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Transition Systems over Continuous Time-Space

# Izumi Takeuti[1](#_bookmark0)

*Faculty of Sciences, Tˆohoˆ University, Miyama, Hunabasi, 274-8510, JAPAN*

Abstract

This paper gives a transition system over continuous time-space, which is a system of functions of continuous time-space into descrete states. This system is situated between the cellular automata and the partial differential equations. This paper shows the reasonable sufficient condition of the uniquness of the solution.

*Keywords:* dynamics, time development, cellular automaton, partial defferential equation, general topology

# Introduction

* 1. *Motivation*

The time development is described as a function *φ* : *X* × *T* → *A* in general, where the set *X* represents the space, or the degree of freedom, *T* is the set of the time, and *A* is the set of attributes. The laws of time development is described with some constraint of such functions *φ* : *X* × *T* → *A*.

According to the purpose of each application, the set *X* is a discrete set of a few elements which represents a small degree of freedom of the system, a set of lots of elements which represents a grid of the space, or a continuous space R. Similarly, the set *T* is sometimes Z or sometimes R. The set of attributes *A* is

1 Email: [takeuti@kuis.kyoto-u.ac.jp](mailto:takeuti@kuis.kyoto-u.ac.jp)

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sometimes a discrete set of states *Q* = {*q*1*, q*2*, .., qn*}, or sometimes continuous

R which represents the displacement of a wave.

As for discrete time *T* = Z, a symbolic dynamics is an example of {·} ×

Z → *Q*. A cellular automaton [[1](#_bookmark14)] is for Z × Z → *Q*, a map dynamics is for

{·} × Z → R, and a map dynamics of higher dimension is for *X* × Z → R for some *X* ⊂ Z. One of the examples of R × Z → R is a map dynamics over a functional space [[5](#_bookmark18),[6](#_bookmark19)].

As for continuous time *T* = R, an ordinary differential equation describes a system of {·}×R → R, an ordinary differential equation with higher dimension describes a system of *X* × R → R for some *X* ⊂ Z, and a partial differential equation describes a system of R × R → R. It is known that a non-linear partial differential equation simulates a kind of cellular automata [[2](#_bookmark15),[3](#_bookmark16),[4](#_bookmark17)].

This work studies transition systems over continuous time-space, which are systems of R*n* × R → *Q*. This system is situated between the cellular automata and the partial differential equations.

* 1. *Determinism and Zeno’s paradox*

We are interested in a deterministic system of R*n* × R → *Q*. That is be- cause, in usual, non-deterministic systems include deterministic systems as the special case and deterministic systems are considered to be simpler than non-deterministic systems. A definition of non-deterministic systems is not intersting unless the definition can describe both determinism and proper non- determinism.

The simplest example of dynamics with higher degree of freedom is a pro- gressive wave. As for a cellular automaton, we define the transition rules for *φ* : Z × {0*,* 1*,* 2*, ...*}→ {0*,* 1} as:

– *φ*(*x, t*)= 1 if either *φ*(*x*−1*, t*−1) = 1, *φ*(*x, t*−1) = 1 or *φ*(*x*+1*, t*−1) = 1,

– *φ*(*x, t*) = 0, otherwise and the initial condition is:

*φ*(0*,* 0) = 1, and *φ*(*x,* 0) = 0 for *x* /= 0.

Then the solution is:

*φ*(*x, t*)= 1 if −*t* ≤ *x* ≤ *t*, and *φ*(*x, t*) = 0 otherwise.

This *φ* denotes the waves progressing both to left and to right.

As for partial differential equation, put the equation of the wave function

*φ* : R × R → R as:

*∂*2*φ*

*∂t*2

*∂*2*φ*

= − .

*∂x*2

Then the solution is *φ*(*x, t*)= *φ*+(*x* − *t*)+ *φ*−(*x* + *t*), which is a summation of

two wave; one progresses to left and the other progresses to right.

The na¨ıve translation of that cellular automaton above into the system of R × R → {0*,* 1} would be as the following. The na¨ıve transition rules for *φ* : [0*,* ∞) × R → {0*,* 1} is written as:

* *φ*(*x, t*) = 1 if, for any *ϵ >* 0, there exists (*x*'*, t*') such that *t* − *ϵ < t*' *< t*,

|*x*' − *x*| *< t* − *t*' and *φ*(*x*'*, t*')= 1,

* *φ*(*x, t*) = 0, otherwise and the initial condition is:

*φ*(0*,* 0) = 1, and *φ*(*x,* 0) = 0 for *x* /= 0.

Then one of its solutions is:

*φ*1(*x, t*)= 1 if −*t* ≤ *x* ≤ *t*, and *φ*1(*x, t*) = 0 otherwise.

