Available online at [www.sciencedirect.com](http://www.sciencedirect.com/)



AASRI Procedia 1 (2012) 299 – 304

AASRI

Procedia

[www.elsevier.com/locate/procedia](http://www.elsevier.com/locate/procedia)

2012 AASRI Conference on Computational Intelligence and Bioinformatics

# The Average Errors for Bernstein-Kantorovich Operators on the r-fold Integrated Wiener Space

Liu Ting, Jiang Yanjie\*

*Department of Mathematics and Physics, North China Electric Power University, Baoding 071003, People's Republic of China*

**Abstract**

In this paper, we discuss the average errors of function weighted approximation by the Bernstein-Kantorovich operators. The strongly asymptotic orders for the average errors of the Bernstein-Kantorovich operators sequence are determined on the r-fold integrated Wiener Space.

© 2012 Published by Elsevier B.V. Open access under [CC BY-NC-ND license.](http://creativecommons.org/licenses/by-nc-nd/3.0/)

Selection and/or peer review under responsibility of American Applied Science Research Institute

*Keywords: Bernstein-Kantorovich operators; weighted Lp-norm; r-fold integrated Wiener space; average error.*

### Introduction

Let *F* be a real separable Banach space equipped with a probability measure ** on the Borel sets of *F*. Let

*X* be another normed space such that *F* is continuously embedded in *X*. By ||.|| we denote the norm in X. Any

*T* : *F*  *X* such that *f* || *f*  *T* ( *f* ) || is a measurable mapping is called an approximation operator. The p-

average error of *T* is defined as

*ep* (*T* ,  , *F*, **)  *F*

 *f*  *T* ( *f* )*p *(*df* )*p* .

1

\* Corresponding author. Tel.: +86-0312-7523326.

*E-mail address:* [jiangyj@126.com.](mailto:jiangyj@126.com)

2212-6716 © 2012 Published by Elsevier Ltd. Open access under [CC BY-NC-ND license.](http://creativecommons.org/licenses/by-nc-nd/3.0/)

doi:10.1016/j.aasri.2012.06.046

Let

*F*0   *f*  *C* 0,1 : *f* (0)  0. For every

*f*  *F*0 set

 *f C*  max0*t* 1

*f* *t*  . Then

(*F*0 ,  *C* ) becomes a

separable Banach space. Denote by

*B*(*F*0 ) *see* 1.

*B*(*F*0 ) the Borel class of (*F*0 ,  *C* ) and by **0 the Wiener measure on

Let *r*  0 be an integer .For all *g*  *F*0 , define *T*0 *g* *t*   *g* *t*  ,and

*t*

Thus we have

*du*, *r*  1.

*Tr g* *t*   0 *g* *u* 

*t*  *u* *r* 1

*r* 1!

*Tr g* *x* *Fr*

  *f* *C**r* 0,1: *f* *k* 0  0, *k*  0,1,⋯, *r*.

It is well known that *Tr* is a bijective mapping from *F*0 to *Fr* . The *r* -fold integrated Wiener measure *r* on

*Fr* is defined by induced measure *r*  *Tr*0 , i.e., for *A*  *Fr* ,

*r*  *A*  **0 *g*,*Tr g*  *A* .

From [1] we know

*f* *s* *f* *t* ** *df*  

1 *s*  *u* *r* *t*  *u* *r du*

  ,

(1)

*F r*

*r*

where *z*  *z* if *z*  0 and *z*  0 otherwise.

0 *r* !2

For **  *L*1[0,1], **  0 , the weighted *Lp* -norm of

*f* *C*[0,1] is defined by

*f*  *f*

1

 1 *f* *t*  *p*  ** *t*  *dt p* .





Let

 *p*, ** 0

*p*  *x*   *n*  *xk* 1 *x* *n* *k*  *Ck xk* 1 *x* *n*  *k* , *k*  0,1,⋯, *n*.

*n*,*k*   *n*

*k*

 

For

*f* *C*[0,1] the well-know Bernstein-Kantorovich polynomials of *f* is given *see*[2] by

*n*

*Kn*  *f* , *x*   *pn*,*k*

*k* 0

 *x**n*  1

*k* 1

*n* 1 *f* *t*  *dt*.

