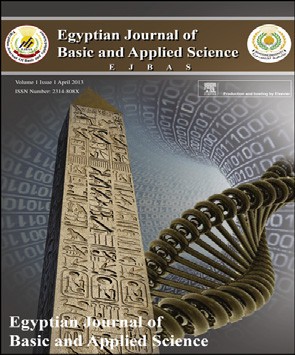
[e g ypti an j o ur nal o f b a sic and a pp l i e d sci en c e s 1 ( 2014) 7 1](http://dx.doi.org/10.1016/j.ejbas.2014.03.001) e[7](http://dx.doi.org/10.1016/j.ejbas.2014.03.001) 6

Available online at [www.sciencedirect.com](http://www.sciencedirect.com/science/journal/2314808X)

ScienceDirect

journal homepage: <http://ees.elsevier.com/ejbas/default.asp>

Two-dimensional time fractional-order biological population model and its analytical solution

*Vineet K. Srivastava*[*a*](#_bookmark0)*,*[*\**](#_bookmark4)*, Sunil Kumar* [*b*](#_bookmark1)*, Mukesh K. Awasthi* [*c*](#_bookmark2)*,*



*Brajesh Kumar Singh* [*d*](#_bookmark3)

a *ISRO Telemetry, Tracking and Command Network (ISTRAC), Bangalore 560058, Karnataka, India*

b *Department of Mathematics, National Institute of Technology, Jamshedpur 831014, Jharkhand, India*

c *Department of Mathematics, University of Petroleum and Energy Studies, Dehradun 247008, Uttarakhand, India*

d *Department of Mathematics, School of Allied Sciences, Graphic Era Hill University, Dehradun 248002, Uttarakhand, India*

## a r t i c l e i n f o

*Article history:*

Received 22 November 2013 Received in revised form

6 March 2014

Accepted 8 March 2014

Available online 27 March 2014

*Keywords:*

GTFBPM FRDTM

Mittag-leffler function Caputo fractional derivative Exact solution

## a b s t r a c t

In this article, a mathematical model has been developed for the generalized time fractional eorder biological population model (GTFBPM). The fractional derivative has been described in the Caputo sense. The model has been solved by a recent approximate analytic method so called the fractional reduced differential transform method (FRDTM). Using this method, it is possible to find the exact solution as well as closed approximate solution of a differential equation. Three numerical examples of GTFBPM have been provided in order to check the effectiveness, accuracy and convergence of the method. The special advantage of using this computational technique is that it is very easy to implement and takes small size of computation contrary to other numerical methods while dealing complex and tedious physical problems arising in various branches of natural sciences and engineering.

Copyright ª 2013, Mansoura University. Production and hosting by Elsevier B.V. All rights

reserved.

# Introduction

Various physical phenomena in natural sciences and engineer- ing can be explained successfully by developing models using the fractional calculus theory. Fractional differential equations have achieved much more attention because fractional order system response ultimately converges to the integer order

\* *Corresponding author*. Tel./fax: +91 8050682145.

equations. The use of fractional differentiation for the mathe- matical modeling of real world physical problems such as the earthquake modeling, the traffic flow model with fractional de- rivatives, measurement of viscoelastic material properties, etc. has been widespread in the recent years. Before the nineteenth century, no analytical solution method was available for such type of equations even for the linear fractional differential

E-mail addresses: [vineetsriiitm@gmail.com](mailto:vineetsriiitm@gmail.com), [vsrivastava107@gmail.com](mailto:vsrivastava107@gmail.com) (V.K. Srivastava).

Peer review under responsibility of Mansoura University

**Production and hosting by Elsevier**

<http://dx.doi.org/10.1016/j.ejbas.2014.03.001>

2314-808X/Copyright ª 2013, Mansoura University. Production and hosting by Elsevier B.V. All rights reserved.

72 [e gypti an j o ur nal o f b a sic and a pp l i ed sci e n c e s 1 ( 2014) 7 1](http://dx.doi.org/10.1016/j.ejbas.2014.03.001) e[7](http://dx.doi.org/10.1016/j.ejbas.2014.03.001) 6

equations. Recently, Keskin and Oturanc [[1]](#_bookmark28) developed the fractional reduced differential transform method (FRDTM) for the fractional differential equations and showed that FRDTM is the most easily implemented analytical method and gives the exact solution for both the linear and nonlinear differential equations. FRDTM is very reliable, efficient and effective powerful computational technique for solving physical prob- lems; see references [[2](#_bookmark29)e[5]](#_bookmark29).