This is the solution which we expect. However, unfortunately, any functions of the following form for 0 *< v <* 1 also satisfy the rules and the initial condition above:

*φv*(*x, t*)=1 if −*vt* ≤ *x* ≤ *vt*, and *φv*(*x, t*) = 0 otherwise.

That is because the rules above cannot write the lower bound of the speed of the wave.

This difficulty looks similar to Zeno’s paradox of the flying arrow. Why *φv*(1*,* 1) /= 1? Because *φv*(*ξ, ξ*) /= 1 for 1*/*2 ≤ *ξ <* 1. Why *φv*(1*/*2*,* 1*/*2) /= 1? Because *φv*(*ξ, ξ*) /= 1 for 1*/*4 ≤ *ξ <* 1*/*2. Why *φv*(1*/*2*n,* 1*/*2*n*) /= 1? Because *φv*(*ξ, ξ*) /= 1 for 1*/*2*n*+1 ≤ *ξ <* 1*/*2*n*. And so forth. In order to escape this difficulty, we write the lower bound *v* of the speed of the wave in the rule. Actually, we write the rule as:

* If (*x, t*) satisfies the condition *p*, then there exists *ϵ >* 0 such that, for any points (*x*'*, t*') such that *t* ≤ *t*' *< t* + *ϵ* and |*x*' − *x*| ≤ *v*(*t*' − *t*), it holds that *φ*(*x*'*, t*')= *q*.

instead of the na¨ıve form:

* If (*x, t*) satisfies the condition *p*, then *φ*(*x, t*)= *q*.

Therefore, we write a transition system for progressing waves as follows.

Example 1.1 The transition rules and the initial condition of *φ* : R × [0*,* ∞)

→ {0*,* 1} are the followings. Transition rules:

* If ∀*δ >* 0*.* ∃(*x*'*, t*')*. t* − *δ < t*' *< t,* |*x*' − *x*| *< t* − *t*'*, φ*(*x*'*, t*')= 1,

then ∃*ϵ >* 0*.* ∀(*x*'*, t*')*. t* ≤ *t*' *< t* + *ϵ,* |*x*' − *x*| *< t*' − *t* ⇒ *φ*(*x*'*, t*')= 1.

* Otherwise, ∃*ϵ >* 0*.* ∀(*x*'*, t*')*. t* ≤ *t*' *< t* + *ϵ,* |*x*' − *x*| *< t*' − *t* ⇒ *φ*(*x*'*, t*')= 0. Initial condition:

*φ*(0*,* 0) = 1, and *φ*(*x,* 0) = 0 for *x* /= 0.

Actually, the function *φ*1 above satisfies these conditions, and the function

*φv* above for *v <* 1 does not.

There is another difficulty on define a deterministic system. We have to keep the development form starting the wave without any cause. Unfortu- nately, the following *φ*' also satisfies Example [1.1](#_bookmark1):

*φ*'(*x, t*)= 1 if 0 *< t* or *x* = 0, and *φ*'(*x,* 0) = 0 for *x* /= 0.

We regard the function as a wave, that is, the wave exists where *φ*'(*x, t*)= 1. At the beginning time *t* = 0, the wave exists only at *x* = 0. But the wave exists everywhere after the beginning time, that is, *t >* 0. This difficulty seems the inverse of Zeno’s paradox. Why does the wave exists without the cause? That is because the wave at *t* = 1 has the cause at time *t* = 1*/*2, the wave at *t* = 1*/*2 has the cause at time *t* = 1*/*4, the wave at *t* = 1*/*2*n* has the cause at time *t* = 1*/*2*n*+1, and so forth.

In order to prevent this difficulty, we define the notions of fundamental functions and speed condition. The intuitive meaning of fundamental function is the follows. In the development described by a fundamental function, one can always trace back the cause of each phenomenon to the initial condition. The formal definition of it is given in Definition [2.10](#_bookmark3) and [4.2](#_bookmark6). The speed condition is a condition for the transition rules. The intuitive meaning of speed condition is that a wave progressing at speed *v* does not make the effect faster than its speed *v*. The formal definition of it is given in Definition [4.5](#_bookmark7).

Through these devices, we prove the uniqness of the solution under a cer- tain kind of restriction. That is Theorem [4.16](#_bookmark10), which is the main theorem of this paper.

As for Example [1.1](#_bookmark1), the set of transition rules actually satisfies speed condition. The function *φ*' is not a fundamental function, and the function *φ*1 is the unique fundamental function which is the solution of the transition rules.

# Topological Operators over Time-Space

Definition 2.1 (Time-Space) Let *X* be a complete metric space. Then the set *U* = *X* × R is called a *time-space*, where the left part *X* is regarded as the space and the right part R is regarded as the time.

Example 2.2 Let *X* be a closed subset of R*n*. Then *X* × R is a time-space.

Definition 2.3 (Accumulation operator) Let *U* be a time-space. We de- fine operators *A*+ and *A*− over *E* ⊂ *U* for a real number *v* ≥ 0.

