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[Electronic Notes in Theoretical Computer Science 333 (2017) 43–61](https://doi.org/10.1016/j.entcs.2017.08.005)

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*s*2-C-continuous Poset

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**Abstract**

In this paper, we introduce the concept of *s*2-C-continuous poset by cut operator. The main results are:

(1) A sup-semilattice is both *s*2-C-continuous and *s*2-continuous if and only if it is *s*2-CD; (2) A sup- semilattice is both *s*2-C-continuous and hypercontinuous if and only if it is *s*2-CD; (3) A sup-semilattice is both *s*2-QC-continuous and *s*2-quasicontinuous if and only if it is *s*2-GCD; (4) A sup-semilattice is both *s*2-QC-continuous and quasi-hypercontinuous if and only if it is *s*2-GCD; (5) A poset is *s*2-C-continuous if and only if it is both *s*2-MC-continuous and *s*2-QC-continuous; (6) A poset is *s*2-CD if and only if its order dual is *s*2-CD; (7) A semi-lattice is *s*2-GCD if and only if its order dual is hypercontinuous; (8) The lattice of all *σ*2-closed subsets of a poset is C-continuous; (9) A poset *P* is *s*2-continuous if and only if the lattice *C*2(*P* ) of all *σ*2-closed subsets of *P* is a continuous lattice if and only if *C*2(*P* ) is a CD lattice; (10) A poset *P* is *s*2-quasicontinuous if and only if the lattice *σ*2(*P* ) of all *σ*2-open subsets of *P* is a hypercontinuous lattice if and only if *C*2(*P* ) is a GCD lattice if and only if *C*2(*P* ) is a quasicontinuous lattice.

*Keywords: s*2-C-continuous poset, *s*2-QC-continuous poset, *s*2-MC-continuous poset, *s*2-CD poset,

*s*2-GCD poset.

# Introduction

Domain theory was introduced by Dana Scott in the late sixties as models for the denotational semantics of programming languages, due to its strong background in computer science, general topology and topological algebra has been extensively studied in various areas. An important approach in the study of domains is to extend the theory of domains to that of posets as much as possible.

In domain theory, there are some categories which gained particularly wide at- tention, that is continuous lattice category, completely distributive (for short, CD) lattice category ([[10](#_bookmark10)]), Domain category, as well as hypercontinuous lattice cate- gory ([[6](#_bookmark6)]) etc. Recently, Ho and Zhao defined the binary relation *≺* by all Scott closed subsets instead of the directed subsets in the definition of the way below

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<https://doi.org/10.1016/j.entcs.2017.08.005>

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relation, and proposed the concept of a C-continuous poset ([[8](#_bookmark8)]). The special case of the C-continuous poset is the C-continuous lattice. To know more about these research objects, the most important studying method is to generalize them. One of the generalizations of the continuous lattices is called quasicontinuous lattice ([[5](#_bookmark5)]), they were introduced by Gierz, Lawson and Stralka in 1983. The basic idea is to generalize the way below relation to that on the collection of all subsets of a complete lattice. Based on this idea: Yang and Xu introduced the concept of a generalized completely distributive (for short, GCD) lattice in 2010 ([[11](#_bookmark11)]); Xu gave the concepts of the quasi-hypercontinuous lattice in 2016; He, Xu and Yang provided the concept of a quasi-C-continuous (for short, QC-continuous) poset in 2016 ([[7](#_bookmark7)]). Another generalizations of the continuous lattices is called *s*2-continuous poset (*s*1-continuous poset) ([[3](#_bookmark1)]), they were proposed by Ern´e in 1981. The basic idea is by making use of the cut operator instead of joins in the definition of the way below relation. The notion of *s*2-continuty admits to generalize most important characterizations of continuity from dcpos to arbitrary posets and has the advantage that the existence of directed joins is not even required. Afterwards Yao gave the concept of the *M*-continuous in 2011 ([[12](#_bookmark12)]) and Zhang and Xu gave the concepts of the *s*2-quasicontinuous in 2015 ([[13](#_bookmark13)]) and *s*1-quasicontinuous poset in 2016 ([[14](#_bookmark14)]) in the manner of Ern´e. In addition, Mao and Xu introduced the concept of meet-C- continuous (for short, MC-continuous) poset by the semi-topological structure ([[1](#_bookmark2)]) and discussed the relation among C-continuous, QC-continuous and MC-continuous in 2016([[9](#_bookmark9)]).

In this paper, we will also generalize the binary relations *≺* and *a* by making use of the cut operator instead of joins, and define *s*2-C-continuous (*s*2-QC-continuous) and *s*2-CD (*s*2-GCD) posets, respectively. We obtain some characterization for the continuity of the poset: a poset is *s*2-CD if and only if its order dual is *s*2-CD; a semilattice is *s*2-GCD if and only if its order dual is hypercontinuous, and a sup- semilattice is both *s*2-C-continuous and *s*2-continuous if and only if it is *s*2-CD. We also prove that the lattice of all *σ*2-closed subsets of a poset is C-continuous. In last section, we will propose the concept of *s*2-meet-C-continuous (for short, *s*2- MC-continuous) poset by semi-topological structure and give the characterization theorem that a poset is *s*2-C-continuous if and only if it is both *s*2-MC-continuous and *s*2-QC-continuous.

# Preliminaries

In this paper, the order dual of the poset *P* is written as *Pop*. For a poset *P* and for all *x ∈ P* , *A ⊆ P* , let *↑x* = *{y ∈ P | x ≤ y}* and *↑A* = *a∈A ↑a*; *↓x* and *↓A* are defined dually. *Al* and *Au* denote the sets of all upper and lower bounds of *A* are defined by

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*Au* = *{x ∈ P | a ≤ x for all a ∈ A} and Al* = *{x ∈ P | x ≤ a for all a ∈ A},*

respectively. Let *Aδ* = (*Au*)*l* be the cut of *A*. Further, *Au* is an up-set and *Al* is a down-set.

**Lemma 2.1** *[*[*2*](#_bookmark3)*] Let P be a poset. For subsets A and B of P, we have*

* 1. *A ⊆ Aδ;*
  2. *if A ⊆ B, then Au ⊇ Bu and Al ⊇ Bl, which implies that Aδ ⊆ Bδ;*
  3. *Au* = *Aulu, i.e., Aδ* = (*Aδ*)*δ;*
  4. (*↓x*)*δ* = *↓x for all x ∈ P;*
  5. *If* sup *A exists in P, then Aδ* = *↓*sup *A.*

**Definition 2.2** [[13](#_bookmark13)] Let *P* be a poset. A subset *U ⊆ P* is called *σ*2-open if it satisfies:

* + 1. *U* = *↑U* .
    2. *Dδ ∩ U /*= *∅* implies *D ∩ U /*= *∅* for all directed sets *D ⊆ P* .

The collection of all *σ*2-open subsets of *P* forms a topology. It will be called *σ*2- topology of *P* and will be denoted by *σ*2(*P* ). The topology *λ*2(*P* )= *σ*2(*P* ) *∨ ω*(*P* ) is called the *λ*2-topology on *P* . Obviously, *υ*(*P* ) *⊆ σ*2(*P* ) *⊆ σ*(*P* ).

The complements of *σ*2-open sets of the poset *P* are the *σ*2-*closed* sets. We use *C*2(*P* ) and *C∗*(*P* ) to denote the set of all *σ*2-closed sets and the set of all nonempty *σ*2-closed sets of *P* , respectively. Thus a subset *S ⊆ P* is *σ*2-closed if and only if *S* = *↓S* and for all directed subsets *D ⊆ S* implies *Dδ ⊆ S*. Both *σ*2(*P* ) and *C*2(*P* ) are distributive complete lattices with respect to the inclusion order. For a poset *P* and *x ∈ P* , let *S*(*x*)= *{S ∈ C∗*(*P* ) *| x ∈ Sδ}*.

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**Remark 2.3** (i) Let *P* be a poset. Then *↓x, Aδ ∈ C*2(*P* ) for all *x ∈ P, A ⊆ P* .

(ii) If *P* is a dcpo, then *σ*2-closed subsets are precisely Scott-closed. We use *C*(*P* ) to denote the set of all Scott-closed sets of *P* .

# *s*2-C-continuous posets

**Definition 3.1** [[13](#_bookmark13)] Let *P* be a poset.

1. The way-below relation2 on *P* is defined by *x* 2*y* for all *x, y ∈ P* , if for all directed subsets *D ⊆ P* with *y ∈ Dδ*, which implies that *x ∈ ↓D*. The set

*{y ∈ P | y* 2*x}* will be denoted *⇓x* and *{y ∈ P | x* 2*y}* denoted *⇑x*.

