holds for *Gn* if *n* ∈ { −1*,* 0 }. Let us suppose that it holds for *Gk* for *k* < *n*, and let us prove that it holds for *Gn*. To this goal let *x, y* be vertices of *Gn*. If *x* = *vn* and *y* = *v*˜*n*, then the property is true. If this is not the case, then { *x, y* } ∈/ *E* implies that *x, y* are vertices of *Gn*−2, and by induction they are related to all the nodes in *Gn*−2 ∖{ *x, y* }. Since they are also both related to *vn* and *v*˜*n*, then they are related to all the vertices of *Gn* ∖{ *x, y* }.

Let us consider a graph *G* which has property *P* and let *n* = card(*A*)− 1. If *G* is a total graph, then clearly it contains a copy of *Gn*. Otherwise, let *x, y* be unrelated vertices, so that they both form a cone over *G*′ = *Gn* ∖{ *x, y* }. Since property *P* is closed under subgraph inclusion, then *G*′ has property *P* , so that it containsa copy of *G**n*−2. Since *x* and *y* are related to all the elements of *Gn*−2, then *G* contains a copy of *Gn*. ◻

* 1. *... are avoided on antichains*

If *P* is a poset, the height *h*(*p*) of an element *p* ∈ *P* is the length of the longest chain of the form *p*0 < *p*1 < *...* < *pn* = *p*. If *A* is a (finite) antichain, then we define its height by *h*(*A*) = ∑*a*∈*A h*(*a*).

Proposition 2.4 *Let* E *be an event structure of degree n. If A* ∈ A(E)*, then the graph* ⟨*A,* =⟩ *does not contain a subgraph of the form Gn.*

Proof. Let us say that a bad antichain of E is an antichain of cardinality *n* + 1 wich contains a copy of *Gn*. Its complexity *ξ*(*A*) is the number of unrelated pairs. We shall show that if such an *A* exists, then there exists another bad antichain *A*′ such that *ξ*(*A*′) < *ξ*(*A*). Since a bad antichain *A* with *ξ*(*A*) = 0 is an *n* + 1-clique, this will show that there are no bad antichains in E.

Let us suppose that a bad antichain *A* exists in E, with *ξ*(*A*) = *n* > 0. We can choose such a bad antichain *A* with *h*(*A*) minimal. Since *ξ*(*A*)= *n* > 0, we can find distinct *x, y* ∈ *A* that are not in the relation =. By the property of *A*, if *z* ∈ *A*∖{ *x, y* } then *x* = *z* = *y*. Since it is not the case that *x* = *y* either we can find an *x*′ < *x* such that not *x*′ = *y*, or we can find an *y*′ < *y* such that not *x* = *y*′. By symmetry, we can consider the first case only. Let *A*′ = { *x*′ }∪ (*A* ∖{ *x* }), we pretend that *A*′ is a bad antichain. *A*′ is an antichain. If *z* ∈ *A* ∖{ *x* }, then *z* ≤/ *x*′, since otherwise *z* ≤ *x*. Also, *x*′ ≤/ *z*: if *z* = *y*, then *x*′ ≤ *y* implies *x*′ = *y*, and otherwise *z* = *y* and *x*′ < *z* implies *x* = *z*. Thus *A*′ is an antichain and card(*A*′) = card(*A*). Moreover all the edges in *A* are inherited in *A*′, thus *A*′ is a bad antichain. Since *h*(*A*′) < *h*(*A*), by minimality *ξ*(*A*′)< *ξ*(*A*). [5](#_bookmark10) ◻

# Topological Properties in Degree 3

From now on we shall consider event structures of degree 3. According to Propo- sition [2.4](#_bookmark8), if *A* ∈ A(E), then ⟨*A,* =⟩ does not contain a subgraph *G*3, which is a one dimensional sphere. The goal of this section is to prove that one dimensional spheres, i.e. cycles, do not occur as subgraphs of ⟨*A,* =⟩ unless they are boundaries. In graph theoretic language, this amount to the following Proposition.