*v* *v*

* *A*+(*E*)

*v*

= {(*x, t*) | ∀*ϵ >* 0*.*∃(*x*'*, t*')*. t*−*ϵ < t*' *< t, d*(*x, x*') ≤ *v*(*t*−*t*')*,* (*x*'*, t*') ∈ *E*}

* *A*−(*E*)= {(*x, t*) | (*x,* −*t*) ∈ *A*+({(*x*'*, t*') | (*x*'*,* −*t*') ∈ *E*})}

*v* *v*

= {(*x, t*) | ∀*ϵ >* 0*.*∃(*x*'*, t*')*.t < t*' *< t*+*ϵ, d*(*x, x*') ≤ *v*(*t*−*t*')*,* (*x*'*, t*') ∈ *E*}

The operator *A*+ is called the *forward accumulation operator at speed limit v*, and *A*− is called the *backward accumulation operator at speed limit v*.

*v*

*v*

Remark 2.4 The operators *A*+ and *A*− are appropriate for the phenomena

*v* *v*

progressing at a constant speed. When we deal with the phenomena with the

speed getting slower, the following operators are appropriate.

*A*+ (*E*)= *A*+(*E*), *A*− (*E*)= *A*−(*E*).

*w>v*

*w<v*

*>v*

*w*

*<v*

*w*

Proposition 2.5 *1. U* = *A*+(*U* ) *2.* ∅ = *A*+(∅)

*v* *v*

*3. E* ⊂ *E*' ⇒ *A*+(*E*) ⊂ *A*+(*E*') *4. A*+(*A*+(*E*)) ⊂ *A*+(*E*)

*v v v v* *v*

*5. A*+(*E* ∪ *E*')= *A*+(*E*) ∪ *A*+(*E*') *6. v* ≤ *w* ⇒ *A*+(*E*) ⊂ *A*+(*E*)

*v v v v w*

*The similar properties hold for A*−*.*

*v*

Definition 2.6 (Closure, interior, boundary) The *closure operators F* +

*v*

and *F* −, the *interior operators G*+ and *G*−, and the *boundary operators ∂*+

*v v v* *v*

and *∂*− are defined as corresponding to *A*+ and *A*−. For *E* ⊂ *U* and a real

*v v* *v*

number *v* ≥ 0, they are defined as:

*F* −(*E*)= *E* ∪ *A*+(*E*), *F* −(*E*)= *E* ∪ *A*−(*E*),

*v v v* *v*

*G*+(*E*)= *U* − *F* +(*U* − *E*), *G*−(*E*)= *U* − *F* −(*U* − *E*),

*v V v* *v*

*∂*+(*E*)= *F* +(*E*) − *G*+(*E*), *∂*−(*E*)= *F* −(*E*) − *G*−(*E*).

*v v v v v* *v*

Remark 2.7 The properties similar to Prop. [2.5](#_bookmark2) hold for these operators

*F* +, *F* −, *G*+, *G*−, *∂*+ and *∂*−.

*v v v v v* *v*

Definition 2.8 (Closed set, open set) The families of *closed sets* and *open sets* are defined for each real number *v* ≥ 0 as follows:

F+ = {*E* ⊂ *U* | *E* = *F* +(*E*)}, F− = {*E* ⊂ *U* | *E* = *F* −(*E*)},

*v v v* *v*

G+ = {*E* ⊂ *U* | *E* = *G*+(*E*)}, G− = {*E* ⊂ *U* | *E* = *G*−(*E*)}.

*v v v* *v*

Remark 2.9 If *v* ≤ *w*, then the following hold: F+

⊂ F+, F−

⊂ F−,

G+ ⊂ G+, and G− ⊂ G−.

*w v w* *v*

*w v w* *v*

Definition 2.10 (Fundamental set) For each real number *v* ≥ 0, we define a family of sets D*v* as:

D*v* = {*E* ⊂ *U* | *E* ∈ F−*, F* −(*∂*−(*E*)) ⊂ *E*}.

0 *v* 0

A set *E* ∈ D*v* is called a *fundamental set at speed limit v*.

Remark 2.11 If *v* ≤ *w* then D*w* ⊂ D*v*.

Remark 2.12 This D*v* is closed under finite union.

# Modal Logic

Definition 3.1 (State) A finite set *Q* = {*q*1*, q*2*, ..., qn*} is called the *set of states*, and an element *qi* ∈ *Q* is called a *state*. States play the roˆle of atomic formulae.

Definition 3.2 (Formula) *Formulae* are defined as the following syntax.

*P* ::= *q* | ¬*P* | *P* ∧ *P* | *vP* (*q* ∈ *Q, v* ≥ 0)

We write *P* for the set of all the formulae. We write *Pv* for the set of all the formulae *p* which do not have any occurrences of modal operators *w* such that *v < w*. Note that *Q* ⊂ *Pv* ⊂ *P* .