1. *P* is called *s*2-continuous if for all *x ∈ P* , the set *⇓x* is directed and *x* = sup *⇓x*.

**Remark 3.2** Let *P* be a poset. Then *x* 2*z* and *y* 2*z* imply *x ∨ y* 2*z* whenever the least upper bound *x ∨ y* exists in *P* .

**Proposition 3.3** *Let P be a poset. If x* 2*z and if z ∈ Dδ for a directed subset D ⊆ P which ⇓d is directed and d* = sup *⇓d for all d ∈ D, then x* 2*d for some element d ∈ D, that is, {⇓z | z ∈ Dδ}* = *{⇓d | d ∈ D}. Further, If P is s*2*- continuous, then ⇑x is σ*2*-open for all x ∈ P.*

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**Proof.** Assume that *x ∈/* S *{⇓d | d ∈ D}*. Let *I* = S*d∈D ⇓d*. Then *I* is a di-

S*d∈D* S*d∈D*

rected subset of *P* and *Iδ* = ( *⇓d*)*δ ⊇* (*⇓d*)*δ ⊇ D*, which implies that

S

*Dδ ⊆* (*Iδ*)*δ* = *Iδ*. Since *x* 2*z* and *z ∈ Dδ*, so *x ∈ ↓I* = *I* = *d∈D ⇓d*, this is a contradiction. Hence *x* 2*d* for some element *d ∈ D*. Further,

*{⇓z | z ∈ Dδ}⊆ {⇓d | d ∈ D}*. Since *D ⊆ Dδ* for a directed subset *D ⊆ P* , we have S *{⇓z | z ∈ Dδ}⊇* S *{⇓d | d ∈ D}*. *2*

S S

**Definition 3.4** [[8](#_bookmark8)] Let *P* be a poset. For any two elements *x* and *y* in *P* , we write *x≺y*, if for each nonempty Scott-closed subset *S ⊆ P* for which sup *S* exists, *y ≤* sup *S* implies *x ∈ S*.

Given a poset *P* , we now define a new binary relation on *P* which is crucial for us to formulate the properties of lattices of *σ*2-closed sets.

**Definition 3.5** Let *P* be a poset. For any two elements *x* and *y* in *P* , we write

*x≺*2*y*, if for each nonempty *σ*2-closed subset *S ⊆ P* , *y ∈ Sδ* implies *x ∈ S*. The set

*{y ∈ P | y≺*2*x}* will be denoted *↓≺x* and *{y ∈ P | x≺*2*y}* denoted *↑≺x*.

**Remark 3.6** (i) If *P* is a poset, then *x≺*2*y* implies *x ≺ y* for all *x, y ∈ P* .

1. If *P* is a complete lattice, then *x≺ y* is equal to *x ≺ y* for all *x, y ∈ P* .

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1. *x≺*2*y* if and only if *x ∈* *S*(*y*) for all *x, y ∈ P* .

Now it is routine to verify the following properties of the relation *≺*2.

**Proposition 3.7** *Let P be a poset and u, v, x, y ∈ P. Then the following statements hold:*

1. *x≺*2*y implies x ≤ y;*
2. *u ≤ x≺*2*y ≤ v implies u≺*2*v;*
3. *if P has a smallest element* 0*, then* 0*≺*2*x always holds.*

**Proof.** This follows immediately from Definition 3.5. *2*

**Proposition 3.8** *Let P be a poset and D a directed subset of P. If D ⊆ ↓≺x, then*

*Dδ ⊆ ↓≺x.*

**Proof.** Suppose *S ∈ C∗*(*P* ) with *x ∈ Sδ*. Since *d≺*2*x* for all *d ∈ D*, it follows that *D ⊆ S*. Because *S* is *σ*2-closed and *D* is directed, we have *Dδ ⊆ S*. Thus *Dδ ⊆ ↓≺x*. *2*

2

Propositions 3.7 and 3.8 together imply the following corollary.

**Corollary 3.9** *Let P be a poset. Then the set ↓≺x is a σ*2*-closed subset of P for all x ∈ P.*

**Definition 3.10** Let *P* be a poset. *P* is called *s*2-C-continuous if for all *x ∈ P* ,

*x* = sup *↓≺x*.

**Remark 3.11** Notice that one of the requirements of a *s*2-continuous poset *P* is that for every *x ∈ P* the set *⇓x* is directed. In contrast, for any poset *P* , the set

*↓≺x* is automatically *σ*2-closed for all *x ∈ P* by virtue of Corollary 3.9.

**Lemma 3.12** *Let P be a poset. Then for any collection {S | i ∈ I} of the nonempty*

*i*

*σ*2*-closed subsets of P,* *i∈I Si /*= *∅ if and only if* ( *i∈I Si*)*δ /*= *∅.*

**Proof.** Since *S ⊆* ( *S* )*δ*, so *S /*= *∅* implies ( *S* )*δ /*= *∅*. Con- versely, assume that *S* = *∅*. Then *P* has no a smallest element, which implies that ( *i∈I Si*)*δ* = *Pl* = *∅*, this is a a contradiction. *2*

*i∈I i*  *i∈I i*  *i∈I i*  *i∈I i*

*i∈I i*

The following proposition is similar to Theorem I-1.10 in [[4](#_bookmark4)].

**Proposition 3.13** *For a poset P, the following conditions are equivalent:*

* 1. *P is s*2*-C-continuous;*
  2. *for each x ∈ P, the set ↓≺x is the smallest nonempty σ*2*-closed set S ⊆ P with*

*x ∈ Sδ;*

* 1. *for each x ∈ P, there is a smallest nonempty σ*2*-closed set S ⊆ P with x ∈ Sδ;*
  2. *for any collection {Si | i ∈ I} of σ*2*-closed subsets of P, the following equation hold:*

*i*

*i∈I*

*Sδ* = (

*i∈I*

*Si*)*δ.*

**Proof.** (1) *⇒* (2): Condition (1) holds if and only if for each *x ∈ P* , *↓≺x ∈ C∗*(*P* ) and *x ∈* (*↓≺x*)*δ* by Definition 3.10. Since for any nonempty *σ*2-closed set *S ⊆ P* with *x ∈ Sδ*, *↓≺x ⊆ S* by Definition 3.5, so (2) hold.

2

Condition (2) trivially implies (3).

* + 1. *⇒* (1): For each *x ∈ P* , if *S*(*x*) has a smallest element *S*, then *S ⊆ S*(*x*) *⊆ S*, and thus *S* = *S*(*x*) = *↓≺x* by Remark 3.6(iii). Since *x ∈ Sδ*, we have *x* = sup *↓≺x*. Hence *P* is *s*2-C-continuous.

1. *⇒* (4): Since *i∈I*

*Si ⊆ Si* for all *i ∈ I*, we have ( *i∈I*

*Si*)*δ ⊆*

*i∈I*

*Sδ*.

For the reverse, suppose *x ∈* *Sδ*. Then *x ∈ Sδ*, which implies that *↓≺x ⊆ Si*

*i*

*i∈I*

*i*

*i*

for all *i ∈ I*, since *Si* is *σ*2-closed for all *i ∈ I*. Thus *↓≺x ⊆* *i∈I Si*. Therefore,

*x ∈* (*↓≺x*)*δ ⊆* ( *i∈I*

*i*

*S∈S*(*x*) *S∈S*(*x*)

*Si*)*δ*, because *P* is *s*2-C-continuous. Hence (

*i∈I*

*Si*)*δ ⊇* *i∈I*

*Sδ*.

* + 1. *⇒* (3): Suppose (4) hold. Then for each *x ∈ P* , ( *S*)*δ* = *Sδ*.

*S∈S*(*x*) *S∈S*(*x*) *S∈S*(*x*)

Therefore *x ∈* ( *S*)*δ*, i.e., ( *S*)*δ /*= *∅* and *S ∈ S*(*x*) by Lemma

3.12. Hence *S∈S*(*x*) *S* a smallest element *S*(*x*). *2*

The following example illustrates that *s*2-continuous and *s*2-C-continuous can be quite different.

**Example 3.14** (1) Consider the poset *P* = *{a, b, x, y, z}*, where the order as defined by *x, y, z < a, b*. Since *P* is a finite poset, so *P* is a *s*2-continuous. Let *S*1 = *{x}* and *S*2 = *{y, z}*. Then *S*1 and *S*2 are *σ*2-closed by Remark 2.3(i), and (*S*1 *∩ S*2)*δ* = (*{x}∩ {y, z}*)*δ* = (*∅*)*δ* = *∅*, *Sδ ∩ Sδ* = *{x}δ ∩ {y, z}δ* = *{x}∩ {x, y, z}* = *{x} /*= *∅*.

