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Weak Domain Models of *T*1 spaces

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**Abstract**

The main objective of this paper is to study some aspects of weak domains. We first show that every meet continuous weak domain is a domain. Then we prove that a dcpo *P* is exact iff the weakly way-below relation is the smallest approximating w-auxiliary relation. It is then shown that for each *T*1 space, Zhao and Xi’s dcpo model is a weak algebraic domain, and hence a weak domain. As a consequence, we have that a weak algebraic domain need not be well-filtered and that every *T*1 space has a weak domain model, which strengthens a result of Mashburn.

*Keywords:* weakly way-below relation, weak domain, meet continuous dcpo, weak algebraic domain, local domain, weakly auxiliary relation

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A poset model of a topological space *X* is a poset *P* such that Max(*P* ), the set of maximal elements of *P* , with the relative Scott topology, is homeomorphic to *X* [[5](#_bookmark29)]. Every space having a poset model must be *T*1. In [[6](#_bookmark30)], Martin proved that if a space is homeomorphic to the maximal point space of a continuous dcpo, then the space is Choquet complete. Thus not every *T*1 space has a domain model.

In 2002, Coecke and Martin [[1](#_bookmark25)] constructed an ordered set to model finite di- mensional quantum states, and it turns out that their model is not a domain, i.e., not continuous with respect to the way-below relation. However, this model is con- tinuous with respect to the weakly way-below relation. Using the weakly way-below relation, Mashburn [[7](#_bookmark31)] defined the weak domains and proved that every first count- able space has a weak domain model. In [[8](#_bookmark32)], Mashburn proved that every linearly ordered topological space is homeomorphic to an open dense subset of a weak do- main representable space, showing that a space with a weak domain model need not be Baire. Then Mashburn [[7](#_bookmark31)] raised the following problem:

* Which topological spaces are weak domain representable?

In this paper, it was shown that meet continuity [[4](#_bookmark27)] plays a crucial role between weak domains and domains. Precisely, we prove that every meet continuous weak domain is a domain. In [[11](#_bookmark35)], Zhao and Xi proved that every *T*1 space *X* has a dcpo model (we will call it Xi-Zhao model of *X*). We show that the Xi-Zhao model of every *T*1 space is a weak algebraic domain (hence a weak domain). As a corollary, every *T*1 space has a weak domain model, strengthening Mashburn’s result in [[7]](#_bookmark31) (every first countable *T*1 space has a weak domain model) and answering one of Mashburn’s problems. Using the main result in [[9](#_bookmark33)], we also deduce that, unlike domains, the Scott space of a weak domain need not be well-filtered.

# Preliminary

In this section, we review some basic notions and results used in this paper. For more details, please refer to [[2,](#_bookmark26)[7,](#_bookmark31)[8](#_bookmark32)].

For a poset *P* and *A ⊆ P* , let *↓A* = *{x ∈ P* : *x* ≤ *a* for some *a ∈ A}* and

*↑A* = *{x ∈ P* : *x* ≥ *a* for some *a ∈ A}*. For *x ∈ P* , we write *↓x* for *↓{x}* and *↑x* for *↑{x}*. A subset *A* is called a *lower set* (resp., an *upper set* ) if *A* = *↓A* (resp., *A* = *↑A*). A nonempty subset *D* of *P* is *directed* if every two elements in *D* have an upper bound in *D*. *P* is called a *directed complete poset*, or *dcpo* for short, if for any directed subset of *D ⊆ P* , *D* exists in *P* .

W

A subset *U* of *P* is *Scott open* if (i) *U* = *↑U* and (ii) for any directed subset *D* for which *D* exists, *D ∈ U* implies *D ∩ U /*= ∅. All Scott open subsets of *P* form a topology, and we call this topology the *Scott topology* on *P* and denote it by *σ*(*P* ). The space (*P, σ*(*P* )) is called the *Scott space* of *P* , and we denote it by Σ*P* . For two elements *x* and *y* in *P* , *x* is *way-below y*, denoted by *x y*, if for any directed subset *D* of *P* for which *D* exists, *y* ≤ *D* implies *D ∩ ↑x /*= ∅. We let

W W

W W

*x* = *{y ∈ P* : *x y}* and *x* = *{y ∈ P* : *y x}*. *P* is *continuous*, if for any *x ∈ P* ,

→

←

the set *x* is directed and *x* = W *x*. A continuous dcpo is also called a *domain*.

←

←

An element *x* in a poset *P* is called *compact* if *x x*. The set of all compact

elements of *P* is denoted by K(*P* ). A poset *P* is *algebraic*, if for any *x ∈ P* , the set

*K*(*P* )*∩ ↓x* is directed and *x* = W *K*(*P* ) *∩ x*.

←

*way-below y*W, denoted by *x w y*, if for any directed subset *D* of *P* for which *D* **Definition 2.1** (Mashburn [[7](#_bookmark31)]) Let *P* be a poset and *x, y ∈ P* . Then *x* is *weakly*

W

exists, *y* =

*D* implies *D ∩ ↑x /*= ∅.

If *P* is continuous, then the relations *w* andon *P* coincide.