Notation 3.3 *p* ∨ *p*' = ¬(¬*p* ∧ ¬*p*'), *vp* = ¬ *v*¬*p*.

Notation 3.4 The powers of connection of ¬, and are strongest. The next is that of ∧, and that of ∨ is the weakest.

Definition 3.5 (Totally Modalised formula) A formula *p* ∈ *P* is *totally modalised* if any occurrences of any *q* ∈ *Q* in *p* are in scopes of ’s. We

write *PM* for the set of all the totally modalised formulae, and write *P M* as

*v*

*PM* = *PM* ∩ *Pv*.

*v*

Definition 3.6 (Interpretation of formulae) For a partial function *φ* of *U*

into *Q* and a formula *p* ∈ *P* , the *interpretation* [[*p*]]*φ* ⊂ *U* is defined as follows. [[*q*]]*φ* = *φ*−1(*q*), **[**¬*p*]]*φ* = *U* − [[*p*]]*φ*, **[***p* ∧ *p*']]*φ* = [[*p*]]*φ* ∩ [[*p*']]*φ*,

[[ *vp*]]*φ* = *A*+([[*p*]]*φ*).

*v*

Notation 3.7 For a function *φ* : *U* → *Q*, we write *φ*|*E* for the function made by restricting the domain into *E*.

Proposition 3.8 *For any v* ≥ 0*, any subset E* ∈ G+*, any functions φ, φ*' : *U* → *Q such that φ*|*E* = *φ*'|*E and any formula p* ∈ *Pv, it holds that* [[*p*]]*φ* ∩ *E* = [[*p*]]*φ*' ∩ *E.*

*v*

Proposition 3.9 *For any v* ≥ 0*, any subset E* ∈ G+*, any functions φ, φ*' :

*v*

*U* → *Q such that φ*|*E* = *φ*'|*E and any formula p* ∈ *P M , it holds that* [[*p*]]*φ* ∩

*v*

*F* +(*E*)= **[***p*]]*φ*' ∩ *F* +(*E*)*.*

*v* *v*

Definition 3.10 (Positive occurrence, negative occurrence) The no- tions of *positive occurrence* and *negative occurrence* of a state *q* ∈ *Q* in a formula *p* ∈ *P* are defined in the usual way.

Remark 3.11 For each formula *p*, there is a formula *p*' which is logically equivalent to *p* and has only positive occurrences of states *q*’s. The logical equivalence of two formula *p* and *p*' is defined as: **[***p*]]*φ* = [[*p*']]*φ* for any total

functions *φ* : *U* → *Q*. Actually, such *p*' is made from *p* by replacing each

negative occurrence of each state *q* with the formula ¬

*q*'/=*q*

*q*'.

Definition 3.12 (Propagating speed in a formula) The *propagating speed* of a state *q* in a formula *p* ∈ *P M* is the greatest number *v* such that the formula *p* has a positive occurrence of *q*, or a negative occurrence of some *q*' other than *q*, in the scope of *v*. If there is no such *v*, then the propagating speed is defined as −∞.

We write *Pr* (*q, p*) for the propagating speed of *q* in *p*.

Example 3.13 It holds that *Pr* (*q,* *vq*)= *v* and *Pr* (*q,* *v*¬*q*)= −∞.

The formula ¬ *vq* has a negative occurrence of *q* in the scope of *v*. Indeed *q* occurs positively in *v*, but it occurs negatively in ¬ *vq*. Thus *Pr* (*q,* ¬ *vq*)= −∞ and *Pr* (*q*'*,* ¬ *vq*)= *v* for a state *q*' /= *q*.

Remark 3.14 The definition of propagating speed of *q* mentions the negative occurrences of the state other than *q*. That is because the negative occurrences of the state other than *q* can be turned into the positive occurrence of *q* by the replacement as in Remark [3.11](#_bookmark5).

Lemma 3.15 *Put φ, φ*' : *U* → *Q and p* ∈ *P M . Suppose that* (*x, t*) ∈ [[*p*]]*φ and*

(*x, t*) /∈ [[*p*]]*φ*' *. Then, for any ϵ >* 0*, there is a point* (*x*'*, t*') *such that:*

0 *< t* − *t*' *< ϵ,*

*d*(*x, x*') *t* − *t*'

≤ *Pr* (*φ*(*x*'*, t*')*, p*)*, and*

*d*(*x, x*') *t* − *t*'

≤ *Pr* (*φ*'(*x*'*, t*')*,* ¬*p*)*.*

# Transition System

Notation 4.1 *U*≥*t*0 = {(*x, t*) ∈ *U*|*t* ≥ *t*0}, *U>t*0 = {(*x, t*) ∈ *U*|*t > t*0}, *U*(*t*1 *,t*2 ) = {(*x, t*) ∈ *U*|*t*1 *< t < t*2}

Definition 4.2 (Fundamental function) For *v* ≥ 0, a function *φ* : *U* → *Q* or *φ* : *U*≥*t*0 → *Q* is a *fundamental function at speed limit v* iff *φ*−1(*q*) ∈ D*v* for each *q* ∈ *Q*.