*n* 1

 *k*

(2)

### Main result

Many mathematicians have investigated the approximation behavior of Bernstein-Kantorovich operators on *Lp* [0,1],1  *p*   . Recently Xu Guiqiao [3] studied the simultaneous approximation average errors for Bernstein operators on the r-fold integrated Wiener space. Motivated by [3], we consider the average errors of function weighted approximation by the Bernstein-Kantorovich operators on the r-fold integrated Wiener space. We obtain the following:

**Theorem 1** Let 1  *p*   , *r*  1 , continuous on 1,1 , then we have

*Kn*  *f* , *x*

be given by (2). If

**  *L*1[0,1], **(*x*)  0 and

**  *x* is

*e* *K* ,  , *F* ,**   *C* *n* 11  *o**n*1 ,

where

*p n r r p*,** ,*r*

 1 



*x*2*r* 1 1 2*x* 2

*x*2*r* 1 1  *x* 2

1

*p p*

*x*2*r*1 1  2*x* 1  *x*   2 



*Cp*,** ,*r*  ** *p* 0  2 

2  2

 **  *x*  *dx* 

  4 2*r* 1*r* 1! 4 2*r*  3*r*  2! 4 *r* 1!  

and

   

**  1



2**

*p* 

 *x p e*

  *x*2

2 *dx*.

Here and in the following the notation



*an*  *o* *bn* 

for sequences *an* 

and *bn* 

means that

lim*n* *an bn*  0.

### Proof of Theorem 1

*Proof of Theorem 1*. From [1] we get

*p n r r p* 



1

*p*

*n r*



*ep* *K* , , *F* ,**   **  

*f*  *x*  *K* ( *f* , *x*) 2 ** *df*  2 **  *x* *dx*.

(3)

0 *Fr*

By (1) and (2), a direct computation shows

*n k* 1 *n*

*k* 1

*K*  *f* , *x*   *f*  *x*    *p*  *x* *n* 1 *n*1 *f* (*t*)*dt*   *p*

 *x* *n* 1

*n*1 *f* (*x*)*dt*

*n n*,*k*

*k* 0

*n*

 *k n*1

*k* 1

*k* 0

*n*,*k*

 *k n*1

 *n* 1  *pn*,*k*

 *k*

*k* 0

 *x* 

*n*1  *f* (*t*)  *f* (*x*)*dt*.

*n*1

For

*r*  1 , by Taylor formula we have

2 *f* ''  *x*

2  *f* '' **   *f* ''  *x* 

*f* *t*   *f*  *x*  *t*  *x* *f* '  *x*  *t*  *x*

where *k* is between in *t* and *x* . Hence

, *x*  *k*  1 

*n*  1 



 *t*  *x*   *k* ,

2  2 

(4)

*f* '' **   *f* ''  *x*   **  *f* '' , max  *x* 

*k*

*n*  1

*k*  

 

 

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *x*   *k*  *n*  1 |  | *x*  *k*  1  *n*  1 |  | *x*   *k*  *n*  1 |  | *x*  *k*  1  *n*  1 |

 

 **  *f* '' , 

 2 

 

 

 3 **  *f* '' , *x*  *k*   3 **  *f* '' , *x*  *k* 1   3 **  *f* '' , 1 ,

2  *n*  1  2  *n*  1  2  *n* 1 

     

where **  *f* , *t*  is the modulus of continuity of *f* in the uniform norm. Hence, by (4) and a simple computation we obtain

*n k* 1

 *k*

*n k* 1

2

 *k*

*f* ''  *x* 

*Kn*  *f* , *x*   *f*  *x*   *n* 1  *pn*,*k*

 *x* 

*n*1 *t*  *x*  *f* '  *x* *dt*  *n*  1  *p*

*n*,*k*

*x* 

*n*1 *t*  *x*  *dt*

2

*k* 0

*n*

*n*1

*k* 0

*k* 1 2  *f* '' **   *f* ''  *x*  

 *k*

*n*1

 *n* 1  *pn*,*k*

 *x* 

*n*1 *t*  *x*   *k*

2

 *dt*

(5)

*k* 0

*n*1  

 *I*1  *x*   *I*2  *x*   *I*3  *x* .