1 2

Thus (*S*1 *∩ S*2)*δ /*= *Sδ ∩ Sδ*, hence *P* is not *s*2-C-continuous by Proposition 3.13.

1 2

*a*

*b*

*x y z*

1. Consider the poset *P* = *{*(*m, n*) *| m ∈ {*0*,* 1*} and n ∈* N*}∪ {ω*0*, ω*1*}* with ordering defined by (*m*1*, n*1) *≤* (*m*2*, n*2) if and only if *m*1 *≤ m*2 and *n*1 *≤ n*2 for all *m*1*, m*2 *∈ {*0*,* 1*}*, *n*1*, n*2 *∈* N, and (*m, n*) *≤ ω*0*, ω*1 for all *m ∈ {*0*,* 1*}*, *n ∈* N. It is easy to prove that
   1. *↓≺*(0*, n*)= *{*(0*, nj*) *| nj ≤ n}* = *⇓*(0*, n*) for all *n ∈* N;
   2. *↓≺*(1*, n*) = *{*(0*, nj*) *| nj ≤ n}∪ {*(1*,* 0)*}*, *⇓*(1*, n*) = *{*(0*, nj*) *| nj ≤ n}* for all

*n ∈* N;

* 1. *↓≺ωi* = *↓ωi* = *⇓ωi*, *i* = 0*,* 1.

Hence *P* is *s*2-C-continuous, but it is not *s*2-continuous.

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**Proposition 3.15** *Let P be a s*2*-C-continuous poset. Then for any collection*

*{Fi | i ∈ I} of ﬁnite subsets of P, the following equation holds:*

*Fδ* =

Q *{f* (*i*) *| i ∈ I}*

*lu* *l*

*i∈I i*

*f∈ i∈I Fi*

**Proof.** For each *i ∈ I* and any *f ∈ F* , *f* (*i*) *∈ F* implies *↓f* (*i*) *⊆ ↓F* .

*i∈I i i* *i*

T hus *{f* (*i* ) *| i ∈ I}l* = *i∈I ↓f* (*i*) *⊆* *i∈I ↓Fi*, which implies that *{f* (*i*)*|i ∈ I}lu ⊇*

*.*

*i∈I*

*↓Fi*

*u*. Therefore,

*f∈*Q

*i∈I Fi*

*{f* (*i*) *| i ∈ I}lu ⊇*

*i∈I*

*↓F* *u,*

that is,

*i*

*f∈*Q

*i∈I Fi*

*{f* (*i*) *| i ∈ I}*

*lu* *l*

*δ*

*⊆* *i∈I ↓Fi* *.*

Since *Fi* is finite for all *i ∈ I*, we have *↓Fi* is *σ*2-closed for all *i ∈ I* by Remark 3.6. Hence by Proposition 3.13, we know that

*f∈*Q

*i∈I Fi*

*{f* (*i*) *| i ∈ I}*

*lu* *l*

*⊆* *i∈I*

(*↓Fi*)*δ* =

*i∈I*

*Fiδ.*

Conversely, suppose *x ∈*

*i∈I*

*Fδ* =

*i∈I*

(*↓Fi*)*δ* and *u≺*2*x*. For each *i ∈ I*, the set

*↓Fi* is *σ*2-closed for all *i ∈ I* by Remark 3.6, since *Fi* is finite for all *i ∈ I*, so there

*i*

exists *yi ∈ Fi* such that *u ≤ yi*. Let *f ∈ i∈I Fi* be defined by *f* (*i*) = *yi* for all

*i ∈ I*. Then *u ∈ {f* (*i*) *| i ∈ I}L*, which implies that *↑u ⊇ {f* (*i*) *| i ∈ I}Lu*, that is,

(*↓≺x*)*u ⊇* *ƒ∈*Q

*i∈I Fi*

*{f* (*i*) *| i ∈ I}Lu*. Therefore,

(*↓≺*

1. *δ ⊆*

Q *{f* (*i*) *| i ∈ I}*

*Lu* *L*

*ƒ∈ i∈I Fi*

Since *P* is *s*2-C-continuous, so *x ∈* (*↓≺x*)*δ* for all *x ∈ P* . Hence

*.*

*ƒ∈*Q

*i∈I Fi*

*{f* (*i*) *| i ∈ I}*

*Lu* *L*

*⊇* *i∈I*

*Fiδ.*

*2*

**Remark 3.16** Let *P* be a *s*2-C-continuous poset which is also a complete lattice. Then for any collection *{Fi | i ∈ I}* of finite subsets of *P* , the following equation holds:

*Fi* = *ƒ∈*Q

*i∈I*

**Definition 3.17** Let *P* be a poset.

*i∈I*

*f* (*i*)*.*

*Fi*

*i∈I*

* 1. The binary relation *a*2 on *P* is defined by *xa*2*y* for all *x, y ∈ P* , if for all subsets *A ⊆ P* with *y ∈ Aδ*, which implies that *x ∈ ↓A*. The set *{y ∈ P | ya*2*x}* will be denoted *↓ax* and *{y ∈ P | xa*2*y}* denoted *↑ax*.
  2. *P* is called *s*2-completely distributive (for short, *s*2-CD) poset, if for all *x ∈ P* ,

*x* = sup *↓ax*.

**Remark 3.18** (i) *ya*2*x* if and only if *x ∈/* (*P\↑y*)*δ*, for all *x, y ∈ P* ;

1. If *P* is a *s*2-CD poset which is also a complete lattice, then *P* is a CD lattice.

**Theorem 3.19** *Let P be a sup-semilattice. Then the following are equivalent:*

* 1. *P is s*2*-C-continuous and s*2*-continuous;*
  2. *P is s*2*-CD.*

**Proof.** (2) *⇒* (1) Follows immediately from Definitions 3.1, 3.10, 3.17 and Remark

3.2. For (1) *⇒* (2): Suppose that *P* is both *s*2-C-continuous and *s*2-continuous. Since *P* is *s*2-continuous, for each *x ∈ P* , *x* = sup*{y ∈ P | y* 2*x}*. Now for each *y* 2*x*, *y* = sup*{z ∈ P | z≺*2*y}* because *P* is *s*2-C-continuous. It follows that

*x* = sup*{z ∈ P | there exists y ∈ P such that z≺*2*y* 2*x}.*

Next, suppose *z≺*2*y* 2*x*, we shall show that *za*2*x*. Let *A ⊆ P* with *x ∈ Aδ*. Construct the set *D* = *{*sup *F | F is a finite subset of A}*. Then *D* is a directed set and *x ∈ Aδ* = *Dδ*. Since *y* 2*x*, so there is a finite subset *F ⊆ A* such that *y ∈ ↓* sup *F* = (*↓F* )*δ*. Note that the last set *↓F* is *σ*2-closed. So it follows from *z≺*2*y* that *z ∈ ↓F ⊆ ↓A*, this implies that *za*2*x*. Hence *P* is *s*2-CD. *2*

**Proposition 3.20** *Let P be a poset and S ∈ C*(*C*2(*P* ))*. Then* *C*2(*P* )*S* = S *S.*

**Proof.** Note that each member of *S* is a *σ*2-closed subset of *P* . So to prove the equation, it suffices to show that

*S ∈ C*2(*P* )*.*

Obviously *S* is a lower subset of *P* . Now let *D* be any directed subset of *P* with *D ⊆ S*. We want to prove that *Dδ ∈ S*. Since *D* = *{↓d | d ∈ D}* is a directed subset of *C* (*P* ). Moreover, *D ⊆ S* because *S* is a lower set in *C* (*P* ). Since *S* is a Scott-closed set of *C*2(*P* ), we have *C* (*P* )*D ∈ S*. But *C* (*P* )*D* is precisely *Dδ* by

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Remark 2.3(i). Hence *Dδ ∈ S*. *2*

**Definition 3.21** (i) An element *x* of a poset *P* is called C-compact if *x≺x*. We use *KC*(*P* ) to denote the set of all the C-compact elements of *P* .

(ii) An element *x* of a poset *P* is called *s*2-C-compact if *x≺*2*x*. We use *KC*2(*P* ) to denote the set of all the *s*2-C-compact elements of *P* .

**Remark 3.22** If *P* is a complete lattice, then *KC*(*P* )= *KC*2(*P* ).

Recall that an element *q /*= 0 of a poset *P* is called co-prime if *P\↑q* is directed in *P* .