We write *Φv* for the set of all the fundamental functions *φ* : *U* → *Q* at speed limit *v*. We write *Φv,*≥*t*0 for the set of all the fundamental functions *φ* : *U*≥*t*0 → *Q* at speed limit *v*.

Remark 4.3 The notion of fundamental functions is a generalisation of the notion of half-open intervals. Actually, for a fundamental function *φ* ∈ *Φv* and a state *q* ∈ *Q*, the set {*t* ∈ *T* | *φ*(*x, t*)= *q*} is a countable union of half-open intervals [*t, t*').

Remark 4.4 If *φ* ∈ *Φv*, then *φ*|*U*≥t

0

∈ *Φv,*≥*t*0 .

The inverse also holds. For each function *φ*' ∈ *Φv,*≥*t* , there is a total

0

function *φ* ∈ *Φv* such that *φ*' = *φ*|*U*

≥t0

. This *φ* is constructed as:

*φ*(*x, t*)= *φ*'(*x, t*) if *t* ≥ *t*0, and *φ*(*x, t*)= *φ*'(*x, t*0) if *t < t*0.

Definition 4.5 (Transition rule) A triple ⟨*q, v, p*⟩ ∈ *Q* × [0*,* ∞) × *P M* is called a *transition rule*. The rule ⟨*q, v, p*⟩ means that, if *p* holds, then the state is changed into *q* at expanding speed *v*.

Definition 4.6 (Satisfaction of rules at a point) A partial function *φ* of *U* into *Q satisﬁes* a transition rule ⟨*q, v, p*⟩ at a point (*x, t*) ∈ *U* iff they satisfy the relation *φ* |=(*x,t*) ⟨*q, v, p*⟩ defined as:

*φ* |=(*x,t*) ⟨*q, v, p*⟩ ⇐⇒ (*x, t*) ∈ *G*−(*φ*−1(*q*)) ∪ [[¬*p*]]*φ*.

*v*

Definition 4.7 (Satisfaction of rules in an area) A partial function *φ* of *U* into *Q satisﬁes* a transition rule ⟨*q, v, p*⟩ at an area *E* ⊂ *U* iff they satisfy the relation *φ* |=*E* ⟨*q, v, p*⟩ defined as:

*φ* |=*E* ⟨*q, v, p*⟩ ⇐⇒ ∀(*x, t*) ∈ *E. φ* |=(*x,t*) ⟨*q, v, p*⟩

⇐⇒ *E* ∩ [[*p*]]*φ*(*x, t*) ⊂ *G*−(*φ*−1(*q*)).

*v*

Definition 4.8 (Transition system) A *transition system* is a finite set *S* of transition rules. A *transition system at speed limit v* is a finite subset *S* ⊂ *Q* × [0*, v*] × *PM* .

*v*

Definition 4.9 (Completeness) A transition system *S* = {⟨*q*1*, v*1*, p*1⟩*,*

⟨*q*2*, v*2*, p*2⟩*, ...,* ⟨*qk, vk, pk*⟩} is *complete at speed limit v* iff, for each function *φ* ∈

*Φv*, it holds that *U* =

1≤*i*≤*k*

[[*pi*]]*φ*, which is equivalent to: *φ* |=*U p*1 ∨ *p*2 ∨ *...* ∨ *pk*.

Conjecture 4.10 *For a transition system S* = {⟨*q*1*, v*1*, p*1⟩*,* ⟨*q*2*, v*2*, p*2⟩*, ...,*

⟨*qk, vk, pk*⟩}*, if S is complete at some speed limit v, then for each function*

*φ* : *U* → *Q, it holds that U* =

1≤*i*≤*k*

[[*pi*]]*φ.*

Definition 4.11 (Propagating speed in a system) The *propagating speed*

of a state *q* in a system *S* is the maximum of *Pr* (*q, p*) for *p*’s such that

⟨*q*'*, v, p*⟩∈ *S*. Note that the state *q*' in ⟨*q*'*, v, p*⟩ may be different to *q*.

We write *Pr* (*q, S*) for the propagating speed of *q* in *S*.

Definition 4.12 (Speed condition) A system *S* satisfies *speed condition* iff there is *q*0 ∈ *Q* such that, for each ⟨*q, v, p*⟩∈ *S*, it holds either

*Pr* (*q, S*) ≤ *v*,

or

*q* = *q*0 and *Pr* (*q*0*, p*) = 0, and moreover, for any other ⟨*q*'*, v*'*, p*'⟩ ∈ *S*,

if *q*' /= *q*0 and *Pr* (*q*'*, p*) ≥ 0, then *Pr* (*q, p*') *<* 0.