Note that

*n n n*

 *p*  *x*   1, *kp*  *x*   *nx*, *k* 2 *p*

*n*,*k n*,*k n*,*k*

 *x*   *n*2 *x*2  *nx* 1 *x* ,

*k* 0

*k* 0

*k* 0

a simple computation we obtain

*n*

*I*1  *x*   *n*  1  *p*

*k* 0

*n*,*k*

 *x* 

*k* 1

*n*1 *t*  *x*  *f* '  *x*  *dt*

 *k*

*n* 1

(6)

*f* '  *x* 

 2 *n*  1

*n*

2*k*  1 *p*

*k* 0

*n*,*k*

 *x*   *xf* '  *x*  

1  2*x*

2 *n*  1

*f* '  *x* ,

*n*

*I*2  *x*   *n*  1  *p*

*k* 0

*n*,*k*

 *x* 

*k* 1

*n*1 *t*  *x* 2

 *k*

*n* 1

*f* ''  *x* 

*dt*

2

 2 2*x*2  2*x*  1 

(7)

   *x*  *x*  3  *f* ''  *x* ,

 2 *n*  1

2 *n*  12 

and

 

 

*n k* 1

*f * 

''

 

*k*

*f x*

''

 

2

2

 *k*

*I*3  *x*  *n*  1  *pn*,*k*

*k* 0

 *x*

*n*1 *t*  *x* *dt*

*n*1

 3*n*  1

## 4

*n*

** 



*k* 0 

*f* '', *x* 



*pn*,*k*

*k*

*n*  1



*k*  1

*n*  1

 *x*

*k* 1

*n*1 *t*  *x*2 *dt*

 *k*

*n*1

 3*n*  1

## 4

** 

*k* 0 

*n*



*f* '', *x* 



*pn*,*k*



 *x*

*k* 1

*n*1 *t*  *x*2 *dt*

 *k*

*n*1

(8)

 3*n*  1

## 4

** 

*f* '' ,

## 1 

*n*  1 

*pn*,*k*

 *x*

*k* 1

*n*1 *t*  *x*2 *dt*

 *k*

*k* 0  

*n*



*n*1

From (5)-(8), we have

*n*





*C*

** *f*



''

1









,

12 *n*5 

 .

  *x*  *x*2 2

 2*x*2  2*x* 

1 2

 

 *x*  *x*2  2*x*2  2*x* 

1     

*K*  *f* , *x*   *f*  *x*  2  





  3    3   *f* '' 2  *x* 

*n*  4 *n*  12

4 *n*  14

2 *n*  13 

 

 

2  2 1  2*x*  2*x*2  2*x*  1  

1  2*x* 

 1 2*x*  *x*  *x*  

3  

 *f* '2  *x*   

    *f* '  *x*  *f* ''  *x* 

4 *n*  12

 2 *n*  12

2 *n*  13 

 

 

 2 2*x*2  2*x*  1 

 *I* 2  *x*   1 2*x f* '  *x*  *I*

 *x*     *x*  *x*  3  *f* ''  *x*  *I*

 *x* .

(9)

3 *n*  1

3  *n*  1

*n*  12  3

 

 

Let *f*  *Tr g* , a direct computation shows



 *x*  *x*2 2

 2*x*2  2*x* 

1 2  

*x*  *x*2  2*x*2  2*x* 

1     

   3    3   *f* ''2  *x* ** *df* 





*F* 2 4 3 *r*

*r*  4 *n*  1

4 *n*  1

2 *n*  1 

 

 



 *x*  *x*2 2

 2*x*2  2*x* 

1 2

 

*x*  *x*2  2*x*2  2*x* 

1     

  3   3  





 

*r* 2





2 4 3 *F*

*f* 2  *x* **

*df* 

(10)