**Proposition 3.23** *Let P be a poset. If k ∈ KC*2(*P* )*, then k is co-prime.*

**Proof.** Assume that *k* is not co-prime. Then there exist *u, v ∈ P\↑k* such that

*{u, v}u ⊆ ↑k*, this implies that *k ∈ {u, v}δ*. Let *S* = *↓u ∪ ↓v*. Then *S ∈ C*2(*P* ) by Remark 2.3(i) and *S ⊆ P\↑k* with *Sδ* = *{u, v}δ*. Since *k ∈ KC*2(*P* ), we have *k ∈ S* implies *k ∈ P\↑k*, this is a contradiction. Hence *k* is co-prime. *2*

**Proposition 3.24** *Let P be a poset and S*0 *∈ C∗*(*P* )*. Then for each x ∈ S*0*,*

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*↓x ≺ S*0 *holds in C*2(*P* )*.*

0 2 0 *C*2(*P* )

**Proof.** Let *x ∈ S* . Suppose *S ∈ C*(*C* (*P* )) with *S ⊆ S*. Then by Proposi- tion 3.20, *S*0 *⊆ S*. Hence there exists *S ∈S* such that *x ∈ S*. Therefore, *↓x ⊆ S*, and thus *↓x ∈ S*. *2*

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**Corollary 3.25** *Let P be a poset. Then for each x ∈ P, it holds that ↓x ∈*

*KC*(*C*2(*P* ))

**Proof.** Since *x ∈ ↓x* and *↓x ∈ C∗*(*P* ), so *↓x ≺ ↓x*, i.e., *↓x ∈ KC*(*C*2(*P* )) for all

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*x ∈ P* by Proposition 3.24. *2*

**Definition 3.26** A poset *P* is said to be *s*2-C-prealgebraic if for each *x ∈ P* ,

*x* = sup*{k ∈ KC*2(*P* ) *| k ≤ x}.*

A *s*2-C-prealgebraic poset *P* is *s*2-C-algebraic if for any *x ∈ P* ,

*↓{k ∈ KC*2(*P* ) *| k ≤ x}∈ C*2(*P* )*.*

Obviously every *s*2-C-prealgebraic poset is *s*2-C-continuous.

**Proposition 3.27** *For any poset P, the lattice C*2(*P* ) *is s*2*-C-prealgebraic. Hence the lattice C*2(*P* ) *is s*2*-C-continuous, i.e., C*2(*P* ) *is C-continuous.*

**Proof.** This follows from Corollary 3.25 and the fact *S* = *C*2(*P* )*{↓x | x ∈ S}* holds for every *S ∈ C*2(*P* ). *2*

It is well-known that a poset *P* is continuous if and only if *C*(*P* ) is completely distributive. From Theorem 3.19, we obtain the following theorem.

**Theorem 3.28** *For any poset P, the following statements are equivalent:*

1. *P is a s*2*-continuous;*
2. *C*2(*P* ) *is a continuous lattice;*
3. *C*2(*P* ) *is a completely distributive lattice.*

**Lemma 3.29** *Let P be a poset. Then for any x ∈ P and H ⊆ P, x ∈/* (*P\↑H*)*δ if and only if there exists u ∈ P such that x ∈/ ↓u and ↓u ∪ ↑H* = *P.*

**Proof.** Suppose *x ∈/* (*P\↑H*)*δ*. Then (*P\↑H*)*u /*= *∅* and there is *u ∈* (*P\↑H*)*u* such that *x ∈/ ↓u*, which implies that *↓u ∪ ↑H* = *P* . Conversely, if *↓u ∪ ↑H* = *P* , then (*P\↑H*)*δ ⊆* (*↓u*)*δ* = *↓u*. *2*

**Proposition 3.30** *A poset P is s*2*-CD if and only if for any x, y ∈ P with x* ¢ *y, there exists v ∈ P such that y ∈/ ↑v and x ∈/* (*P\↑v*)*δ.*

**Proof.** Suppose *P* is *s*2-CD and *x, y ∈ P* with *x* ¢ *y*. Then there exists *v ∈ P* such that *y ∈/ ↑v* and *va*2*x*. Thus *x ∈/* (*P\↑v*)*δ* by Remark 3.18(i).

Conversely, we now need to prove that *x* = sup *↓ax* for all *x ∈ P* . Suppose there exists *x ∈ P* such that *x /*= sup *↓ax*, i.e., there is *y ∈* (*↓ax*)*u* such that *x* ¢ *y*. Then there exists *v ∈ P* such that *y ∈/ ↑v* and *x ∈/* (*P\↑v*)*δ*, i.e., *va*2*x*. Thus *y ∈ ↑v*, this is a contradiction. *2*

Proposition 3.30 and Lemma 3.29 together imply the following corollary.

**Corollary 3.31** *A poset P is s*2*-CD if and only if for any x, y ∈ P with x* ¢ *y, there exist u, v ∈ P such that x ∈/ ↓u, y ∈/ ↑v and ↓u ∪ ↑v* = *P. Further, a poset P is s*2*-CD if and only if so is Pop.*

For a poset *P* , we define *x* “ *y* if and only if *y ∈* int*υ↑x*. A poset *P* is called *hypercontinuous* if and only if for all *x ∈ P* , the set *↓*“*x* = *{y ∈ P | y* “ *x}* is directed and *x* = sup *↓*“*x*.

**Proposition 3.32** *Let P be a poset. The following statements are equivalent:*

1. *P is hypercontinuous;*
2. *P is s*2*-continuous in which y* 2*x if and only if y* “ *x.*

**Proof.** Suppose *P* is a hypercontinuous poset and *y* 2*x*. Then there exists *z ∈ P* such that *y ≤ z* “ *x*. If *y* “ *x*, then *x ∈* int*υ↑y ⊆* int*σ*2 *↑y*, i.e., *y* 2*x*. Hence *P* is *s*2-continuous.

Conversely if (2) holds, then “ is approximating since2 is. Hence *P* is hyper- continuous. *2*

**Proposition 3.33** *Let P is a hypercontinuous poset. Then for any x, y ∈ P with*

*x* ¢ *y, there exist u ∈ P and a ﬁnite subset F of P such that y ∈/ ↑u, x ∈/ ↓F and*

*↑u ∪ ↓F* = *P.*

**Proof.** Suppose *P* is hypercontinuous and *x, y ∈ P* with *x* ¢ *y*. Then there exists *u ∈ P* such that *y ∈/ ↑u* and *u* “ *x*. Since *u* “ *x* if and only if *x ∈* int*υ↑u*, so there is a finite subset *F ⊆ P* such that *x ∈ P\↓F ⊆ ↑u*, i.e., *x ∈/ ↓F* and *↑u ∪ ↓F* = *P* .*2*

**Proposition 3.34** *A sup-semilattice P is hypercontinuous if and only if for any x, y ∈ P with x* ¢ *y, there exist u ∈ P and a ﬁnite subset F of P such that y ∈/ ↑u, x ∈/ ↓F and ↑u ∪ ↓F* = *P.*

**Proof.** If *P* is hypercontinuous, it is obvious that for any *x, y ∈ L* with *x* ¢ *y*, there exist *u ∈ P* and a finite subset *F* of *P* such that *y ∈/ ↑u, x ∈/ ↓F* and *↑u ∪ ↓F* = *P* by Proposition 3.33.

Conversely, for any fixed *x ∈ P* , if *x ≤ y* for all *y ∈ P* , then *x* is the least element of *P* , and thus *x* “ *x*; If *x* is not the least element of *P* , then there exists *y ∈ P* such that *x* ¢ *y*, thus there exist *u ∈ P* and a finite subset *F* of *P* such that *y ∈/ ↑u, x ∈/ ↓F* and *↑u ∪ ↓F* = *P* , implies *x ∈ P\↓F ⊆ ↑u*, i.e., *u* “ *x*. Hence

*↓*“*x /*= *∅* for all *x ∈ P* . Since *P* is a sup-semilattice, we have *↓*“*x* is directed for all

*x ∈ P* .

We now need to prove that *x* = sup *↓*“*x* for all *x ∈ P* . Suppose there exists *x ∈ P* such that *x /*= sup *↓*“*x*, i.e., there is *y ∈* (*↓*“*x*)*u*, such that *x* ¢ *y*. Then there exist *u ∈ P* and a finite subset *F* of *P* such that *y ∈/ ↑u, x ∈/ ↓F* and *↑u ∪ ↓F* = *P* , i.e., *x ∈ P\↓F ⊆ ↑u* implies *u* “ *x*. Thus *y ∈ ↑u*, this is a contradiction. *2*

From Theorem 3.19, Propositions 3.32, 3.34 and Corollary 3.31 we have the following.