# Generalized time fractional-order biological population model (GTFBPM)

Biological scientists believe that the dispersal or emigration plays a crucial role in the regulation of population of the species. The diffusion of a biological species in a region C is described by the three functions of position →*x* = (*x*; *y*) in region C and time *t*

[[6]](#_bookmark32) namely population density *p*(→*x* ; *t*), diffusion velocity *v*(→*x* ; *t*),

and the population supply, *f* (→*x* ; *t*). The population density *p*(→*x* ; *t*) gives the number of individuals, per unit volume, at po- sition →*x* and time *t*; it’s integral over any sub region D of region C gives the total population of D at time *t*. The entity *f* (→*x* ; *t*) gives the rate at which individuals are supplied, per unit volume, at position →*x* by births and deaths. The diffusion velocity *v*(→*x* ; *t*) represents the average velocity of those individuals who occupy

rectangular mesh, in which at each step an animal may either stay at its present location or may move in the direction of the lowest population density. The probability distribution among these two possibilities being determined by themagnitude of the population density gradient at the mesh side concerned. This model leads to Eq. [(3)](#_bookmark8) with *ɸ* (*p*)=*p*2, to the following equation

va*p* v2*p*2 v2*p*2

v*t*a = v*x*2 + v*y*2 + *f* (*p*); *t* ≥ 0; x; y˛R; (4)

with the given initial condition *p*(*x*,*y*,0). For a = 1, Eq. [(4)](#_bookmark5) re- duces to the normal biological population model (NBPM)

v*p* v2*p*2 v2*p*2

v*t* = v*x*2 + v*y*2 + *f* (*p*); *t* ≥ 0; x; y˛R. (5)

Some properties of Eq. [(4)](#_bookmark5) such as Holder estimates and its solutions have been studied in [[8]](#_bookmark34).

Three examples of constitutive equations for *f*(*p*) can be given as

1. *f*(*p*) = *cp*, c = constant, Malthusian Law [[6]](#_bookmark32).
2. *f*(*p*) = *c*1*p* — *c*2*p* , *c*1,*c*2 = positive constants, Verhulst Law [[8]](#_bookmark34).

2

1. *f*(*p*) = *cp* , *(c* > *0, 0* < q < *1)*, Porous media [[9,10]](#_bookmark35).

q

Let us consider a more general form of *f*(*p*) as

*f*(*p*) = *hpa*(1 — l*pb*), so that Eq. [(4)](#_bookmark5) becomes

the position →*x* at time *t*, and it describes the flow of population

*a*

from point to point. The entities *p*, →*v* and *f* must be consistent

v*t*a = v*x*2 + v*y*2 + *hp*

1 — l*pb*

; *t* ≥ 0; x; y˛R; (6)

va*p*

v2*p*2

v2*p*2

with the following law of population balance, for every regular sub region *D* of *C* and for all time *t*

where *h*, *a*, l, *b*, are real numbers. If *h* = *c*, *a* = 1, l = 0 and *h* = *c*1, *a* = *b* = 1, l = *c*2/*c*1, then Eq. [(6)](#_bookmark6) leads to Malthusian Law and Verhulst Law. The reason behind using the fractional-order

*d*a

Z

*dt*a

*D*

*pdV* +

v*D*

Z

*p*→*v* $*ndA* =

*D*

Z

b

*fdV*; (1)

differential equation is that it is naturally related to systems with memory which exists in most of the biological systems.

where *n* is the outward unit normal to the boundary v*D* of *D*. The derivative in Eq. [(1)](#_bookmark7) has been taken in the Caputo deriv- ative sense. From the Eq. [(1)](#_bookmark7), it is obvious that the rate of change of population of *D* plus the rate at which the in- dividuals leave *D* across its boundary must be equal to the rate at which the individuals are supplied directly to *D*. By making the assumptions [[8]](#_bookmark34)

b

*f* = *f* (*p*); →*v* = —l(*p*)V*p*; (2)

where l(*p*) > 0 for *p* > 0, and V is the Laplace operator, the following two-dimensional nonlinear degenerate parabolic partial differential equation for the population density *p* can be obtained as

Also they are closely related to fractals which are abundant in biological systems. The linear and nonlinear population sys- tems were solved in [[11,12]](#_bookmark36) using variational iteration method (VIM), adomian decomposition method (ADM), homotopy analysis method (HAM), and homotopy perturbation method (HPM). The major disadvantage of these methods is that they require complex and large size of computations. To overcome from the complex computational efforts, FRDTM has been employed for solving the problem.