 4 *n*  1

4 *n*  1

2 *n*  1

 *r*2

 

 

*x*2 *r* 1 1  *x* 2

 4 2*r*  3*r*  2!2 *n*  1

1 2*x*2

 *o*  1  ,

 

2  *n*2 

1 2*x*2 2

*F* 4 *n* 12

*r*

*r*

0

*f* '2  *x*** *df*  

4 *n* 12 *F*

*Tr* 1*g*  *x* **0 *dg* 

and

*x*2*r* 1 1 2*x*2

 4 2*r* 1*r* 1! *n* 1

2 2 ,

(11)

 2 1  2*x*  2*x*2  2*x*  1  

 1  2*x*  *x*  *x*  

3  

     *f* '  *x*  *f* ''  *x* ** *df* 

*F* 

*r*

2 *n*  12

2 *n*  13  *r*

  (12)

 

*x*2*r*1 1  2*x* 1  *x*   1 

 

 4 *r* 1!2 *n*  1

2  *o*  *n*2 .

From [4] we know

**  *g*, 1 **

1

*dg*   *C*  ln *n* 2 .

*F*  *n*  0

 *n* 

By a simple computation we get

0    

*I* 2  *x* ** *df*   *C*

**  *f* '' ,

2

1  ** *df* 



12 *n*5



*r*

*F* 3 *r*

*r*

*n*2 *F*  

*C*  



*r*

1 2

 2 *F*

 2*r*2** *T g* *r*  ,

 **0 *dg* 

*n* 0 

*r*

 

*C*  22*r* 4 2



12 *n*5



**  *g*, 1

*F* 12

 

2

** *dg*

 0  

5

*n* 0  *n* 

5

 *C*  22*r*4  ln *n*12 

 1 

*n*2 5

*n*12

*o*   ,

 

*n*2

(13)

1  2*x f* '  *x*  *I*



*F n*  1 3

*r*

 *x* *r* *df*  

 *f* '  *x*  *I*  *x* ** *df* 

*r*

1  2*x*

*n*  1

1  2*x*

*n*  1

*r*

*F*

3



 *f* '2  *x* *r* *df* 2  *I*  *x* ** *df* 

2 2

3 *r*

1 1

*Fr Fr*

1

(14)



1  2*x*

*n*  1

  

*x*2*r* 1

2  1 

2  

 *o*



 2*r* 1*r* 1! 

 *n*2 

 *n* 

and

 

 *o*  1  ,

 

 2 2*x*2  2*x*  1 

  *x*  *x*  3  *f* ''  *x*  *I*



 *x* ** *df* 

*F* 

*r* 

*n*  1

*n*  12  3 *r*

 

|

2



3

3

| *x*  *x*

2*x*2  2*x*  1

2 *F*

*f* ''  *x*  *I*

 *x* *r* *df* 

*n*  1

2

*n*  1 *r*

2*x*2  2*x*  1 1 1

(15)

|  *x*  *x*  3 | 

*f* '' 2  *x* *r* *df* 2 

*I*32  *x* *r* *df* 2

*n*  1

*n*  12 *Fr Fr*

2 1 1

*x*  *x*2

2*x*  2*x*  

3

*x*2 *r* 3

2  1   1 

| *n*  1 

*n*  12

|  

 2*r*  3*r*  2!2 

 *o*    *o* 

  

*n n*

2 .



 

From (3) and (9)-(15), we obtain the desired estimate of Theorem 1.

### Acknowledgements

This work was supported by a grant from Hebei province higher school science and technology research

(Z2010160).

### References

1. Klaus Ritter, Average-case analysis of numerical problems. New York : Springer-Verlag Berlin

Heidelberg; 2000.

1. Devore R A, Lorentz G G. Constructive Approximation. Berlin: Springer-Verlag;1993.
2. XU Guiqiao. The Simultaneous Approximation Average Errors for Bernstein Operators on the r-fold

Integrated Wiener Space. 2012; to appear.

1. Sun Yongsheng, Wang Chengyong. Average error bounds of best approximation of continuous functions

on the Wiener space. J. Complexity 1995;11: 74-104.