**Theorem 3.35** *Let P be a sup-semilattice. Then*

1. *P is s*2*-CD implies P is hypercontinuous.*
2. *P is s*2*-C-continuous and hypercontinuous if and only if P is s*2*-CD.*

# *s*2-QC-continuous posets

For a set *X*, we use *P*(*X*) to denote the power set of *X*. We consider the order between subsets *G, H* of a poset *P* by *G ≤ H* if *H ⊆ ↑G*. This implies that a family *F* of subsets is *directed* if the corresponding family *{↑F | F ∈ F}* is a filter base. Generalizing the way below relation2 on points of a poset *P* to the nonempty subsets of *P* , one obtains the following concept of *s*2-quasicontinuous poset.

**Definition 4.1** [[13](#_bookmark13)] Let *P* be a poset.

1. The way-below relation2 on *P*(*P* )*\{∅}* is defined by *G* 2*H* for all *G, H ⊆ P* , if for all directed subsets *D ⊆ P* with *↑H ∩ Dδ /*= *∅*, which implies that *↑G ∩ D /*= *∅*. We write *G* 2*x* for *G* 2*{x}*. The set *{x ∈ P | H* 2*x}* will be denoted *⇑H*.
2. *P* is called *s*2-quasicontinuous if for all *x ∈ P* , the family

*Qƒin*(*x*)= *{F ⊆ P | F is finite and F* 2*x}*

is a directed family and whenever *x* ¢ *y*, there exists a finite subset *F ∈ Qƒin*(*x*) with *y ∈/ ↑F* , i.e., *↑x* = *{↑F | F ∈ Qƒin*(*x*)*}*.

1. *P* is called *s*2-quasialgebraic if for all *x ∈ P* , the family

*KQƒin*(*x*)= *{F ⊆ P | F is finite, F* 2*F and x ∈ ↑F }*

is a directed family and whenever *x* ¢ *y*, there exists *F ∈ KQƒin*(*x*) with *y ∈/ ↑F* , i.e., *↑x* = *{↑F | F ∈ KQƒin*(*x*)*}*.

**Remark 4.2** Let *P* be a sup-semilattice. Then *F*12*x* and *F*22*x* imply *F*1 *∨ F*22*x*, where *F*1*, F*2 are the nonempty finite subsets of *P* and *F*1 *∨ F*2 = *{y*1 *∨ y*2 *| yi ∈ Fi, i* = 1*,* 2*}*.

**Lemma 4.3** *[*[*13*](#_bookmark13)*] Let F be a directed family of nonempty ﬁnite sets in a poset. If*

*H* 2*x and F∈F ↑F ⊆ ↑x, then F ⊆ ↑H for some F ∈ F.*

**Proposition 4.4** *Let P be a poset. If H* 2*x and if x ∈ Dδ for a directed subset D ∈ P which Qƒin*(*d*) *is directed and ↑d* = *{↑F | F ∈ Qƒin*(*d*)*} for all d ∈ D, then H* 2*d for some element d ∈ D. Further, If P is s*2*-quasicontinuous, then ⇑H is σ*2*-open for all H ⊆ P.*

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**Proof.** Let *F* = *{Qƒin*(*d*) *| d ∈ D}*. Then for any *F*1*, F*2 *∈ F*, there exist *d*1*, d*2 *∈ D* such that *Fi* 2*di,i* = 1*,* 2. Since *D* is directed, so there is *d ∈ D* such that *F*1*, F*22*d*, i.e., *F*1*, F*2 *∈ Qƒin*(*d*), which implies that *F ⊆ ↑F*1 *∩ ↑F*2 for some *F ∈ Qƒin*(*d*). Thus *F* is a directed family. Since *F∈F ↑F* = *d∈D ↑d* = *Du ⊆ ↑x* and *H* 2*x*, so there exists *F ∈F* such that *F ⊆ ↑H* by Lemma 4.3. For *F* , there is *d ∈ D* such that *F* 2*d*, and thus *H* 2*d*. *2*

**Definition 4.5** [[7](#_bookmark7)] Let *P* be a poset. For any two subsets *G* and *H* of *P* , we write *G≺H*, if for each nonempty Scott-closed subset *S ⊆ P* for which sup *S* exists, sup *S ∈ ↑H* implies *S ∩ ↑G /*= *∅*. We write *G≺x* for *G≺{x}*.

Given a poset *P* , we now define a new binary relation on *P*(*P* ) which is crucial for us to formulate the properties of lattices of *σ*2-closed sets.

**Definition 4.6** Let *P* be a poset. For any two subsets *G* and *H* of *P* , we write *G≺*2*H*, if for each nonempty *σ*2-closed subset *S ⊆ P* , *Sδ∩↑H /*= *∅* implies *S∩↑G /*= *∅*. We write *G≺*2*x* for *G≺*2*{x}* and *y≺*2*H* for *{y}≺*2*H*. The set *{x ∈ P | H≺*2*x}* will be denoted *↑≺H*.

**Remark 4.7** (i) If *P* is a poset, then *G≺*2*H* implies *G ≺ H* for all *G, H ⊆ P* .

(ii) If *P* is a complete lattice, then *G≺*2*H* is equal to *G ≺ H* for all *G, H ⊆ P* .

The next proposition is basic and the proof is omitted.

**Proposition 4.8** *Let P be a poset and E, F, G, H ⊆ P. Then the following state- ments hold:*

1. *G≺*2*H implies G ≤ H;*
2. *G≺*2*H if and only if G≺*2*h for all h ∈ H;*
3. *E ≤ G≺*2*H ≤ F implies E≺*2*F;*
4. *{x}≺*2*{y} if and only if x≺*2*y;*
5. *if P has a smallest element* 0*, then* 0*≺*2*H always holds.*

**Definition 4.9** Let *P* be a poset.

* 1. *P* is called *s*2-quasi-C-continuous (for short, *s*2-QC-continuous), if for all *x ∈ P* ,

*↑x* = *{↑F | F ∈ Cƒin*(*x*)*},*

where *Cƒin*(*x*)= *{F ⊆ P | F is finite and F≺*2*x}*;

* 1. *P* is called *s*2-quasi-C-prealgebraic (for short, *s*2-QC-prealgebraic), if for all

*x ∈ P* ,

*↑x* = *{↑F | F ∈ KCƒin*(*x*)*},*

where *KCƒin*(*x*)= *{F ⊆ P | F is finite, F≺*2*F and x ∈ ↑F }*.

**Proposition 4.10** *Every s*2*-C-continuous (resp., s*2*-C-prealgebraic) poset is a s*2*- QC-continuous (resp., s*2*-QC-prealgebraic) poset.*

**Proof.** Suppose *P* is a *s*2-C-continuous poset and *x ∈ P* . Then *{{y}⊆ P | y≺*2*x}⊆*

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*C* (*x*) and *{↑y | y≺ x}* = *↑x*. Therefore, *↑x ⊆ {↑F | F ∈ C* (*x*)*} ⊆*

*{↑y | y≺*2*x}* = *↑x*. Hence *P* is *s*2-QC-continuous. The *s*2-C-prealgebraic case can be similarly proved. *2*

**Definition 4.11** Let *P* be a poset.

1. The binary relation *a*2 on *P*(*P* ) is defined by *Ga*2*H* for all *G, H ⊆ P* , if for all subsets *A ⊆ P* with *Aδ ∩ ↑H /*= *∅*, which implies that *↑G ∩ A /*= *∅*. We write *Ga*2*x* for *Ga*2*{x}* and *ya*2*H* for *{y}a*2*H*. The set *{x ∈ P | Ha*2*x}* will be denoted *↑aH*.
2. *P* is called *s*2-generalized completely distributive (for short, *s*2-GCD), if for all

*x ∈ P* ,

*↑x* = *{↑F | F ∈ Gƒin*(*x*)*},*

where *Gƒin*(*x*)= *{F ⊆ P | F is finite and Fa*2*x}*;

1. *P* is called *s*2-strongly pseudoalgebraic (for short, *s*2-SPA), if for all *x ∈ P* ,

*↑x* = *{↑F | F ∈ KGƒin*(*x*)*},*

where *KGƒin*(*x*)= *{F ⊆ P | F is finite, Fa*2*F and x ∈ ↑F }*.