The purpose of this work is to solve the generalized time fractionaleorder biological population model (GTFBPM) using the fractional reduced differential transformation method. The obtained results are compared well with those obtained

by VIM, ADM, HPM and HAM.

v f(*p*)

va*p* v*t*a =

v2f *p*

v*x*2 +

( )

2

v*y*2 + *f* (*p*); *t* ≥ 0; x; y˛R. (3)

Eq. [(3)](#_bookmark8) is known as the time fractional-order biological popula- tion model (TFBPM). Gurney and Nisbet [[7]](#_bookmark33) employed *ɸ* (*p*), as a special case for the modeling of the population of animals. The movements are made generally either by mature animalsdriven by mature invaders or by young animals just reaching maturity moving out of their parental territory to establish breeding ter- ritory of their own. In both cases, it is much more plausible to assume that they will be directed towards nearby vacant terri- tory. Therefore, in this model, movement takes place almost exclusively down the population density gradient and will be more rapid at high population densities than at low ones. To model this scenario, they considered a walk through a

# Fractional calculus theory

In this section, we present some notations, definitions and preliminary facts that will be used further in this study. Fractional calculus theory is more than 200 years old theory present in the literature. Several definitions of fractional in- tegrals and derivatives have been proposed but the first major contribution to give proper and most meaningful definition is due to Liouville as follows.

Definition 3.1. A real function *f*(x), x > 0 is said to be in the space *C*m, m ˛ *ℝ* if there exists a real number q (>m), such that

[e g ypti an j o ur nal o f b a sic and a pp l i e d sci en c e s 1 ( 2014) 7 1](http://dx.doi.org/10.1016/j.ejbas.2014.03.001) e[7](http://dx.doi.org/10.1016/j.ejbas.2014.03.001) 6 73

*f*(x) = xqg(x), where g(x) ˛ C[0,N), and it is said to be in the space*Cm* if *f*(m) ˛ *C*m, m ˛ ℕ.

m

Definition 3.2. For a function *f*, the Riemann-Liouville frac- tional integeral operator [[15]](#_bookmark40) of order a ≥ 0, is defined as

Definition 4.1. If *w*(*x*,*t*) is analytic and continuously differen- tiable with respect to space variable *x* and time variable *t* in the domain of interest, then the *t*-dimensional spectrum function

1 " v*k* #

8> Z*x*

*Wk*(*x*)= G(*k*a + 1)

v*tk w*(*x*, *t*)

*t*=*t*0

(11)

1

>< a ( )=

*J f x*

G(a)

>>: *J*0*f* (*x*)= *f* (*x*).

0

(*x* — *t*)a—1*f* (*t*)*dt*, a > 0, *x* > 0,

(7)

is the fractional reduced transformed function of *w*(*x*,*t*), where a is a parameter which describes the order of time-fractional derivative. In this paper, (lowercase) *w*(*x*,*t*) represents the

The Riemann-Liouville derivative has major drawbacks while modeling the real world problems with fractional dif- ferential equations. To overcome this discrepancy, Caputo and Mainardi [[16]](#_bookmark41) proposed a modified fractional differentia- tion operator *D*a in his work on the theory of viscoelasticity.

original function while (uppercase) *Wk*(*x*) stands for the frac- tional reduced transformed function. The differential inverse transform of *Wk*(*x*) is defined as

N

X

*k*a*w*(*x*, *t*)= *Wk*(*x*)(*t* — *t*0) . (12)

The Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order de-

*k*=0

combining Eqs. [(11) and (12)](#_bookmark9), it can be found that

rivatives, which have clear physical interpretations.

XN 1

" v*k* # *k*a

Definition 3.3. The fractional derivative of *f*(*x*) in the Caputo

*w*(*x*, *t*) =

*k*=0

G(*k*a + 1)

v*tk w*(*x*, *t*)

*t*=*t*0

(*t* — *t*0)

. (13)

sense [[17]](#_bookmark42) is defined as

1 Z*x*

when *t* = 0, Eq. [(13)](#_bookmark10) reduces to

" #

X

*D*a*f* (*x*) = *Jm*—a*Dmf* (*x*)=

G(*m* — a)

0

(*x* — *t*)*m*—a—1*f* (*m*)(*t*)*dt*, (8)

N

*w*(*x*, *t*)=

*k*=0

1

G(*k*a + 1)

v*k* v*tk*

*w*(*x*, *t*)

*t*=0

*tk*a. (14)

for *m* 1 < a *m*, *m*˛N, *x* > 0, *f* ˛*Cm* .