This *q*0 is called the *exception state*.

Remark 4.13 If it holds *Pr* (*q, S*) ≤ *v* for each ⟨*q, v, p*⟩ ∈ *S*, then the system

*S* satisfies speed condition.

Definition 4.14 (Solution at the initial condition with positive thick- ness) For a transition system *S* and a function *φ*0 : *U*(−*є,*0) → *Q*, a function *φ* : *U*≥−*є* → *Q* is a *solution* of *S* at the initial condition *φ*0 iff the followings hold:

*φ*|*U*(−*є,*0) = *φ*0, and *φ* |=*U*≥0 *R* for each *R* ∈ *S*.

Definition 4.15 (Solution at Hagiya’s initial condition) For a transition system *S* and a function *φ*0 : *X* → *Q*, a function *φ* : *U*≥0 → *Q* is a *solution* of *S* at the initial condition *φ*0 iff the followings hold:

*φ*(*x,* 0) = *φ*0(*x*), and *φ* |=*U*>0 *R* for each *R* ∈ *S*.

Theorem 4.16 (Main theorem) *Let S be a transition system at speed limit*

*v. Suppose that S is complete at speed limit v and S satisﬁes speed condition. Let φ*0 *be a function X* → *Q. Then, the solution of S at the initial condition φ*0 *in Φv,*≥0 *is unique if it exists. On the other words, if both of functions φ, φ*' ∈ *Φv,*≥0 *are the solution of S at the initial condition φ*0*, then φ* = *φ*'*.*

We will show the proof of this theorem latter. First, we put the corollary of this theorem.

Corollary 4.17 *Let S be a transition system at speed limit v. Suppose that s is complete at speed limit v and S satisﬁes speed condition. Let φ*0 *be a function in U*(−*є,*0) → *Q. Then, the solution of S at the initial condition φ*0 *in Φv,*≥−*є is unique if it exists.*

Outline of the proof. The function *φ*|*X*×{0}, which is the solution at *t* = 0, is uniquely determined from the initial condition *φ*0, because of Proposition

[3.9](#_bookmark4). The function *φ*|*U*≥0 is the solution of *S* at Hagiya’s initial condition *φ*|*X*×{0}. Therefore, the whole function *φ* is uniquely determined because of the main theorem (Thm. [4.16](#_bookmark10)).

Hereafter we define several notions in order to prove the main theorem (Thm. [4.16](#_bookmark10)).

Definition 4.18 (Difference set) Let *S* be a transition system at speed limit *v* which is complete at speed limit *v*, and satisfies speed condition. For

two functions *φ, φ*' ∈ *Φv,*≥0 such that both *φ* |=*U*

>0

>0

*S* and *φ*' |=*U*

*S*. Then,

the *difference set ∆* is the set {(*x, t*) ∈ *U*≥0 | *φ*(*x, t*) /= *φ*'(*x, t*)}.

Definition 4.19 (Cause-effect relation) For two point (*x, t*)*,* (*x*'*, t*') ∈ *∆*, the point (*x, t*) is a *cause* of (*x*'*, t*') iff the followings hold:

1. *t < t*', 2.

*d*(*x, x*') *t*' − *t*

≤ *Pr* (*φ*(*x, t*)*, S*), 3.

*d*(*x, x*') *t*' − *t*

≤ *Pr* (*φ*'(*x, t*)*, S*),

4. Either (*x, t*) ∈ *∂*−(*φ*−1(*φ*(*x, t*))) or (*x, t*) ∈ *∂*−(*φ*'−1(*φ*'(*x, t*))).

0 0

The relation of these (*x, t*) and (*x*'*, t*') are called the *cause-effect relation*.

Lemma 4.20 *For each* (*x, t*) ∈ *∆, there exists* (*x*'*, t*') ∈ *∆ which is a cause of* (*x, t*)*.*

Remark 4.21 The cause-effect relation is not transitive.

Definition 4.22 (Cause-effect chain) Let (*D, <*) be an ordered set. A function *f* : *D* → *∆* is a *cause-effect chain* iff the followings hold:

* *f* is injective.
* For any *e, e*' ∈ *D*, if *e < e*', then there is a finite sequence *e* = *e*0 *< e*1 *< e*2 *< ... < en* = *e*' in *D* such that *f* (*ei*+1) is a cause of *f* (*ei*) for each *i* ≤ *n* − 1.

Notation 4.23 We write ℵ1 for the least uncountable ordinal, as is usual. We write *Ord* (*α*) for the set of all the ordinals *β < α*, which is sometimes identified to *α* itself in the set theory.