**Proposition 2.2** *(Mashburn [*[*7*](#_bookmark31)*,*[*8*](#_bookmark32)*]) In a poset P, the following statements hold for all x, y, z ∈ P:*

1. *x y ⇒ x w y;*
2. *x w y ⇒ x* ≤ *y;*
3. *x* ≤ *y w z ⇒ x w z;*
4. *⊥ w x whenever P has a smallest element T.*

One property owned byis that *x y* ≤ *z* implies *x z*. This property may not be true for *w*. It is true for *w* if and only if *w* andcoincide, as Coecke and Martin pointed out in [[1](#_bookmark25)].

←

For *x ∈ P* , let

→

*wx* = *{y ∈ P* : *x w y}* and

*wx* = *{y ∈ P* : *y w x}*.

**Definition 2.3** (Mashburn [[7,](#_bookmark31)[8](#_bookmark32)]) A poset *P* is called *exact* if for any *x ∈ P* , *wx* is

←

directed and W *wx* = *x*. *P* is a *weak domain* if *P* is an exact dcpo and the relation

←

*w* is *weakly increasing* : for any *x, y, z, u ∈ P* , *x w y* ≤ *z w u* implies *x w z*.

**Proposition 2.4** *A dcpo P is exact iff for each x ∈ P, there exists a directed subset*

*D of wx such that* W *D* = *x.*

←

**Proof.** We only prove the Sufficiency. Assume *x ∈ P* and there is a directed subset

←

←

←

that is, *y*1*, y*2 *w x*. Since *x* =

*D*, there exist *d*1*, d*2 *∈ D* such that *y*1 ≤ *d*1 and

*D* of *wx* such that W *D* = *x*. TWo verify the directedness of *wx*, let *y*1*, y*2 *∈ wx*,

*y*2 ≤ *d*2. Since *D* is directed, there is *d ∈ D* such that *d*1*, d*2 ≤ *d*, so that *y*1*, y*2 ≤ *d*.

Note that *d ∈ wx* because *D ⊆*

←

←

←

←

*wx*. This implies *wx* is directed. In addition,

since *x* = W *D* ≤ W

←

*wx* ≤ *x*, we have *x* = W *wx*. Hence, *P* is an exact dcpo. *2*

Note that every domain is a weak domain because

←

*x* is a directed subset of *wx*

with W *x* = *x*.

←

←

Like the way-below relation on domains, the weakly way-below relation on do-

mains has the important interpolation property.

**Theorem 2.5** *(Mashburn [*[*7*](#_bookmark31)*, Theorem 3.6]) If P is a weak domain, then w is interpolative in P, that is, for any x, y ∈ P, x w y implies the existence of z ∈ P such that x w z w y.*

# More relationships between domains and weak do- mains

In this section, we provide some more relationships among certain types of domains. We call a dcpo in which any two elements have an infimum a *directed complete semilattice*. A directed complete semilattice *P* is called *meet continuous* if for any

W W

*x ∈ P* and any directed subset *D* of *P* , *x* ≤ *D* implies *x* = *{x ∧ d* : *d ∈ D}*.

**Lemma 3.1** *If P is a meet continuous directed complete semilattice, then the rela- tions w and on P coincide.*

**Proof.** Suppose *a w*W*b* and *D* is a directed subset of *P* with *b* ≤ W *D*. Since *P* is

meet continuous, *b* =

*{b ∧ d* : *d ∈ D}*. From *a w b*, it follows that *a* ≤ *b ∧ d*0

for some *d*0 *∈ D*, and hence *a b*. Trivially, we have that *a b* implies *a w b*.

Hence *w* andcoincide. *2*

As an immediate result of Lemma [3.1](#_bookmark3), we have the following result.

**Proposition 3.2** *A directed complete semilattice P is a domain if and only if it is a meet continuous weak domain.*

In 2001, Kou, Liu and Luo extended the notion of meet continuity to the general dcpos [[4](#_bookmark27)]. A dcpo *P* is *meet continuous* if for any *x ∈ P* and any directed set *D* with *x* ≤ *D*, *x ∈* cl*σ*(*↓D ∩ ↓x*), where cl*σ* is the closure operator for the Scott topology. A well-known result is that every domain is meet continuous.

W

It is natural to ask whether Proposition [3.2](#_bookmark4) holds for any dcpo. In the following, we will answer this question in the affirmative.

**Lemma 3.3** *If P is a meet continuous weak domain, then the relations w and*

*on P coincide.*

W

**Proof.** Let *y w x*. Assume that *D* is a directed subset of *P* such that *x* ≤ *D*. By Theorem [2.5](#_bookmark2), there exists *z ∈ P* such that *y w z w x*. Since *P* is meet continuous, it follows that *z ∈* cl*σ*(*↓z ∩ ↓D*), or equivalently *↓z* = cl*σ*(*↓z ∩ ↓D*).

Now for every ordinal *α*, we define a subset *Aα* of *P* inductively:

*A*0 := *↓z ∩ ↓D,*

*Aα* := *↓ {*W *E* : *E* is a directed subset of *Aβ} ,* if *α* = *β* + 1 for some ordinal *β, Aα* := *β<α Aα,* if *α* is a limit ordinal*.*

Then cl*σ*(*A*0) = *{Aα* : *α* is an ordinal*}*. Since the cardinality of cl*σ*(*A*0) is less than that of the power set of *P* , there exists a smallest ordinal number *γ* such that *Aγ* = *Aγ′* for all *γj ≥ γ*, and hence cl*σ*(*A*0)= *α≤γ Aα* = *Aγ*.