— ≤

—1

The fundamental properties of the Caputo fractional de- rivative are given in the following lemma.

Lemma. If *m* — 1 < a ≤ *m*, *m* ˛ ℕ and *f* ˛*Cm*, m ≥ —1, then

m

From the above definition, it can be found that the concept of the fractional reduced differential transform is derived from the power series expansion of a function.

Definition 4.2. If *u*(*x*, *t*)= *R*—1[*Uk*(*x*)], *v*(*x*, *t*)= *R*—1[*Vk*(*x*)] and the

*D*

*D*

8>< *D*a*J*a*f* (*x*)= *f* (*x*), *x* > 0,

> *J*a*D*a*f x f x* P*m f* (*k*) 0+

: ( )= ( )— *k*=0 ( )

*x* , *x* > 0.

*k*!

*k*

(9)

convolution 5 denotes the fractional reduced differential transform version of the multiplication, then the fundamental operations of the fractional reduced differential transform have been expressed in [Table 1](#_bookmark12).

In this study, the Caputo fractional derivative has been chosen because it allows traditional initial and boundary conditions to be included in the formulation of the physical

In [Table 1](#_bookmark12) G, represents the Gama function, which is defined as

N

Z

problems. Some other important characteristics of fractional derivatives can be seen in [[17,18]](#_bookmark42).

G(*z*) :=

0

*e*—*ttz*—1*dt*, *z*˛C. (15)

# Fractional reduced differential transform method (FRDTM)

In this section, the basic definitions and properties of the fractional reduced differential transform method have been described.

here we notice that the Gama function is the continuous extension to the factorial function [[3]](#_bookmark30).

The recursive relationship G(*z* + 1) = *z*G(*z*), *z* > 0 can be used

to compute the value of the gamma function of all real numbers (except the non-positive integers) by knowing only the value of the gamma function between 1 and 2.

Consider a function of two variables *w*(*x*,*t*), and assume

that it can be represented as a product of two single-variable functions, i.e. *w*(*x*,*t*) = *F*(*x*)*G*(*t*). On the basis of the properties of the one-dimensional differential transform, the function *w*(*x*,*t*) can be represented as

*i*

*j*

*i j*

|  |
| --- |
| Table 1 e Fundamental operations of the fractional reduced differential transform method. |
| Original function Fractional reduced differential  transformed function |
| *RD* [*u*(*x*,*t*)*v*(*x*,*t*)] *Uk*(*x*)5*Vk*(*x*)= P*k Ur*(*x*)*Vk*—*r*(*x*)  *r*=0  *RD* [a*u*(*x*,*t*) b*v*(*x*,*t*)] a*Uk*(*x*) b*Vk*(*x*)  *RD* [(v*l*/v*xl*) *u*(*x*,*t*)] [(*k* + *l*)!/*k*!] (v*l*/v*xl*) *Uk*+*l*(*x*)  *RD* [(v*N*a/v*tN*a) *u*(*x*,*t*)] [(G(*k*a + *N*a + 1))/(G(*k*a + 1))] *Uk*+*N*(*x*) *RD* [*xmtnu*(*x*,*t*)] *xmUk*—*n*(*x*)  *RD* [*e*l*t*] l*k*/*k*!  *k*  *RD* [sin (*wt* + a)] (*w* /*k*!)*sin*[(p*k*/2!) + a]  *k*  *RD* [cos (*wt* + a)] (*w* /*k*!)*cos*[(p*k*/2!) + a] |

*w*(*x*, *t*) =

N

*i*=0

X

*F*(*i*)*x*

N

*j*=0

X

*G*(*j*)*t* =

N

*i*=0

X

N

*j*=0

X

*W*(*i*, *j*)*x t* , (10)

where *W*(*i*,*j*) = *F*(*i*)*G*(*j*) is called the spectrum of *w*(*x*,*t*).

Let *R*D denotes the reduced differential transform operator and *R*—1 the inverse reduced differential transform operator

D

[[4]](#_bookmark31). The basic definitions and operations of the reduced dif- ferential transform are introduced as follows.