Lemma 4.24 *If* {(*x*1*, t*1)*,* (*x*2*, t*2)*,* (*x*3*, t*3)*, ...*} *which is regarded as a function of Ord* (*ω*) → *∆ is a cause-effect chain, then* lim*n*→∞(*xn, tn*) ∈ *∆. Moreover, there is* (*xω, tω*) ∈ *∆ such that* {(*x*1*, t*1)*,* (*x*2*, t*2)*,* (*x*3*, t*3)*, ...,* (*xω, tω*)} *which is regarded as a function of Ord* (*ω* + 1) → *U is a cause-effect chain.*

Proof of the main theorem (Thm. [4.16](#_bookmark10)). Suppose that there are two distinct solutions *φ, φ*' ∈ *Φv,*≥0 of *S* at the initial condition *φ*0, and we will derive the contradiction from this assumption. Note that *∆* ∩ (*X* × {0})= ∅, because both *φ* and *φ*' follow the same initial condition *φ*0.

We will construct a cause-effect chain *f* : *Ord* (ℵ1) → *U>*0.

For zero: Because *φ* and *φ*' are distinct, the difference set *∆* is not empty. Thus there is a point (*x, t*) ∈ *∆*. We put *f* (0) = (*x, t*) for this (*x, t*).

For successors: We will define *f* (*α* + 1) for a successor *α* + 1. As the induction hypothesis, we have already defined *f* (*α*). By Lemma [4.20](#_bookmark11), there is (*x, t*) ∈ *∆* ⊂ *U>*0 which is a cause of *f* (*α*). We put *f* (*α* + 1) = (*x, t*) for this (*x, t*).

We have to check that *f* |{*β*|*β*≤*α*+1} is a cause-effect chain. As the induction hypothesis, we already have that *f* |{*β*|*β*≤*α*} is a cause-effect chain. Hence it suffices to show that there is a finite cause-effect chain from *f* (*β*) to *f* (*α* + 1) for each *β* ≤ *α*. That is shown because we already have a finite cause-effect

chain from *f* (*β*) to *f* (*α*).

For limits: We will define *f* (*α*) for a limit ordinal *α*. There is a increas- ing sequence *α*1*, α*2*, α*3*, ...* which converges into *α*. As the induction hy- pothesis, we have already defined *f* (*β*) for *β < α* and *f* |{*β*|*β<α*} is a cause- effect chain. Thus, we can choose the sequence {*f* (*α*1)*,f* (*α*2)*,f* (*α*3)*, ...*} as a cause-effect chain. By Lemma [4.24](#_bookmark12), there is (*x, t*) ∈ *∆* such that the chain

{*f* (*α*1)*,f* (*α*2)*,f* (*α*3)*, ...,* (*x, t*)} is a cause-effect chain. we put *f* (*α*)= (*x, t*) for this (*x, t*).

We have to check that *f* |{*β*|*β*≤*α*} is a cause-effect chain. It suffices to show that there is a finite cause-effect chain from *f* (*β*) to *f* (*α*) for each *β* ≤ *α*. That is shown because we already have a finite cause-effect chain from *f* (*β*) to *f* (*αn*), for some *αn > β*.

Thus we have constructed a cause-effect chain *f* : *Ord* (ℵ1) → *U>*0. For *α <* ℵ1, put (*xα, tα*) as (*xα, tα*)= *f* (*α*). Then 0 *< tα < tβ* for any *β < α <* ℵ1. Therefore, the chain {*tα*}*α<*ℵ1 is an uncountable bounded monotone chain. However, there is no uncountable bounded monotone chain in real numbers. That is contradiction.

Remark 4.25 The main theorem (Thm. [4.16](#_bookmark10)) gives a reasonable sufficient condition which ensures the uniqueness of the solution. It is similar to the theorem of Cauchy and Kovalevskaja for differential equations. Unfortunately, it seems very much difficult to give a reasonable sufficient condition which ensures the existence of the solution, although the condition in the theorem of Cauchy and Kovalevskaja also ensures the existence of the solution.

# Examples

Example 5.1 Example [1.1](#_bookmark1) is formalised as below. The space is a line *X* = R, and the set of states is *Q* = {0*,* 1}. The transition system *S* and Hagiya’s initial condition *φ*0 are defined as follows:

*S* = {⟨1*,* 1*,* 11⟩*,* ⟨0*,* 1*,* ¬ 11⟩},

*φ*0(0) = 1, and *φ*0(*x*)=0 for *x* /= 0.

This *S* is a transition system at speed limit 1, is complete at speed limit 1, and satisfies speed condition. The function *φ*1 in Section 1 is the unique solution of *S* at the initial condition *φ*0 in *Φ*1*,*≥0.

Example 5.2 (Hagiya’s Example) The space is a plane *X* = R2, and the set of states is *Q* = {white*,* red*,* blue*,* black}, which consists of four colours. The transition system *S* and Hagiya’s initial condition *φ*0 are defined as follows.