We note that

1. *β < α* implies *Aβ ⊆ Aα*;
2. for any *α* ≤ *γ*, *Aα ⊆ ↓z* because *Aα ⊆* cl*σ*(*A*0)= *↓z*;
3. for any ordinal *α*, *z ∈ Aα* iff *Aα* = cl*σ*(*A*0)= *↓z*.

We assert that *γ* is not a limit ordinal. Otherwise, *z ∈ Aγ* = *α<γ Aα* implies that *z ∈ Aα*0 for some *α*0 *< γ*, and hence *Aα*0 = cl*σ*(*A*0)= *Aγ*, a contradiction.

We say that an ordinal *α* has **F** property if *Aα ∩*

→

*wy ∩ ↓z /*= ∅ implies *y ∈ ↓D*.

Now we prove that every ordinal has **F** property by transfinite induction.

* 1. If there exists *u ∈ A*0 with *y w u* ≤ *z*, then *y* ≤ *u ∈ ↓D* and hence *y ∈ ↓D*. Thus 0 has **F** property.
  2. Assume *α* has **F** property. Let *u ∈ Aα*+1 *∩ wy ∩ ↓z*. Then there exists a

→

such that *y w v w u*. Since *v w u* ≤

*E* ≤ *z w x*, it holds that *v w*

*E*,

directed subset *E* of *Aα* such that *u ≤* WW*E*. By Theorem [2.5](#_bookmark2), there exists *v* W*∈ P*

and hence there exists *e ∈ E ⊆ Aα* such that *v* ≤ *e*. Note that *y w v* ≤ *e* ≤ *z w*

implies *y w e*. Now since *e ∈ Aα ∩*

→

*y ∈ ↓D*. Thus *α* +1 has **F** property.

*wy ∩ ↓z* and *α* has **F** property, it follows that

* 1. Assume *α* is a limit ordinal and *β* has **F** property for all *β < α*. Let

*u ∈ Aα ∩ wy ∩↓z*. Since *u ∈ Aα* = *β<α Aβ*, there exists *β*0 *< α* such that *u ∈ Aβ*0 .

→

→

Since *β*0 has **F** property and *u ∈ Aβ*0 *∩ wy ∩ ↓z*, it follows that *y ∈ ↓D*. So *α* has

**F** property.

By transfinite induction, *α* has **F** property for all *α* ≤ *γ*. In particular, *γ* has **F**

property. Note that *z ∈ Aγ ∩*

→

*wy ∩ ↓z /*= ∅, so that *y ∈ ↓D*.

All above results together show that *y x*. The converse implication is trivial.*2*

As a direct consequence of Lemma [3.3](#_bookmark5), the following result shows that meet continuity forces weak domains to domains.

**Theorem 3.4** *Every meet continuous weak domain is a domain.*

**Definition 3.5** A dcpo *P* is called a *local domain* (resp., a *local algebraic domain*) if for each *x ∈ P* , *↓x* is a domain (resp., an algebraic domain).

A dcpo is called a *local weak domain* if for each *x ∈ P* , *↓x* is a weak domain.

The following proposition is trivial since every element in a dcpo *P* is below some maximal point of *P* .

**Proposition 3.6** *A dcpo P is a local domain (resp., a local algebraic domain) iff for each a ∈* Max(*P* )*, ↓a is a domain (resp., an algebraic domain).*

In [[3](#_bookmark28)], Jung has proved the following result.

**Theorem 3.7** *[*[*3*](#_bookmark28)*, Corollary 1.7]* (1) *Local domains are exactly domains.*

(2) *Local algebraic domains are exactly algebraic domains.*

By Proposition [3.6,](#_bookmark7) Theorems [3.7](#_bookmark8), we have

**Corollary 3.8** (1) *A dcpo P is a domain iff ↓a is a domain for each a ∈* Max(*P* )*.*

(2) *A dcpo P is an algebraic domain iff ↓a is an algebraic domain for each*

*a ∈* Max(*P* )*.*

The following result is trivial.

**Lemma 3.9** *Let P be a dcpo and x, y ∈ P. Then y w x in P iff y w x in ↓x.*

By Lemma [3.9](#_bookmark9), we have

**Corollary 3.10** *A dcpo is a weak domain iff it is a local weak domain.*

For subsets *G* and *H* of a dcpo *P* , *G* is way-below *H*, denoted by *G H*, if for any directed set *D*, *D ∈ ↑H* implies *D ∩ ↑G /*= ∅. A dcpo *P* is *quasicontinuous* if for any *x ∈ P* , the family

W

*fin*(*x*)= *{↑F ⊆ P* : *F* is finite and *F x}*

is filtered and *↑x* = *fin*(*x*) (see [[2](#_bookmark26), Definition III-3.2]). Note that every domain is a quasicontinuous domain. A dcpo *P* is *quasialgebraic* if for any *x ∈ P* , the family

*comp*(*x*)= *{F ⊆ P* : *F* is finite and *F F x}*

is filtered and *↑x* = *comp*(*x*) (see [[2](#_bookmark26), Definition III-3.23]). It is clear that every quasialgebraic domain is quasicontinuous and every algebraic domain is quasialge- braic.