74 [e gypti an j o ur nal o f b a sic and a pp l i ed sci e n c e s 1 ( 2014) 7 1](http://dx.doi.org/10.1016/j.ejbas.2014.03.001) e[7](http://dx.doi.org/10.1016/j.ejbas.2014.03.001) 6

# Numerical examples

The same result was obtained by Y. Liu et al. [[13]](#_bookmark38) using HPM and Arafa et al. [[14]](#_bookmark39) using HAM. When a / 1 in Eq. [(22)](#_bookmark20), we get

In this section, we describe the method explained in the Section [4](#_bookmark11) by taking three numerical examples to validate the

N

,ﬃﬃﬃﬃﬃ X

*p*(*x*, *y*, *t*)= *xy*

*k*

(*ht*)

= ,ﬃ*x*ﬃﬃ*y*ﬃﬃﬃ *eht*, (23)

efficiency and reliability of FRDTM for the GTFBPM.

Example 5.1. Consider the following linear time fractional- order biological population model

va*p* v2*p*2 v2*p*2

v*t*a = v*x*2 + v*y*2 + *hp*, (16)

subject to the initial condition

*p*(*x*, *y*, 0)= ,ﬃ*x*ﬃﬃ*y*ﬃﬃ. (17)

*k*=0 G(1 + *k*)

as the exact solution for the standard form of the biological population equation (SBPE), same result was obtained for this standard form by Roul [[12]](#_bookmark37) using HPM and Shakeri et at [[11]](#_bookmark36). using VIM and ADM with parameter *h* = 1/5.

Example 5.2. Consider the following linear time fractional- order biological population model

va*p*

v2*p*2

v2*p*2

Applying the FRDTM to Eq. [(16)](#_bookmark13), we obtain the following recurrence relation

v*t*a = v*x*2 + v*y*2 + *p*, (24)

G(*k*a + a + 1)

v2 " X*k* #

with the initial condition

G(*k*a + 1) *Pk*+1(*x*, *y*) =v*x*2

*Pr*(*x*, *y*)*Pk*—*r*(*x*, *y*)

qﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ

*r* 0

=

v2 " X*k*

# *p*(*x*, *y*, 0)= sin *x* sinh *y*. (25)

+ v*y*2

*r*=0

*Pr*(*x*, *y*)*Pk*—*r*(*x*, *y*)

+ *hPk*(*x*, *y*).

(18)

Applying the FRDTM to Eq. [(24)](#_bookmark15), we obtain the following recurrence equation:

Using the FRDTM to the initial condition [(17)](#_bookmark14), we get

G(*k*a + a + 1)

v2 " X*k* #

*P* (*x*, *y*)= ,[ﬃ](#_bookmark18)*x*[ﬃﬃ](#_bookmark18)*y*[ﬃﬃﬃ](#_bookmark18). (19)

G(*k*a + 1) *Pk*+1(*x*, *y*) =v*x*2

*r*=0

*Pr*(*x*, *y*)*Pk*—*r*(*x*, *y*)

0

using Eq. [(19)](#_bookmark18) in Eq. [(18)](#_bookmark16), we get the following *Pk*(x,y) values successively

v2 *k*

+ v*y*2

*r*=0

*Pr*(*x*, *y*)*Pk*—*r*(*x*, *y*)#

+ *Pk*(*x*, *y*).

(26)

,ﬃﬃﬃﬃﬃ ,ﬃﬃﬃﬃﬃﬃ

" X

*P*1(*x*, *y*)= G(1 + a)

*xy*; *P*2(*x*, *y*)= G(1 + 2a)

*xy*; *P*3(*x*, *y*)= G(1 + 3a)

*h*

*h*2

*h*4

*hk*

*P*4(*x*, *y*)= G(1 + 4a)

*xy*; .; *Pk*(*x*, *y*)= G(1 + *k*a)

,ﬃﬃﬃﬃﬃﬃ

*xy*.

(20)

,ﬃﬃﬃﬃﬃﬃ ,ﬃﬃﬃﬃﬃﬃ

*xy*;

*h*3

Using the differential inverse transform of *Pk*(*x*,*y*), *k* = 1,2,3,., we get

*p x y t*

( ,

PN *P*

*x y tk*a *P x y*

*k*( ,

)

=

0( ,

)+ 1( ,

+ G(1 + 2a *t*

*P x y t*a

)

+ . + G(1 + *k*a *t*

*P x y t*2a

+

2( ,

)

*P x y t*3a .