Proposition 3.1 *If* E *is an event structure of degree* 3 *and A* ∈ A(E)*, then A*

*contains no cycle of the form p*0 = *p*1 = *... pn*−1 = *pn* = *p*0 *with n* ≥ 4*.*

Proof. We shall show that if *A* ∈ A(E) contains a cycle of length *n* > 4, then we can find an antichain *A*′ ∈ A(E) containing a cycle of length *n*′ with *n* > *n*′ ≥ 4. Since by Proposition [2.4](#_bookmark8) an *A* ∈ A(E) cannot contain a cycle of length 4, we shall have found a contradiction.

Thus let *n* > 4 and among all the cycles *p*0 = *p*1 = *p*2 *... pn* = *pn*+1 = *p*0 lying on antichain, chose a cycle *C* of minimal height. If *p*0 = *p*2, then *p*0*p*2 *... pn* is a cycle of shorter length lying on an antichain, and we have reached our goal. Otherwise

*p*0 = *p*2 does not hold, and either we can find *p*′

0

< *p*0 such that *p*′

=/ *p*0, or we

can find *p*′ < *p*2 such that *p*0 =/ *p*′ . By symmetry, we can assume the first case

0

2

2

holds. As in the proof of Proposition [2.4](#_bookmark8) { *p*′ *, p*1*, p*2*, p*3 } form an antichain, and

0

*p*′ *p*1*p*2*p*3 is a path. By minimality of *C*, *p*′ *p*1*,... pn*−1*p*′

is not an antichain and thus

0 0 0

the set { *j* ∈ { 4*,... ,n* − 1 }∣ *pj* ≥ *p*′ } is not empty. Let *i* be the minimum in this

0

0

0

set, and observe that *pi*−1 = *pi* and *p*′

0

≤ *pi* but *p*′

≤/ *pi*−1 implies *pi*−1 = *p*′ . Thus

*p*′ *p*1*p*2*p*3 *... pi*−1*p*′

is an antichain and a cycle of lenght at least 4 and strictly less

0 0

than *n*. ◻

5 More precisley we have *x*′ = *y* and *h*(*A*′)= *h*(*A*)− 1.

The Proposition can be used to prove that every graph ⟨*A,* =⟩, *A* ∈ A(E), can be colored with at most three colors. We shall skip on this point and proceed instead to a somewhat straightforward computation of the homology groups of antichains of event structures of degree 3. More precisely, if *A* ∈ A(E), then we let *Hn*(*A*) be the *n*-th homology group of the clique complex of the graph ⟨*A,* =⟩.

Corollary 3.2 *Let* deg(E) = 3 *and A* ∈ A(E)*. Then*

*H*0(*A*)= *an arbitrary ﬁnitely freely generated abelian group*

*Hn*(*A*)= 0*, for n* ≥ 1*.*

Proof. We observe firstly that it is not difficult to construct an event structure E and an *A* ∈ A(E) with an arbitrary number of connected components. Therefore we shall be interested to the groups *Hn*(*A*) with *n* > 0.

Since E contains no clique of cardinality greater than 3, the groups *Cn*(*A*) [6](#_bookmark11) are trivial (hence *Hn*(*A*) = 0) for *n* > 2. On the other hand, let *γ* = ∑*i αiγi* be a chain in dimension 2, where *αi* ∈ Z and the *γi* are 2-dimensional oriented simplices, that is, each *γi* is a clique of cardinality 3 together with an orientation. If *δ*2(*γ*)= 0 and *γ* ≠ 0, then we can find distinct *i, j* such that *γi* and *γj* share a common 2-face, but this implies an occurrence of *G*3 as a subgraph of *A*. Therefore ker *δ*2 = 0 and *H*2(*A*) = 0.

Finally, let *γ* = ∑*n*

*i*=1

*αiγi* be a chain in dimension 1 such that *αi* ≠ 0 and *δ*1(*γ*)= 0.