The next two examples show that quasicontinuous domains need not be weak domains, and weak domains need not be quasicontinuous domains. In the following, the notation N means the set of natural numbers.

**Example 3.11** Let *P* = *{⟨n, i⟩* : *n ∈* N*,i* = 0*,* 1*} ∪ {T}*. Define the order on *P* by the following rules:

1. *⟨n, i⟩* ≤ *⟨m, j⟩* iff *n* ≤ *m* and *i* = *j*;
2. *∀p ∈ P* , *p* ≤ *T*.

Then *P* can be represented by Figure 1. It is easy to check that *P* is a quasicontin-

uous domain. However, it is not a weak domain because

←

*w T* = ∅.

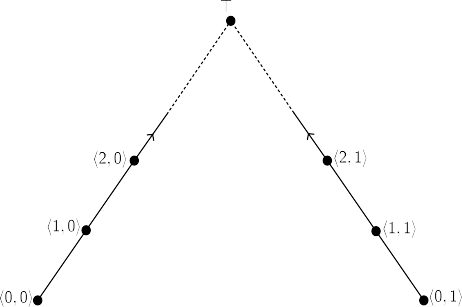


Fig. 1. A quasicontinuous domain but not a weak domain

**Example 3.12** A weak domain need not be a quasicontinuous domain. Let *Q* =

*{⟨m, n⟩* : *m, n ∈* N*}∪ {⟨ω, n⟩* : *n ∈* N*} ∪ {T}*. Define an order on *Q* by the following rules: *∀m, mj, n, nj ∈* N,

1. *∀q ∈ Q*, *q* ≤ *T*;
2. *⟨ω, m⟩* ≤ *⟨ω, n⟩* iff *m* ≤ *n*;
3. *⟨m, n⟩≤ ⟨ω, n⟩*;
4. *⟨m, n⟩* ≤ *⟨mj, nj⟩* iff *m* ≤ *mj* and *n* = *nj*.

The order on *Q* can be represented by Figure 2. We have the following facts:

(q1)

(q2)

(q3)

*w T* = *Q \ {T}*;

*w⟨ω, n⟩* = *{⟨m, n⟩* : *m ∈* N*}*;

← ← ←

*w⟨m, n⟩* = *↓⟨m, n⟩*.

Thus, *Q* is a weak domain. But it is not quasicontinuous. Considering *⟨ω,* 0*⟩*, for any finite subset *F* of *Q*, there exists *n*0 *∈* N such that *x < ⟨ω, n*0*⟩* and *x* ¢ *⟨m, n*0*⟩* for all *x ∈ F* and *m ∈* N. Note that *⟨ω,* 0*⟩* ≤ *⟨ω, n*0*⟩* = *{⟨m, n*0*⟩* : *m ∈* N*}*, but

W

*{⟨m, n*0*⟩* : *m ∈* N*}∩ ↑F* = ∅. So *F / ⟨ω,* 0*⟩*. This means *fin*(*⟨ω,* 0*⟩*)= ∅. Hence,

*Q* is not a quasicontinuous domain.



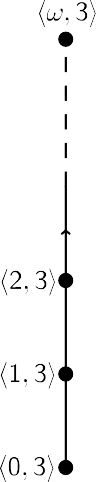
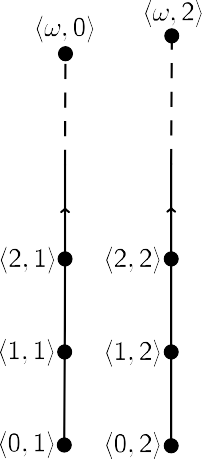


Fig. 2. A weak domain but not a quasicontinuous domain

A dcpo *P* is called a *local quasicontinuous domain* (resp., *a local quasialgebraic domain*) if for each *x ∈ P* , *↓x* is a quasicontinuous domain (resp., *a quasialgebraic domain*).

**Remark 3.13** (1) Every quasicontinuous domain is local quasicontinuous.

* 1. Observe that the weak domain (= *↓T*) of Figure 2 is not quasicontinuous, thus is not local quasicontinuous. It follows that weak domains need not be a local quasicontinuous domains.
  2. In [[9](#_bookmark33)], it was proved that every *T*1 space *X* has a local quasicontinuous domain model, denoted by Zh(*X*). An important property is that *X* is sober iff Zh(*X*) is sober. So if *X* is not sober, then Zh(*X*) cannot be quasicontinuous. Hence, there exists a local quasicontinuous domain which is not quasicontinuous.

^ ^

^

**Theorem 3.14** *Every meet continuous local quasicontinuous domain is a domain.*

**Proof.** Assume *P* is a meet continuous local quasicontinuous domain. Let *x ∈ P* . Then *↓x* is a meet continuous quasicontinuous domain, so it is a domain. By [3.7](#_bookmark8), *P* is a domain. *2*

By Theorem [3.7,](#_bookmark8) Corollary [3.10](#_bookmark10), Examples [3.11](#_bookmark11), [3.12](#_bookmark12) and Remark [3.13](#_bookmark13), the relations among domains, local domains, quasicontinuous domains, local quasicon- tinuous domains, weak domains and local weak domains are shown in Figure 3.