,ﬃﬃﬃﬃﬃ

=

*xy* 1 + G(1 + a *t*

*h* a *h*2 2a

)=

,

*k*=0

)

)

a *k*

*hk* 3a

+

3( ,

)

+

)

,ﬃﬃﬃﬃﬃ PN

+ ...

(21)

=

*xy*

.

G(1 + *k*a)

(*ht* ) #

*k*=0

The exact solution can be expressed as

*p*(*x*, *y*, *t*) = ,ﬃ*x*ﬃﬃ*y*ﬃﬃﬃ *E*a(*ht*a), (22)

using the FRDTM to the initial condition [(25)](#_bookmark17), we get

qﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ

where *E*a(*ht*a) is Mittag-leffler function, defined as

*P*0(*x*, *y*) = sin *x* sinh *y*. (27)

*E* (*z*)= PN *zk* , which is a advanced form of *exp* (*z*), and

a

*k*=0G(1+*k*a)

reduces to *exp* (*z*) when a / 1.

using Eq. [(27)](#_bookmark21) in Eq. [(26)](#_bookmark19), we get the following *Pk*(x,y) values

successively

[e g ypti an j o ur nal o f b a sic and a pp l i e d sci en c e s 1 ( 2014) 7 1](http://dx.doi.org/10.1016/j.ejbas.2014.03.001) e[7](http://dx.doi.org/10.1016/j.ejbas.2014.03.001) 6 75

*P* (*x*, *y*  1 qsﬃﬃﬃiﬃnﬃﬃﬃﬃﬃ*x*ﬃﬃﬃﬃsﬃﬃiﬃnﬃﬃﬃﬃhﬃﬃﬃﬃ*y*ﬃﬃﬃ; *P* (*x*, *y*  1 qsﬃﬃﬃiﬃnﬃﬃﬃﬃﬃ*x*ﬃﬃﬃﬃsﬃﬃiﬃnﬃﬃﬃﬃhﬃﬃﬃﬃ*y*ﬃﬃﬃ;

)= )=

1

G(1 + a)

2

G(1 + 2a)

*P* (*x*, *y*  1 qsﬃﬃﬃiﬃnﬃﬃﬃﬃﬃ*x*ﬃﬃﬃﬃsﬃﬃiﬃﬃnﬃﬃﬃhﬃﬃﬃﬃ*y*ﬃﬃ; *P* (*x*, *y*  1 qsﬃﬃﬃiﬃnﬃﬃﬃﬃﬃ*x*ﬃﬃﬃﬃsﬃﬃiﬃﬃnﬃﬃﬃhﬃﬃﬃﬃ*y*ﬃﬃﬃ;

)= )=

3

G(1 + 3a)

4

G(1 + 4a)

(28)

*P* (*x*, *y*  1 qsﬃﬃﬃiﬃnﬃﬃﬃﬃﬃ*x*ﬃﬃﬃﬃsﬃﬃiﬃnﬃﬃﬃﬃhﬃﬃﬃﬃ*y*ﬃﬃﬃ.

)=

*k*

G(1 + *k*a)

using the differential inverse reduced transform of *Pk*(*x*,*y*), *k* = 1,2,3,., we have

*p x y t*

( ,

PN *P*

*x y tk*a

*P x y*

*P x y t*a

*P x y t*2a

*P x y t*3a .

= psﬃﬃﬃiﬃnﬃﬃﬃﬃﬃ*x*ﬃﬃﬃﬃsﬃﬃiﬃﬃnﬃﬃﬃhﬃﬃﬃﬃ*y*ﬃﬃﬃ 1 + 1 a + 1 2a + . + 1 *k*a + .

)=

*t t t*

,

*k*=0

*k*( ,

)

=

0( ,

)+ 1( ,

)

+

2( ,

)

+

3( ,

)

+

G(1 + a)

G(1 + 2a)

G(1 + *k*a)

(29)

pﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ PN

=

sin *x* sinh *y*

.