If *n* > 3, then there is a cycle of length greater than 3. Thus *n* = 3, and *γ* is the

boundary of some 2-dimensional simplex. ◻

# The Homology of Event Structures

In the previous section we have isolated a class of graphs – the cycles – that are some kind of topological transformation of the graph *G*3. We have proved then that a graph in this class does not occur as a subgraph of an antichain of an event structure of degree 3. Given the results of Section [2](#_bookmark7) it is tempting to conjecture that analogous properties hold in higher dimensions and degrees. W.r.t. a given class *Rn*, the conjecture could take the following form: *if G* ∈ *Rn, then G does not occur as a subgraph of an antichain A* ∈ A(E) *with E of degree* 2*n* + 1*.* The class *Rn* might consists of those graphs whose clique complex geometric realization is homeomorphic (or homotopic) to a *n*-sphere. It could also be the class of graphs that are contracticble transformations [[8](#_bookmark18)] of the graph *G*2*n*+1. A computational ap- proach suggests instead to investigate the homology groups *Hn*(*A*) and ask whether *Hk*(*A*) = 0 if *A* ∈ A(E), deg(E) = 2*n* + 1, and *k* ≥ *n*. We shall develop some con- sideration in this direction. The reader may have noticed that part of the proof of Proposition [3.1](#_bookmark9) amounts to pushing down a cycle from an antichain *B* to an antichain *A* whenever *B* >> *A*. In the rest of the paper we argue that this is possible in higher dimensions as well, since it is a consequence of the functorial properties

6 *C*n(*A*) is the group freely generated by simplices, up to a choice of an orientation, cf. [[9](#_bookmark19)], of the clique complex of ⟨*A,* =⟩.

of the correspondence which takes an antichain (an element of a partially ordered set) to its graph, and then to its simplicial complex, and finally to the sequence of its homology groups.

Recall from Section [2](#_bookmark7) the definition of the poset ⟨A(E)*,* >>⟩ and observe that this poset is isomorphic to the lattice of upper sets of *P*E . For *B, A* such that *B* >> *A*, define the relation *RB,A* ∶ *B* —→ *A* as the order restricted to *B* and *A*; that is, for *b* ∈ *B* and *a* ∈ *A*, *bRB,Aa* if and only if *b* ≥ *a*. Observe that *RA,A* = *IdA*, but that only the inclusion *RC,B* ○ *RB,A* ⊆ *RC,A* holds. The lax-functor that we have defined lands in a 2-category richer than the one of sets and relations. Every antichain carries the structure of a graph with its associated clique complex and, as we shall see, the cliques are sent to cliques.

Lemma 4.1 *Let A, B* ∈ A(E) *with B* >> *A. Let γ be a clique in* ⟨*B,* =⟩ *and deﬁne*

Φ*B,A*(*γ*)= { *a* ∈ *A* ∣ ∃*b* ∈ *γ, bRB,Aa* }*. Then* Φ*B,A*(*γ*) *is a clique in* ⟨*A,* =⟩*.*

The proof is straightforward. The above observation allows to define a the functorial action *HB,A* ∶ *H*∗(*B*) —→ *H*∗(*A*) for *B* >> *A*. The relevant observation is that the mapping Φ*B,A* is an acyclic carrier, cf. [[9](#_bookmark19), §13]. To realize the maps *H*∗, for each pair *B, A* with *B* >> *A*, choose a function *f B,A* ∶ *B* —→ *A* such that *f B,A*(*b*)∈ Φ*B,A*({ *b* }) (such a function exists because of the definition of the relation

∗

>>).

Lemma 4.2 *The map f B,A is simplicial from the clique complex of* ⟨*B,* =⟩ *to the clique complex of* ⟨*A,* =⟩*. Moreover the induced map f B,A* ∶ *Cn*(*B*) —→ *Cn*(*A*) *is*

♯

*carried by* Φ*B,A.*

As a consequence of the acyclic carrier theorem, the choice of different maps

*f B,A* does not affect the induced map at the level of homology groups. This implies that *H*∗ is a functor. Indeed *f B,A*(*f C,B*(*c*)) ∈ Φ*C,A*({ *c* }) so that *f B,A* ○ *f C,B* and

♯ ♯

*f C,A* are both carried by Φ*C,A* and there exists a homotopy between them.

♯

Proposition 4.3 *There is a well deﬁned homology functor H*∗ *from the poset* ⟨A(E)*,* >

>⟩ *to the category of inﬁnite sequences of abelian groups.*

The above Proposition is certainly a simple consequence of existing theory, but certainly it is an unavoidable step toward further understanding of the topological properties of event structures.

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