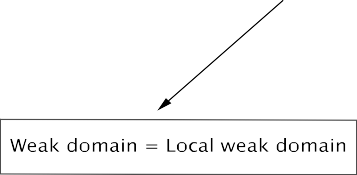


Fig. 3. Relations among (local) domains, (local) quasicontinuous domains and (local) weak domains

**Proposition 3.15** *[*[*2*](#_bookmark26)*, Lemma III-2.10] If F is a ﬁnite subset of a meet continuous*

*dcpo P, then* int*σ*(*↑F* ) *⊆* *{ x* : *x ∈ F}.*

→

**Lemma 3.16** *Let P be a domain and let F be a ﬁnite subset of P with F F.*

1. *Every minimal element in F is compact.*
2. *If F x, then there exists y ∈ F ∩ K*(*P* ) *such that y x.*

**Proof.** (1) Let *x* be a minimal element in *F* . By *F F* , *F x*. Since *P* is a

domain, *x ∈ F* = int*σ*(*↑F* ). By Proposition [3.15](#_bookmark14), there exists *y ∈ F* such that

→

*y x*. Then *y* = *x* because *x* is minimal in *F* . Thus, *x x*.

(2) As *x ∈ F* if and only if there exists a minimal (compact) element *y* in *F*

such that *y* ≤ *x*. Then by (1), it is trivial. *2*

**Proposition 3.17** *Every meet continuous quasialgebraic domain is an algebraic domain.*

**Proof.** Suppose *P* is a meet continuous quasialgebraic domain. By [[2](#_bookmark26), Theorem III-3.10], *P* is a domain. Let *x ∈ P* . By Lemma [3.16](#_bookmark15), for any *F ∈ comp*(*x*), there exists *y ∈ F ∩ K*(*P* ) such that *y x*. Let *GF* = *{y ∈ F ∩ K*(*P* ) : *y x}*. Since *P* is quasialgebraic, the family *{↑GF* : *F ∈ comp*(*x*)*}* is filtered. Then, by Rudin’s Lemma, there exists a directed set *D ⊆ {GF* : *F ∈ comp*(*x*)*}* such that *D ∩ GF /*= ø for all *F ∈ comp*(*x*).

NowWwe show that W *D* = *x*. It is immediate that W *D* ≤ *x* since *D ⊆ K*(*P* )*∩↓x*.

If *x* A

*D ∩ F* = ø. Thus *D ∩ GF* = ø, a contradiction. Hence, *x* ≤

*D*.

*D*, then there exists *F ∈ comp*(*x*) such that W *D* W*∈/*

*↑F* , which implies

Note that *D ⊆ ↓x ∩ K*(*P* ) iWs directed with W *D* = *x*. Then one can deduce that

*↓x∩K*(*P* ) is directed and *x* =

*↓x∩K*(*P* ). Therefore, *P* is an algebraic domain.*2*

**Theorem 3.18** *Every meet continuous local quasialgebraic domain is a quasialge-*

*braic domain.*

**Proof.** Assume *P* is a meet continuous local quasialgebraic domain. Let *x ∈ P* . Then *↓x* is a meet continuous quasialgebraic domain, so by Proposition [3.17](#_bookmark16), it is an algebraic domain. By [3.7](#_bookmark8), *P* is an algebraic domain. *2*

# Locating the relation *w* within the weakly auxiliary relations

In this section, we define a new auxiliary relation corresponding to the weakly way- below relation, and use it characterize exact dcpos.

**Definition 4.1** A binary relation *≺* on a poset *P* is called a *weakly auxiliary rela- tion*, or *w-auxiliary relation* for short, if for all *x, y, z ∈ P* :

1. *x ≺ y* implies *x* ≤ *y*;
2. *x* ≤ *y ≺ z* implies *x ≺ z*;
3. *⊥≺ x* whenever the smallest element *⊥* exists in *P* .

The set of all w-auxiliary relations on *P* is denoted by WAux(*P* ).

**Remark 4.2** (1) The only difference between w-auxiliary relations and auxiliary relations (see [[2](#_bookmark26), Defintion I-1.11]) is that w-auxiliary relations may not be increasing: *x ≺ y* ≤ *z* may not imply *x ≺ z*.

1. Weakly way-below relations are w-auxiliary.
2. The set WAux(*P* ) is a poset relative to the containment of graphs as subsets of

*P ×P* . The largest element is the relation ≤ itself. If *P* has a smallest element

*⊥*, then WAux(*P* ) has a smallest element *≺*0 given by *x ≺*0 *y* if and only if *x* = *⊥*. Moreover, WAux(*P* ) is closed under arbitrary nonempty intersection in the powerset of *P × P* . Hence, WAux(*P* ) is a complete lattice whenever the smallest element of *P* exists.

For a poset *P* , we use Low(*P* ) to denote the set of all lower sets in *P* .

**Proposition 4.3** *Let P be a poset and* Φ(*P* ) *be the set of all mappings s* : *P −→*

Low*P satisfying s*(*x*) *⊆ ↓x for all x ∈ P. Then the assignment*

*≺'→ s≺* = (*x '→ {y* : *y ≺ x}*)

*is a well-deﬁned isomorphism from* WAux(*P* ) *to* Φ(*P* )*, whose inverse sends every mapping s ∈* Φ(*P* ) *to the relation ≺s given by*

*x ≺s y if and only if x ∈ s*(*y*)*.*

**Proof.** Let *≺* be a w-auxiliary relation on *P* . Then *s≺*(*x*) is a lower set by Definition

* 1. (ii) and is contained in *↓x* by Definition [4.1](#_bookmark17) (i). Thus *s≺* is in Φ(*P* ) and the

assignment *≺ '→ s≺* is clearly order-preserving.