G(1 + *k*a)

*tk*a #

*k*=0

thus, the exact solution can be given as

*p x*, *y*, 0

exp"1 rﬃlﬃﬃ *x*

*y* #. (33)

*p*(*x*, *y*, *t*)= qsﬃﬃﬃiﬃnﬃﬃﬃﬃﬃ*x*ﬃﬃﬃﬃsﬃﬃiﬃﬃnﬃﬃﬃhﬃﬃﬃﬃ*y*ﬃﬃﬃ *E*a(*t*a). (30)

( )= 2 2( + )

the same solution was obtained by Arafa et al. [[14]](#_bookmark39) using HAM. When a / 1 in Eq. [(30)](#_bookmark23), we get

Applying the FRDTM to Eq. [(32)](#_bookmark25), we obtain the following iteration formula

qﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ

*p*(*x*, *y*, *t*)=

sin *x* sinh *y*

*et*. (31)

G(*k*a + a + 1)

v2 " X*k* #

The same exact solution had been obtained by Roul [[12]](#_bookmark37) using HPM for the standard biological population model (SBPM).

*r*=0

v2 *k*

" X

G(*k*a + 1) *Pk*+1(*x*, *y*) =v*x*2

*Pr*(*x*, *y*)*Pk*—*r*(*x*, *y*)

+ v*y*2

*r*=0

*Pr*(*x*, *y*)*Pk*—*r*(*x*, *y*)# X*k*

*r*=0

(34)

Example 5.3. Consider the following generalized nonlinear time fractional-order biological population model

— *Pk*(*x*, *y*)— l

*Pr*(*x*, *y*)*Pk*—*r*(*x*, *y*).

using the FRDTM to the initial condition [(33)](#_bookmark22), we get

*P* (*x*, *y*)= exp"1 rﬃlﬃﬃ(*x* + *y*)#. (35)

va*p* v2*p*2 v2*p*2

0 2 2

v*t*a = v*x*2 + v*y*2 + *p*(1 — l*p*), (32)

subject to the initial condition

using Eq. [(35)](#_bookmark26) in Eq. [(34)](#_bookmark24), the following *Pk*(x,y) values are ob- tained successively

*P* (*x*, *y*)= 1 exp"1 rﬃlﬃﬃ(*x* + *y*)#; *P* (*x*, *y*)= 1 exp"1 rﬃlﬃﬃ(*x* + *y*)#;

1

2

G(1 + a) 2 2 G(1 + 2a) 2 2

*P* (*x*, *y*  1 exp"1 rﬃlﬃﬃ(*x* + *y*)#; *P* (*x*, *y*  1 exp"1 rﬃlﬃﬃ(*x* + *y*)#;

)= )=

3

4

(36)

G(1 + 3a) 2 2 G(1 + 4a) 2 2

*P* (*x*, *y*)= 1 exp"1 rﬃlﬃﬃ(*x* + *y*)#.

*k*

G(1 + *k*a) 2 2

76 [e gypti an j o ur nal o f b a sic and a pp l i ed sci e n c e s 1 ( 2014) 7 1](http://dx.doi.org/10.1016/j.ejbas.2014.03.001) e[7](http://dx.doi.org/10.1016/j.ejbas.2014.03.001) 6

Using the differential inverse reduced transform of *Pk*(*x*,*y*), *k* = 1,2,3,., we get

*p x y t*

( ,

PN *P*

*x y tk*a

*k*( ,

)

*P x y*

=

0( ,

)+ 1( ,

*P x y t*a

)

*P x y t*2a

+

2( ,

)

*P x y t*3a .

= exp"1 rﬃlﬃﬃ(*x* + *y*)# 1 + 1

)=

,

*k*=0

*t*a + 1

*t*2a + . + 1

*tk*a + .

(37)

2 2 G(1 + a)

+

3( ,

)

+

G(1 + 2a)

G(1 + *k*a)

"1 rﬃlﬃﬃ

a *k*

= exp 2

2(*x* + *y*)

.

G(1 + *k*a)

#" XN

(*t* ) #

*k*=0

Hence the closed form solution is given by

*p*(*x*, *y*, *t*)= exp"1 rﬃlﬃﬃ(*x* + *y*)#*E* (*t*a). (38) The same result was obtained by Arafa et al. [[14]](#_bookmark39) using

2 2 a

HAM. As a / 1 in Eq. [(38)](#_bookmark27), we have

*p*(*x*, *y*, *t*)= exp "1 rﬃlﬃﬃ(*x* + *y*)# + *t*!. (39)

2 2

which is the same exact solution obtained by Roul [[12]](#_bookmark37) using HPM, and also by Shakeri et at [[11]](#_bookmark36). using VIM and ADM.