**Remark 4.12** (i) *Ga*2*H* if and only if *↑H ∩* (*P\↑G*)*δ* = *∅*, for all *G, H ⊆ P* ;

1. If *P* is a *s*2-GCD poset which is also a complete lattice, then *P* is a GCD lattice ([[11](#_bookmark11)]).

**Theorem 4.13** *Let P be a sup-semilattice. Then the following are equivalent:*

* 1. *P is s*2*-QC-continuous and s*2*-quasicontinuous;*
  2. *P is s*2*-GCD.*

**Proof.** (2) *⇒* (1) follows immediately from Definitions 4.1, 4.9, 4.11 and Re- mark 4.2. For (1) *⇒* (2): Suppose that *P* is both *s*2-QC-continuous and *s*2- quasicontinuous. Since *P* is *s*2-quasicontinuous, for each *x ∈ P* , *↑x* = *{↑F | F ∈*

*Qƒin*(*x*)*}*. Now for each *F ∈ Qƒin*(*x*),

*↑F* = *x∈F* *{↑Ex | Ex ∈ Cƒin*(*x*)*}* = *{↑*( *x∈F Ex*) *| Ex ∈ Cƒin*(*x*) *and x ∈ F}*

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because *P* is *s*2-QC-continuous. By Proposition 4.8, we have *{ x∈F Ex | Ex ∈*

*Cƒin*(*x*) *and x ∈ F} ⊆ {E | E is finit and E≺*2*F}*. Therefore, *↑F* =

*{↑E | E is finit and E≺*2*F}*. It follows that

*↑x* = *{↑E | there exists F ∈ Qƒin*(*x*) *such that E≺*2*F* 2*x}.*

Next, suppose *E≺*2*F* 2*x*, we shall show that *Ea*2*x*. Let *A ⊆ P* with *x ∈ Aδ*. Construct the set *D* = *{*sup *G | G is a finite subset of A}*. Then *D* is a directed set and *x ∈ Aδ* = *Dδ*. Since *F* 2*x*, so there is a finite subset *G ⊆ A* such that sup *G ⊆ ↑F* , i.e., (*↓G*)*δ ∩ ↑F /*= *∅*. Note that the last set *↓G* is *σ*2-closed. So it follows from *E≺*2*F* that *↑E ∩ ↓G /*= *∅*, i.e., *↑E ∩ A ⊇ ↑E ∩ G /*= *∅*, this implies that *Ea*2*x*. Hence *P* is *s*2-GCD. *2*

For the algebraic case, similarly, we have the following statements:

**Theorem 4.14** *Let P be a sup-semilattice. Then the following are equivalent:*

* + 1. *P is s*2*-QC-prealgebraic and s*2*-quasialgebraic;*
    2. *P is s*2*-SPA.*

**Proposition 4.15** *A poset P is s*2*-GCD if and only if for any x, y ∈ P with x* ¢ *y, there exists a ﬁnite subset F of P such that y ∈/ ↑F and x ∈/* (*P\↑F* )*δ.*

**Proof.** Suppose *P* is *s*2-GCD and *x, y ∈ P* with *x* ¢ *y*. Then there exists a finite subset *F ⊆ P* such that *y ∈/ ↑F* and *F a*2*x*. Thus *x ∈/* (*P\↑F* )*δ* by Remark 4.12(i).

Conversely, for any fixed *x ∈ P* , if *x ≤ y* for all *y ∈ P* , then *x* is the least element of *P* , and thus *{x} ∈ Gƒin*(*x*); If *x* is not the least element of *P* , then there exists *y ∈ P* such that *x* ¢ *y*, thus there exists a finite subset *F* of *P* such that *y ∈/ ↑F* and *x ∈/* (*P\↑F* )*δ*, implies *F ∈ Gƒin*(*x*) by Remark 4.12(i). Hence *Gƒin*(*x*) */*= *∅* for all *x ∈ P* .

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We now need to prove that *↑x* = *{↑F | F ∈ G* (*x*)*}* for all *x ∈ P* . Suppose there exists *x ∈ P* such that *↑x /*= *{↑F | F ∈ G* (*x*)*}*, i.e., there exists *y ∈*

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*{↑F | F ∈ Gƒin*(*x*)*}*, such that *x* ¢ *y*. Then there exists a finite subset *F* of *P* such that *y ∈/ ↑F* and *x ∈/* (*P\↑F* )*δ*, i.e., *F ∈ Gƒin*(*x*) by Remark 4.12(i). Thus *y ∈ ↑F* , this is a contradiction. *2*

Proposition 4.15 and Lemma 3.29 together imply the following corollary.

**Corollary 4.16** *A poset P is s*2*-GCD if and only if for any x, y ∈ P with x* ¢ *y, there exist u ∈ P and a ﬁnite subset F of P such that x ∈/ ↓u, y ∈/ ↑F and ↓u∪ ↑F* = *P.*

From Proposition 3.34 and Corollary 4.16 we have the following.

**Corollary 4.17** *A semi-lattice P is s*2*-GCD if and only if Pop is hypercontinuous.*

It is easy to see that for a finite poset *P* , *Pop* is hypercontnious, hence by Corollary 4.17, *P* is *s*2-GCD, especially *P* is a *s*2-QC-continuous poset by Definitions

4.9 and 4.11. By this observation, we have

**Corollary 4.18** *Every ﬁnite poset is a s*2*-QC-continuous poset.*

Observe that a *s*2-QC-continuous poset is generally not actually a *s*2-C- continuous poset.

**Example 4.19** Consider the poset *P* = *{a, b, x, y, z}*, where the order is defined by *x, z ≤ a, b* and *y ≤ x*. Since *P* is a finite poset, so *P* is *s*2-QC-continuous by Corollary 4.18. Let *S*1 = *↓x* and *S*2 = *↓y ∪ ↓z*. Then *S*1 and *S*2 are *σ*2-closed by Remark 2.3(i), and

(*S*1 *∩ S*2)*δ* = (*↓x ∩* (*↓y ∪ ↓z*))*δ* = (*↓y*)*δ* = *↓y,*

as well as

*Sδ ∩ Sδ* = (*↓x*)*δ ∩* (*↓y ∪ ↓z*)*δ* = *↓x ∩* (*↓x ∪ ↓z*)= *↓x.*

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Thus (*S*1 *∩ S*2)*δ /*= *Sδ ∩ Sδ*, hence *P* is not *s*2-C-continuous by Proposition 3.13.

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*a b*

*x z*

*y*

For a poset *P* , a binary relation “ on the set of all subsets of *P* is defined as follows: *G* “ *H* if and only if *H ⊆* int*υ↑G*. We write *G* “ *x* for *G* “ *{x}* and *y* “ *H* for *{y}* “ *H*. Note that *x* “ *y* is unambiguously defined. A poset *P* is called *quasi*-*hypercontinuous* if for each *x ∈ P* the family

*Hƒin*(*x*)= *{F ⊆ P | F is finite and F* “ *x}*

is a directed family and whenever *x* ¢ *y*, then there exists a finite subset *F ∈ Hƒin*(*x*) with *y ∈/ ↑F* , i.e., *↑x* = *∩{↑F | F ∈ Hƒin*(*x*)*}*.

**Proposition 4.20** *Let P be a poset. The following statements are equivalent:*

1. *P is quasi-hypercontinuous;*
2. *P is s*2*-quasicontinuous in which H* 2*x implies H* “ *x.*

**Proof.** Suppose *P* is a quasi-hypercontinuous poset and *H* 2*x*. Then there exists a finite subset *F ∈ P* such that *H ≤ F* “ *x*. If *H* “ *x*, then *x ∈* int*υ↑H ⊆* int*σ*2 *↑H*, i.e., *H* 2*x*. Hence *P* is *s*2-quasicontinuous.

Conversely if (2) holds, then “ is approximating since2 is. Hence *P* is hyper- continuous. *2*

**Proposition 4.21** *Let P is a quasi-hypercontinuous poset. Then for any x, y ∈ P*

*with x* ¢ *y, there exist ﬁnite subsets F*1*, F*2 *of P such that y ∈/ ↑F*1*,x ∈/ ↓F*2 *and*

*↑F*1 *∪ ↓F*2 = *P.*

**Proof.** Suppose *P* is quasi-hypercontinuous and *x, y ∈ P* with *x* ¢ *y*. Then there exists a finite subset *F*1 *⊆ P* such that *y ∈/ ↑F*1 and *F*1 “ *x*. Since *F*1 “ *x* if and only if *x ∈* int*υ↑F*1, so there is a finite subset *F*2 *⊆ P* such that *x ∈ P\↓F*2 *⊆ ↑F*1, i.e., *x ∈/ ↓F*2 and *↑F*1 *∪ ↓F*2 = *P* . *2*

**Proposition 4.22** *A sup-semilattice P is quasi-hypercontinuous if and only if for any x, y ∈ P with x* ¢ *y, there exist ﬁnite subsets F*1*, F*2 *of P such that y ∈/ ↑F*1*, x ∈/ ↓F*2 *and ↑F*1 *∪ ↓F*2 = *P.*

**Proof.** Suppose that *P* is quasi-hypercontinuous. It is obvious that for any *x, y ∈ P*

with *x* ¢ *y*, there exist finite subsets *F*1, *F*2 of *P* such that *y ∈/ ↑F*1, *x ∈/ ↓F*2 and

*↑F*1 *∪ ↓F*2 = *P* by Proposition 4.21.