Conversely, if *s ∈* Φ(*P* ), then *s*(*x*) *⊆ ↓x* implies that *≺s* satisfies Definition [4.1](#_bookmark17)

* 1. Furthermore, if *x* ≤ *y ≺s z*, then *y ∈ s*(*z*). Since *s*(*z*) is a lower set, it follows that *x ∈ s*(*z*), implying *x ≺s z*. Thus Definition [4.1](#_bookmark17) (ii) is satisfied. Condition (iii) of Definition [4.1](#_bookmark17) is immediate. Therefore, the assignment *s '→≺s* is well-defined, and it is obviously order-preserving.

Note that *x ≺s≺ y* if and only if *x ∈ s≺*(*y*) if and only if *x ≺ y*. In addition, *s≺s* (*x*) = *{y ∈ P* : *y ≺s x}* = *{y ∈ P* : *y ∈ s*(*x*)*}* = *s*(*x*). Therefore, the two assignments are inverse from each other. *2*

**Lemma 4.4** *Let P be a dcpo and x ∈ P.*

* + 1. *wx* = *{I ∈* Id(*P* ): *x* = W *I};*

←

←

* + 1. *the assignment x '→ wx is a member of* Φ(*P* ) *deﬁned in Proposition* [*4.3*](#_bookmark18)*;*
    2. *for every ideal I ∈* Id(*P* )*, the mapping mI* : *P −→* Low(*P* ) *given by*

*m* (*x*) := ⎧⎨ *I,* if *x* = W *I,*

*I*

* + 1. ​

*is in* Φ(*P* )*;*

*wx* = Φ(*P* )*{mI* (*x*): *I ∈* Id(*P* )*}.*

←

⎩ *↓x,* otherwise

**Proof.** We only check (4), as (1)-(3) are obvious. In fact,

Φ(*P* )*{mI* (*x*): *I ∈* Id(*P* )*}*

= *{mI* (*x*): *I ∈* Id(*P* )*}*

= *{mI* (*x*): *I ∈* Id(*P* )*,x* = W *I}∩* *{mI* (*x*): *I ∈* Id(*P* )*,x /*= W *I}*

= *{I ∈* Id(*P* ): *I ∈* Id(*P* )*,x* = W *I}∩ ↓x*

←

= *wx,*

completing the proof. *2*

**Definition 4.5** A w-auxiliary relation *≺* on a dcpo *P* is *approximating* if for each

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*x* in *P* , the set *s≺*(*x*)= *{y ∈ P* : *y ≺ x}* is directed and *x* = *s≺*(*x*).

**Lemma 4.6** *Let P be a dcpo and I ∈* Id(*P* )*. Then the relation ≺I deﬁned below is an approximating w-auxiliary relation:*

*∀x, y ∈ P, y ≺I x ⇔ y ∈ mI* (*x*)*.*

**Proof.** It is easy to check that *≺I* defined above is a w-auxiliary relation. We now check that it is approximating. Let *x ∈ P* . There are two cases:

1. If *x* = W *I*, then W*{y ∈ P* : *y ≺I x}* = W *mI* (*x*)= W *I* = *x*.
2. If *x /*= W *I*, then W*{y ∈ P* : *y ≺I x}* = W *mI* (*x*)= W *↓ x* = *x*.

Thus, for all *x ∈ P* , W*{y ∈ P* : *y ≺I x}* = *x*, hence *≺I* is approximating. *2*

**Proposition 4.7** *For a dcpo P, the weakly way-below relation on P is the inter-*

*section of all the approximating w-auxiliary relations on P.*

**Proof.** Suppose *x w y* and *≺∈* WAux(*P* ) is approximating. Since *{z ∈ P* : *z ≺ y}* is directed and *y* = *{z ∈ P* : *z ≺ y}*, there exists *z ∈ P* such that *x* ≤ *z ≺ y*, implying *x ≺ y*. Therefore, *w* is contained in *≺*.

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Additionally, by Lemma [4.6](#_bookmark19), we have

*wx* = *{mI* (*x*): *I ∈* Id(*P* )*}⊇* *{s≺*(*x*) :*≺* is approximating*}*,

←

where *s≺*(*x*) = *{y ∈ P* : *y ≺ x}*. Therefore, the relation *w* is the intersection of all approximating w-auxiliary relations on *P* . *2*

**Theorem 4.8** *For a dcpo P, the following statements are equivalent:*

1. *P is exact.*
2. *The relation w is the smallest approximating w-auxiliary relation on P.*
3. *There is a smallest approximating w-auxiliary relation on P.*

# Weak domain models of *T*1 spaces

Now let’s go back to Mashburn’s question [[7](#_bookmark31)] as indicated in the introduction:

* Which topological spaces have weak domain models?