# Conclusions

In this work, the fractional reduced differential transform method has been implemented for a degenerate Caputo time- fractional order parabolic partial differential equation arising in the spatial diffusion biological populations. The solution obtained by the FRDTM is an infinite power series for appro- priate initial condition which finds the solution without any discretization, transformation, perturbation, or restrictive conditions. We have also illustrated three numerical exam- ples considering the situations of both linear as well nonlinear phenomenon of GTFBPM to study the effectiveness and accurateness of the method. The solutions obtained by the FRDTM are in excellent agreement with those obtained by the HPM, HAM, VIM and ADM. However, computations show that the FRDTM is very easy to implement and needs small size of computation contrary to HPM, HAM, VIM and ADM.

## references

1. [Keskin Y, Oturanc G. Reduced differential transform method:](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref1) [a new approach to factional partial differential equations.](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref1) [Nonlinear Sci Lett A 2010;1:61](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref1)e[72](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref1).
2. [Gupta PK. Approximate analytical solutions of fractional](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref2) [Benney-Lin equation by reduced differential transform](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref2) [method and the homotopy perturbation method. Comp Math](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref2) [Appl 2011;58:2829](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref2)e[42](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref2).
3. [Srivastava VK, Awasthi MK, Tamsir M. RDTM solution of](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref3) [Caputo time fractional-order hyperbolic telegraph equation.](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref3) [AIP Adv 2013;3:032142](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref3).
4. [Srivastava VK, Awasthi MK, Chaurasia RK, Tamsir M. The](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref4) [telegraph equation and its solution by reduced](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref4) [differential transform method. Model Simul Eng](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref4) [2013;2013. Article ID 746351](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref4).
5. Srivastava VK, Awasthi MK, Kumar S. Analytical approximations of two and three dimensional time- fractional telegraphic equation by reduced differential transform method. Egypt J Basic Appl Sci; 2014. [http://](http://dx.doi.org/10.1016/j.ejbas.2014.01.002) [dx.doi.org/10.1016/j.ejbas.2014.01.002](http://dx.doi.org/10.1016/j.ejbas.2014.01.002).
6. [Gurtin ME, Maccamy RC. On the diffusion of biological](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref6) [population. Math Biol Sci 1977;54(33):35](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref6)e[49](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref6).
7. [Gurney WSC, Nisbet RM. The regulation of inhomogenous](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref7) [populations. J Theor Biol 1975;52:441](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref7)e[57](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref7).
8. [Lu YG. Ho¨ lder estimate of solutions of biological population](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref8) [equations. Appl Math Lett 2000;13:123](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref8)e[6](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref8).
9. [Bear J. Dynamics of fluids in porous media. New York:](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref9) [American Elsevier; 1972](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref9).
10. [Okubo A. Diffusion and ecological problem, mathematical](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref10) [models. Biomathematics 10. Berlin: Springer; 1980](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref10).
11. [Shakeri F, Dehghan M. Numerical solution of a biological](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref11) [population model using He’s variational iteration method.](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref11) [Comput Math Appl 2006;54:1197](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref11)e[209](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref11).
12. [Roul P. Application of homotopy perturbation method to](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref12) [biological population model. Appl Appl Math](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref12) [2010;10:1369](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref12)e[78](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref12).
13. [Liu Y, Li Z, Zhang Y. Homotopy perturbation method to](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref13) [fractional biological population equation. Fract Diff Calc](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref13) [2011;1:117](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref13)e[24](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref13).
14. [Arafa AA, Rida SZ, Mohamed H. Homotopy analysis method](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref14) [for solving biological population model. Commun Theor](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref14) [Phys 2011;56:797](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref14)e[800](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref14).
15. [Millar KS, Ross B. An introduction to the fractional calculus](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref15) [and fractional differential equations. New York: Wiley; 1993](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref15).
16. [Caputo M, Mainardi F. Linear models of dissipation in](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref16) [anelastic solids. Rivista del Nuovo Cimento 1971;1:161](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref16)e[98](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref16).
17. [Podlubny I. Fractional differential equations. San Diego:](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref17) [Academic Press; 1999](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref17).
18. [Hilfer R. Applications of fractional calculus in physics.](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref18) [Singapore: World Scientific; 2000](http://refhub.elsevier.com/S2314-808X(14)00010-4/sref18).