Conversely, for any fixed *x ∈ P* , if *x ≤ y* for all *y ∈ P* , then *x* is the least element of *P* , and thus *{x} ∈ Hƒin*(*x*); If *x* is not the least element of *P* , then there exists *y ∈ P* such that *x* ¢ *y*, thus there exist finite subsets *F*1, *F*2 of *P*

such that *y ∈/*

*↑F*1*,x ∈/*

*↓F*2 and *↑F*1 *∪ ↓F*2 = *P* , implies *x ∈ P\↓F*2 *⊆ ↑F*1, i.e.,

*F*1 *∈ Hƒin*(*x*). Hence *Hƒin*(*x*) */*= *∅* for all *x ∈ P* . Since *P* is a sup-semilattice, we have *{↑F | F ∈ Hƒin*(*x*)*}* is filtered for all *x ∈ P* .

*ƒin*

We now need to prove that *↑x* = *{↑F | F ∈ H* (*x*)*}* for all *x ∈ P* . Sup- pose there exists *x ∈ P* such that *↑x /*= *{↑F | F ∈ H* (*x*)*}*, i.e., there is *y ∈* *{↑F | F ∈ Hƒin*(*x*)*}*, such that *x* ¢ *y*. Then there exist finite subsets *F*1,

*F*2 of *L* such that *y ∈/*

*↑F*1*,x ∈/*

*↓F*2 and *↑F*1 *∪ ↓F*2 = *P* , i.e., *x ∈ P\↓F*2 *⊆ ↑F*1

*ƒin*

implies *F*1 *∈ Hƒin*(*x*). Thus *y ∈ ↑F*1, this is a contradiction. *2*

From Theorem 4.13, Propositions 4.20, 4.22 and Corollary 4.16 we have the following.

**Theorem 4.23** *Let P be a sup-semilattice. Then*

1. *P is s*2*-GCD implies P is quasi-hypercontinuous.*
2. *P is s*2*-QC-continuous and quasi-hypercontinuous if and only if P is s*2*-GCD.*

**Lemma 4.24** *[*[*13*](#_bookmark13)*]A poset P is s*2*-quasicontinuous if and only if the lattice σ*2(*P* )

*of all σ*2*-open sets is hypercontinuous.*

So, in view of Corollary 4.17 and Remark 4.12(ii), a poset *P* is *s*2-quasicontinuous if and only if *C*2(*P* ) is a *GCD* lattice.

**Theorem 4.25** *For any poset P, the following statements are equivalent:*

1. *P is a s*2*-quasicontinuous poset;*
2. *σ*2(*P* ) *is a hypercontinuous lattice;*
3. *C*2(*P* ) *is a GCD lattice;*
4. *C*2(*P* ) *is a quasicontinuous lattice.*

**Proof:** (1) *⇔* (2) By Lemma 4.24. (2) *⇔* (3) By Remark 4.12(ii) and Corollary

4.17. (3) *⇔* (4) By Propositions 3.27, 4.10, 4.15 and Remark 4.12(ii). *2*

# *s*2-MC-continuous posets

We first give a definition of the *σ*2-C-set of a poset *P* , and then introduce the notion of *s*2-meet-C-continuous (for short, *s*2-MC-continuous) poset by the *σ*2-C-set.

**Definition 5.1** Let *P* be a poset. A subset *A ⊆ P* is called *σ*2-C-set if it satisfies:

1. *A* = *↓A*;
2. *S ⊆ A* implies *Sδ ⊆ A* for all *S ∈ C∗*(*P* ).

2

The collection of all *σ*2-C-sets of *P* will be denoted by *SC*2(*P* ) and let *SO*2(*P* )=

*{U ⊆ P | P\U ∈ SC*2(*P* )*}*.

**Proposition 5.2** *Let P be a poset. Then the following statements hold:*

1. *L, ∅∈ SC*2(*P* )*;*
2. *↓x ∈ SC*2(*P* ) *for all x ∈ P;*

1. *For any the family {Ai | i ∈ I}⊆ SC*2(*P* )*, i∈I Ai ∈ SC*2(*P* )*;*
2. *Aδ ∈ SC*2(*P* ) *for all A ⊆ P;*
3. *U ∈ SO*2(*P* ) *if and only if U* = *↑U and Sδ ∩ U /*= *∅ implies S ∩ U /*= *∅ for all*

*S ∈ C∗*(*P* )*.*

2

**Proof.** This follows immediately from Definition 5.1 and Lemma 2.1. *2*

**Remark 5.3** For a poset *P* and *A*1*, A*2 *∈ SC*2(*P* ), in general *A*1 *∪ A*2 *∈ SC*2(*P* ) does not hold. Thus the dual *SO*2(*P* ) of *SC*2(*P* ) cannot compose a topology on *P* . But *SO*2(*P* ) can compose a semi-topology of *P* by Proposition 5.2.

**Definition 5.4** A poset *P* is *s*2-meet-C-continuous (for short, *s*2-MC-continuous), if for any *x ∈ P* and any *S ∈ C∗*(*P* ) with *x ∈ Sδ*, then *x ∈ {A ∈ SC*2(*P* ) *| ↓x∩S ⊆*

2

*A}*.

**Proposition 5.5** *If a poset P is s*2*-C-continuous, then it is also s*2*-MC-continuous.*

**Proof.** Suppose that *P* is a *s*2-C-continuous poset. Then for any *x ∈ P* and *S ∈ C∗*(*P* ) with *x ∈ Sδ*, we know that *↓≺x ⊆ S∩↓x*. Therefore, for any *A ∈ SC*2(*P* ) with *↓x ∩ S ⊆ A*, we have *↓≺x ⊆ A*. Hence by Corollary 3.9 and Definition 5.1, we have *x ∈* (*↓≺x*)*δ ⊆ A*. Thus *x ∈ {A ∈ SC*2(*P* ) *| ↓x ∩ S ⊆ A}*, that is, *P* is *s*2-MC-continuous. *2*

2

**Proposition 5.6** *Let P be a s*2*-MC-continuous poset. Then for any x, y, z ∈ P, the following equation hold:*

(*↓x ∩* (*↓y ∪ ↓z*))*δ* = *↓x ∩* (*↓y ∪ ↓z*)*δ.*

**Proof.** Since *↓x ∩* (*↓y ∪ ↓z*) *⊆ ↓y ∪ ↓z* and *↓x ∩* (*↓y ∪ ↓z*) *⊆ ↓x*, we know that

(*↓x ∩* (*↓y ∪ ↓z*))*δ ⊆ ↓x ∩* (*↓y ∪ ↓z*)*δ,*

by Lemma 2.1. Conversely, for any *u ∈ ↓x ∩* (*↓y ∪ ↓z*)*δ*, we know that *↓u ⊆ ↓x* and

*u ∈* (*↓y ∪ ↓z*)*δ*. Thus *↓u ∩* (*↓y ∪ ↓z*) *⊆ ↓x ∩* (*↓y ∪ ↓z*) *⊆* (*↓x ∩* (*↓y ∪ ↓z*))*δ*. Since

*↓y ∪ ↓z ∈ C∗*(*P* ) and (*↓x ∩* (*↓y ∪ ↓z*))*δ ∈ SC*2(*P* ), so *u ∈* (*↓x ∩* (*↓y ∪ ↓z*))*δ*. *2*

2

**Corollary 5.7** *Let P be a s*2*-MC-continuous poset. If P is also a lattice, then P*

*is a distributive lattice.*

**Proof.** For any *x, y, z ∈ P* ,

*↓x ∩* (*↓y ∪ ↓z*)= (*↓x ∩ ↓y*) *∪* (*↓x ∩ ↓z*)= *↓*(*x ∧ y*) *∪ ↓*(*x ∧ z*)*.*

Thus (*↓x ∩* (*↓y ∪ ↓z*))*δ* = *↓*((*x ∧ y*) *∨* (*x ∧ z*)) by Lemma 2.1(5). Since *↓x ∩* (*↓y ∪ ↓z*)*δ* = *↓*(*x ∧* (*y ∨ z*)) by Lemma 2.1(5), and (*↓x ∩* (*↓y ∪ ↓z*))*δ* = *↓x∩*(*↓y ∪ ↓z*)*δ* by Proposition 5.6, we know that *↓*((*x ∧ y*) *∨* (*x ∧ z*)) = *↓*(*x ∧* (*y ∨ z*)), that is, ((*x ∧ y*) *∨* (*x ∧ z*)) = (*x ∧* (*y ∨ z*)). *2*

**Theorem 5.8** *A poset P is s*2*-MC-continuous if and only if for any U ∈ SO*2(*P* )

*and any x ∈ P, ↑*(*U ∩ ↓x*) *∈ SO*2(*P* )*.*

**Proof.** Suppose that *P* is a *s*2-MC-continuous poset, *x ∈ P* and *U ∈ SO*2(*P* ) with *Sδ ∩ ↑*(*U ∩ ↓x*) */*= *∅* for some *S ∈ C∗*(*P* ). Then there exists *z ∈ U ∩ ↓x* such that *z ∈ Sδ*. Assume that *S ∩ ↓z ∩ U* = *∅*. Then *S ∩ ↓z ⊆ P\U ∈ SC*2(*P* ). Since *P* is *s*2-MC-continuous, so *z ∈ P\U* , this is a contradiction. Thus *S ∩ ↓z ∩ U /*= *∅*, which implies that *S ∩ ↑*(*U ∩ ↓x*) *⊇ S ∩ ↑*(*U ∩ ↓z*) */*= *∅*. Therefore, *↑*(*U ∩ ↓x*) *∈ SO*2(*P* ).