In this section, we will show that every *T*1 space has a weak domain model, which answers the above question. The crucial tool that we make use of is the Xi-Zhao model, which was introduced by Dongsheng Zhao and Xiaoyong Xi [[11](#_bookmark35)].

In [[10](#_bookmark34)], Zhao proved that every *T*1 space *X* has a bounded complete algebraic poset model. This model is constructed by using the set of all filtered families of

poset *P* , Xi and Zhao [[11](#_bookmark35)] constructed a dcpo *P*^ as: open sets of *X* with a nonempty intersection. For each bounded complete algebraic

*P*^ = *{*(*x, d*): *x ∈ P, d ∈* Max(*P* ) and *x* ≤*P d},*

where ≤*P* is the partial order on *P* , and (*x, d*) ≤ (*y, e*) in *P*^ iff either *d* = *e*

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and *x* ≤*P y*, or *y* = *e* and *x* ≤*P e*. They proved that Max(*P* ) and Max(*P* ) are

homeomorphic, showing that every *T*1 space has a dcpo model (the Xi-Zhao model).

**Definition 5.1** Let *P* be a poset.

1. An element *x ∈ P* is called *weakly compact*, if *x w x*. Denote by K*w*(*P* ) the set of all weakly compact elements.
2. *P* is called a *weak algebraic poset*, if *w* is weakly increasing and for any *x ∈ P* ,

the set K*w*(*P* ) *∩ wx* is directed and *x* = W K*w*(*P* ) *∩ wx*.

←

←

A weak algebraic dcpo is also called a *weak algebraic domain*.

Note that every weak algebraic domain is a weak domain and that every meet continuous weak algebraic dcpo is algebraic by Lemma [3.3](#_bookmark5).

Next, we will show that the Xi-Zhao model constructed for each *T*1 space is a weak algebraic domain (and hence a weak domain).

**Remark 5.2** The following facts on the dcpo *P* constructed from a bounded com- plete algebraic poset *P* will be used subsequently.

^

* 1. If *D* is a directed subset of *P* and it does not have a largest element, then there exists *d ∈ Max*(*P* ) and a directed subset *{xi* : *i ∈ I}* of *P* such that *D* = *{*(*xi, d*): *i ∈ I}*, and in this case *D* = ( *P {xi* : *i ∈ I}, d*).

^

W W

* 1. The set of maximal points of *P*^ equals *{*(*d, d*): *d ∈ Max*(*P* )*}*.

^

**Lemma 5.3** *Let P be a weak algebraic poset P and* (*x, d*) *∈ P .*

* + 1. *x ∈ Kw*(*P* ) *if and only if* (*x, d*) *∈ Kw*(*P* )*;*

^

* + 1. *if x ∈/ Kw*(*P* )*, then* (*y, e*) *w* (*x, d*) *if and only if y w x and d* = *e;*
    2. *for any* (*y, e*) *∈ P,* (*x, d*) *w* (*y, e*) *implies x w y.*

^ W

**Proof.** (1) Suppose that *x w x* and *D* is a directed subset of *P* with (*x, d*)= *D*. If (*x, d*) *∈/ D*, then, by Remark [5.2](#_bookmark20), there is a directed subset *{xi* : *i ∈ I}* of *P* such that *D* = *{*(*xi, d*): *i ∈ I}* and *D* = ( *P {xi* : *i ∈ I}, d*). Thus *x* = *P {xi* : *i ∈ I}*. Since *x w x*, it follows that *x* ≤*P xi*0 for some *i*0 *∈ I*. This means (*x, d*) ≤ (*xi*0 *, d*) *∈ D*, and hence (*x, d*) *w* (*x, d*). The other direction, (*x, d*) *w* (*x, d*) implies *x w x*, is trivial.

W W W

* + - 1. Suppose that (*y, e*) *w* (*x, d*). First, let *D* = *Kw*(*P* ) *∩*

←

*wx*. By the assump-

poset, *D* Wis directed aWnd *P D* = *x*. Thus, the set *{*(*z, d*): *z ∈ D}* is directed and tion that *x ∈/ Kw*(*P* ), we have *x ∈/ D*, and hence *d ∈/ D*. As *P* is a weak algebraic

W

(*x, d*)=(

*P D, d*)=

*{*(*z, d*): *z ∈ D}.* Since (*y, e*) *w* (*x, d*), there exists *z*0 *∈ D*

such that (*y, e*) ≤ (*z*0*, d*). Thus *e* = *d* because *z*0 */*= *d*. Now suppose that *S* be a directed subset of *P* with *x* = *P S*. Then *{*(*z, d*) : *z ∈ S}* is a directed subset of *P* and (*x, d*)= *{*(*z, d*): *z ∈ S}*. By (*y, e*) *w* (*x, d*), there exists *z*0 *∈ S* such that (*y, e*)= (*y, d*) ≤*P* (*z*0*, d*). Thus *y* ≤*P z*0. Hence, *y w x*.

W

^ W

The converse is straightforward by using (i) in Remark [5.2](#_bookmark20).