*R*1 = ⟨red*,* 1*, p*1⟩*, p*1 = 1red ∧ 1(white ∨ red),

*R*2 = ⟨blue*,* 1*, p*2⟩*, p*2 = 1blue ∧ 1(white ∨ blue),

*R*3 = ⟨black*,* 0*, p*3⟩*, p*3 = 1red ∧ 1blue,

*R*4 = ⟨black*,* 0*, p*4⟩*, p*4 = 0black,

*R*5 = ⟨white*,* 1*, p*5⟩*, p*5 = 1white,

*R*6 = ⟨white*,* 1*,* ¬(*p*1 ∨ *p*2 ∨ *p*3 ∨ *p*4 ∨ *p*5)⟩, *S* = {*R*1*, R*2*, R*3*, R*4*, R*5*, R*6}

 red*,* (*x, y*)= (1*,* 0)



*φ*0 : *X* → *Q, φ*0(*x, y*)= blue*,* (*x, y*)= (−1*,* 0)

 white*,* otherwise

This *S* is a transition system at speed limit 1, is complete at speed limit 1, and satisfies speed condition. The function *φ* ∈ *Φ*1*,*≥0 defined as follows is the solution of *S* at the initial condition *φ*0.

 red*, x >* 0*, t* ≥ √(*x* − 1)2 + *y*2



 blue*, x <* 0*, t* ≥ √(*x* + 1)2 + *y*2

*φ* : *U*≥0 → *Q, φ*(*x, y, t*)=



black*, x* = 0*, t* ≥

√*y*2 +1

 white*,* otherwise

The rule *R*6 is a dummy rule; this rule is not applied at any points actually, that is, **[**¬(*p*1 ∨ *p*2 ∨ *p*3 ∨ *p*4 ∨ *p*5)]]*φ* = ∅ for this *φ*. The system *S* has the rule *R*6 because it makes the system complete.

Example 5.3 The space is a line *X* = R, and the set of states is *Q* =

Z*/*5Z = {¯0*,* ¯1*,* ¯2*,* ¯3*,* ¯4} which is the cyclic group of period 5. It holds in *Q* that

−1 = ¯4 = ¯9= *...*, −2 = ¯3 = ¯8= *...* and so forth. The transition system *S* is

defined as below.

For ¯*i* ∈ *Q*,

*R*1*,*¯*i* = ⟨¯*i,* 1*, P*1*,*¯*i*⟩*, P*1*,*¯*i* = 1¯*i* ∧ 1(¯*i* ∨ *i* − 1), *R*2*,*¯*i* = ⟨¯*i,* 1*, P*2*,*¯*i*⟩*, P*2*,*¯*i* = 1¯*i* ∧ 1(¯*i* ∨ *i* − 2), *R*3*,*¯*i* = ⟨¯*i,* 1*, P*3*,*¯*i*⟩*, P*3*,*¯*i* = 1*i* − 1 ∧ 1*i* − 2, *R*4*,*¯*i* = ⟨¯*i,* 1*, P*4*,*¯*i*⟩*, P*4*,*¯*i* = 1¯*i*,

*R*5 = ⟨¯0*,* 1*,* ¬(*P*1*,*¯0 ∨ *...* ∨ *P*4*,*¯4)⟩,

*S* = {*R*1*,*¯0*, ..., R*4*,*¯4*, R*5}

This *S* is a transition system at speed limit 1, is complete at speed limit 1, and satisfies speed condition.

The initial condition is defined as below. The functions *φ* ∈ *Φ*1*,*≥−*є* and

*φ* : *U*(−*є,*0) → *Q* are defined as:

*m, n* ∈ Z*, n* ≤ *t* + *x < n* + 1*, m* ≤ *t* − *x < m* +1 ⇒ *φ*(*x, t*)= *n* + 2*m*.

The area {(*x, t*) | *n* ≤ *t* + *x < n* + 1*, m* ≤ *t* − *x < m* + 1} is an oblique square whose lowest point is (*m*+*n/*2*, n*−*m/*2). The set of such squares fills the whole plain. Therefore this *φ* is well-defined. The initial condition *φ*0 is defined as: *φ*0 = *φ*|*U*(−c,0) .

Then, the function *φ* is the solution of *S* at the initial condition *φ*0.

Remark 5.4 This *φ* in this example makes periodic intervals on the one- dimensional line. In general, on *n*-dimensional space R*n*, we can construct a transition system and an initial condition such that the solution makes periodic cells.

The set of states in this example represents only the phases of cells. If we put the set of states as the direct product of the set of colours and the set of phases, such as *Q* = {white*,* black}× Z*/*5Z, then the system can re- alise a cellular automaton. This can be generalised into the spaces of higher dimensions.

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