2

Conversely, assume that there exist *x ∈ P, S ∈ C∗*(*P* ) with *x ∈ Sδ* and *A ∈ SC*2(*P* ) with *↓x ∩ S ⊆ A* such that *x ∈/ A*. Then *x ∈ P\A* = *U ∈ SO*2(*P* ), and thus *x ∈ Sδ ∩ U ∩ ↓x* implies *Sδ ∩ ↑*(*U ∩ ↓x*) */*= *∅*. Since *↑*(*U ∩ ↓x*) *∈ SO*2(*P* ), so *S ∩ ↑*(*U ∩ ↓x*) */*= *∅* implies *S ∩ U ∩ ↓x /*= *∅*, this is a contradiction. Therefore, for any *x ∈ P* and any *S ∈ C∗*(*P* ) with *x ∈ Sδ*, *x ∈ {A ∈ SC*2(*P* ) *| ↓x ∩ S ⊆ A}*, that is, *P* is *s*2-MC-continuous. *2*

2

2

**Corollary 5.9** *A poset P is s*2*-MC-continuous if and only if for any U ∈ SO*2(*P* )

*and any nonempty subset A ⊆ P, ↑*(*U ∩ ↓A*) *∈ SO*2(*P* )*.*

**Proof.** This follows immediately from Theorem 5.8. *2*

**Theorem 5.10** *For any poset P, the following statements are equivalent:*

* 1. *P is a s*2*-MC-continuous poset;*
  2. *↓x ∩ Sδ* = (*↓x ∩ S*)*δ for all x ∈ P and S ∈ C∗*(*P* )*;*

2

* 1. *Sδ ∩ Sδ* = (*S*1 *∩ S*2)*δ for all S*1*, S*2 *∈ C∗*(*P* )*;*

1 2 2

* 1. *x ∈ Sδ implies x* = sup(*↓x ∩ S*) *for all x ∈ P and S ∈ C∗*(*P* )*.*

2

**Proof.** (1) *⇒* (2) Since *↓x ∩ S ⊆ S* and *↓x ∩ S ⊆ ↓x*, we know that (*↓x ∩ S*)*δ ⊆*

*↓x ∩ Sδ*, by Lemma 2.1. Conversely, for any *u ∈ ↓x ∩ Sδ*, we have *↓u ⊆ ↓x* and

*u ∈ Sδ*. Thus *↓u∩S ⊆ ↓x∩S ⊆* (*↓x∩S*)*δ*. Since *S ∈ C∗*(*P* ) and (*↓x∩S*)*δ ∈ SC*2(*P* ),

2

so *u ∈* (*↓x ∩ S*)*δ*. Hence *↓x ∩ Sδ* = (*↓x ∩ S*)*δ*.

1. *⇔* (3) Suppose (3) holds. Then (2) is also holds, because *↓x ∈ C∗*(*P* ) and

2

(*↓x*)*δ* = *↓x* for all *x ∈ P* . Suppose (2) holds. Then

*δ δ*

*S ∩ S* =

* 1. 2 *x∈Sδ*

1

=

*x∈Sδ*

*↓x* *∩ Sδ* = (*↓x ∩ Sδ*)

1

*δ*

1

(*↓x ∩ S*2)*δ ⊆*

2 *x∈Sδ* 2

*x∈Sδ*

(*↓x ∩ S*2)

=

1

*x∈Sδ*

1

*↓x*

*δ*

*∩ S*2

= *Sδ ∩*

*x∈S*2

1

*↓x* *δ*

=

(*Sδ ∩ ↓x*) *δ* =

(*S ∩ ↓x*)*δ* *δ*

*x∈S*2 1

*x∈S*2

*⊆*

*S*1 *∩*

*δ* *δ*

*↓x*

*x∈S*2 1

= ((*S*1 *∩ S*2)*δ*)*δ* = (*S*1 *∩ S*2)*δ.*

Since *S*1 *∩S*2 *⊆ Si, i* = 1*,* 2, so (*S*1 *∩S*2)*δ ⊆ Sδ, i* = 1*,* 2. Thus (*S*1 *∩S*2)*δ ⊆ Sδ ∩Sδ*.

*i* 1 2

(2) *⇒* (4) For any *x ∈ P* and *S ∈ C∗*(*P* ) with *x ∈ Sδ*, by (2), we have *↓x* =

2

*↓x ∩ Sδ* = (*↓x ∩ S*)*δ*. Thus *x* = sup(*↓x ∩ S*).

(4) *⇒* (1) For any *x ∈ P, S ∈ C∗*(*P* ) with *x ∈ Sδ*, by Lemma 3.12 we have

2

*↓x ∩ S ∈ C∗*(*P* ) because *↓x* = (*↓x ∩ S*)*δ /*= *∅*. Thus for every *A ∈ SC*2(*P* ) with

2

*↓x ∩ S ⊆ A*, we know that *x ∈* (*↓x ∩ S*)*δ ⊆ A*. Hence *P* is *s*2-MC-continuous. *2*

**Lemma 5.11** *Let P be a s*2*-MC-continuous poset, x ∈ P and F ⊆ P ﬁnite. Then*

*F≺*2*x if and only if there exists y ∈ F such that y≺*2*x.*

**Proof:** Suppose *F≺*2*x*. Assume that *x ∈/ y∈F ↑≺y*. Then by Definition 3.5, we know that for each *y ∈ F* , there exists *Sy ∈ C∗*(*P* ) such that *x ∈ Sδ* and *y ∈/ Sy*.

S

* 1. *y*

Let *S* = *Sy*. Then *x ∈* *Sδ* = *Sδ* by Theorem 5.10, and *S ∈ C∗*(*P* ) by

*y∈F*

*y∈F*

*y*

2

Lemma 3.12. Since *F≺*2*x*, so *↑F ∩ S /*= *∅*. Thus there exists *y*0 *∈ F* such that

*y*0 *∈ S ⊆ Sy*0 , this is a contradiction. Therefore, there exists *y ∈ F* such that *y≺*2*x*.

Conversely, if there exists *y ∈ F* such that *y≺*2*x*, then for any *S ∈ C∗*(*P* ) with

2

*x ∈ Sδ*, *y ∈ S*, which implies that *↑F ∩ S /*= *∅*. Thus *F≺*2*x*. *2*

**Theorem 5.12** *Let P be a poset. Then P is s*2*-C-continuous if and only if P is both s*2*-MC-continuous and s*2*-QC-continuous.*

**Proof.** Suppose that *P* is *s*2-C-continuous. Then *P* is both *s*2-MC-continuous and

*s*2-QC-continuous by Propositions 4.10 and 5.5.

Suppose that *P* is both *s*2-MC-continuous and *s*2-QC-continuous. Assume that there exist *x ∈ P* and *z ∈* (*↓≺x*)*u* such that *x* ¢ *z*. Since *P* is *s*2-QC-continuous,

so there exists a finite subset *F* of *P* such that *F≺*2*x* and *z ∈/ ↑F* . By Lemma

5.1, there exists *y ∈ F* such that *y≺*2*x*, because *P* is *s*2-MC-continuous. Therefore *y ≤ z*, this is a contradiction. Hence *x* = sup *↓≺x* since *x ∈* (*↓≺x*)*u* for all *x ∈ P* , i.e., *P* is *s*2-C-continuous. *2*

# Acknowledgements

This work is supported by National Natural Science Foundation of China (No.11371130, 11611130169) and the Research Fund for the Doctoral Program of Higher Education of China (No.20120161110017).

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