* + - 1. It is immediate by using (i) in Remark [5.2](#_bookmark20). *2*

**Lemma 5.4** *Let P be an algebraic poset.*

1. *If x w y* ≤*P z on P, then x w z.*
2. *If* (*x, d*) *w* (*y, e*) ≤ (*u, f* ) *w* (*v, g*) *on P*^*, then* (*x, d*) *w* (*u, f* )*.*

**Proof.** (1) Suppose that *D* is a directed subset of *P* such that *z* = W*P D*. As *P* is

W

algebraic, *y* =

*P {u ∈ K*(*P* ) : *u* ≤*P y}*. By *x w y*, *x* ≤*P u* for some *u ∈ K*(*P* )

W

and *u* ≤*P y*. Note that *u u* ≤*P z* = *P D*, and hence there exists *v ∈ D* such

that *u* ≤*P v*, implying *x* ≤*P v*. Therefore *x w z*.

(2) First, by Lemma [5.3](#_bookmark21) (3), *x w y* ≤*P u*. By (1), *x w u*. We consider the following cases:

**Case 1:** *u ∈ Kw*(*P* ).

By Lemma [5.3](#_bookmark21) (1), (*u, f* ) *w* (*u, f* ). By (*x, d*) ≤ (*u, f* ), we have (*x, d*) *w*

(*u, f* ) by Proposition [2.2](#_bookmark1) (3).

**Case 2:** *u* = *f* .

In this case, (*u, f* ) = (*f, f* ) *∈ Max*(*P* ), and hence (*v, g*) = (*u, f* ), (*u, f* ) *w*

^

(*u, f* ). By Proposition [2.2](#_bookmark1) (3), we have (*x, d*) *w* (*u, f* ).

**Case 3:** *x* = *d* or *y* = *e*.

A similar argument of Case 2 applies.

**Case 4:** *u ∈/ Kw*(*P* ), *u /*= *f* , *y /*= *e* and *x /*= *d*.

By (*x, d*) ≤ (*y, e*) ≤ (*u, f* ), we have *d* = *e* = *f* . By the fact that *x w u* and Lemma [5.3](#_bookmark21) (2), (*x, d*) *w* (*u, f* ).

*2*

As a corollary of Lemmas [5.3](#_bookmark21), [5.4](#_bookmark22), we obtain the following.

**Corollary 5.5** *For an algebraic poset P, P*^ *is a weak algebraic domain.*

Applying Corollary [5.5](#_bookmark23) to Xi-Zhao model for *T*1 space, we obtain our main result.

**Theorem 5.6** *Every T*1 *topological space has a weak algebraic domain model.*

# Conclusion

1. Let’s note that the analogue of Figure 3 for the algebraic case hold too, that is,

algebraic domain *e* local algebraic domain

*⇒* ⎧⎨

⎩

local weak algebraic domain *e* weak algebraic domain quasialgebraic domain *⇒* local quasialgebraic domain.

All above notions coincide in a meet continuous dcpo.

1. A dcpo *P* is called *well-ﬁltered* if its Scott space is well-filtered (see [[2](#_bookmark26)] for the definition of well-filteredness). In [[9](#_bookmark33)], it was proved that a *T*1 space *X* is well-filtered iff its Xi-Zhao model is well-filtered. As was pointed out in [[9](#_bookmark33)], there exists a *T*1 space that is not well-filtered, thus its Xi-Zhao model is a weak domain (even a weak algebraic domain) that is not well-filtered.
2. In [[7](#_bookmark31)], Mashburn proved that every first countable space has a weak domain model. Our Theorem [5.6](#_bookmark24) shows that the condition “first countable” in Msh- burn’s result is surplus. Now we can say that a space has a weak domain model iff it is a *T*1 space, answering Mashburn’s problem, as indicated in the introduction.
3. By Theorem [3.4,](#_bookmark6) we prove that every meet continuous weak domain is a do- main. However, the following question is still unknown:
   * Is a meet continuous exact dcpo a domain?

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# References

1. B. Coecke and K. Martin, *A partial order on classical and quantum states*, Oxford University Computing Laboratory, Research Report PRG-RR-02- 07, August 2002.
2. G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *Continuous Lattices* *and Domains*, Cambridge University Press, 2003.
3. A. Jung, *Cartesian Closed Categories of Domains*, CWI Tracts vol. 66, Centrum voor Wiskunde en Informatica, Amsterdam 1989.
4. H. Kou, Y. M. Liu and M. K. Luo, *On meet continuous dcpo*, Domains and Processes II, Semantic Structures in Computation, Kluwer, 2001.
5. J. D. Lawson, *Spaces of maximal points*, Math. Structures. Comput. Sci. **7** (5) (1997), 543–555.
6. K. Martin, *Topological games in domain theory*, Topology Appl. **129** (2) (2003), 177–186.
7. J. Mashburn, *A comparison of three topologies on ordered sets*, Topology Proceedings. Auburn University **31** (2007), 1–21.
8. J. Mashburn, *Linearly Ordered Topological Spaces and Weak Domain Representability*, Topology Proc.

**35** (2007), 149–164.

1. X. Xi and D. Zhao, *Well-filtered spaces and their dcpo models*, Math. Struct. Comput. Sci. **27** (2017), 507–515.
2. D. Zhao, *Poset models of topological spaces*, Proceeding of International Conference on Quantitative Logic and Quantification of Software, GlobalLink Publisher, pp. 229–238, 2009.
3. D. Zhao, X. Xi, *Directed complete poset models of T*1 *spaces*, Math. Proc. Camb. Phil. Soc. **164** (2018), 125–134.