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*Aα*-Spectrum of a Firefly Graph

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**Abstract**

Let *G* be a connected graph of order *n, A*(*G*) is the adjacency matrix of *G* and *D*(*G*) is the diagonal matrix of the row-sums of *A*(*G*)*.* In 2017, Nikiforov [[8](#_bookmark17)] defined the convex linear combinations *Aα*(*G*) of *A*(*G*) and *D*(*G*) by

*Aα*(*G*)= *αD*(*G*)+ (1 *− α*)*A*(*G*)*,* 0 *≤ α ≤* 1*.*

In this paper, we obtain a partial factorization of the *Aα*-characteristic polynomial of the firefly graph which explicitly gives some eigenvalues of the graph.

*Keywords:* Eigenvalues, *Aα* matrix, firefly Graphs

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# 1 Introduction

Let *G* be a simple graph of order *n* with vertex set *V* (*G*) and edge set *E*(*G*), such that *|E*(*G*)*|* = *m*. We denote the complete graph with *n* vertices by *Kn*. The set of *neighbours* of a vertex *v* in *G* is denoted by *NG*(*v*) and *NG*[*v*] = *NG*(*v*) *∪ {v}.* The *degree* of a vertex *v* of *G, d*(*v*), is defined by *|NG*(*v*)*|*. Two distinct vertices *u* and *v* are called *true twins* in *G* if *NG*[*u*] = *NG*[*v*] and are called *false twins* if *NG*(*u*)= *NG*(*v*) and *u* is not adjacent to *v*, see [[7](#_bookmark16)]. The *signless Laplacian* matrix of *G* is defined by *Q*(*G*) = *A*(*G*)+ *D*(*G*), where *D*(*G*) is the diagonal matrix of the degrees and *A*(*G*) = [*aij*] is the *adjacency* matrix of *G*, where *aij* = 1 if *vi* is adjacent to *vj* and *aij* = 0 otherwise. Recently Nikiforov [[8](#_bookmark17)] defined for any real *α ∈* [0*,* 1], the convex linear combinations *Aα*(*G*) of *A*(*G*) and *D*(*G*) by

*Aα*(*G*)= *αD*(*G*)+ (1 *− α*)*A*(*G*)*.*

It is easy to see that *A*(*G*)= *A*0(*G*), *D*(*G*)= *A*1(*G*) and *Q*(*G*)= 2*A* 1 (*G*). The *Aα-*

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*characteristic polynomial* of *G* is defined by *PAα*(*G*)(*x*) = det(*xI − Aα*(*G*)) and its

roots are called the eigenvalues of *Aα*(*G*). As usual, we shall index the eigenvalues of *Aα*(*G*) in non-increasing order and denote them as *λ*1(*Aα*(*G*)) *≥ λ*2(*Aα*(*G*)) *≥*

*··· ≥ λn*(*Aα*(*G*))*.* The spectrum of *Aα*(*G*) is defined as the multiset of eigenvalues with their algebraic multiplicity and denoted by *Spec*(*Aα*(*G*))*.* To simplify we use the *Aα* and *λi*(*Aα*) notation when there is no risk of ambiguity.

As defined by Aouchiche *et al*. [[1](#_bookmark10)], a *ﬁrefly graph Fs,r,t* is a graph on *n* = 2*r* + *s* + 2*t* + 1 vertices that consists of *s* pendant edges, *r* triangles, and *t* pendant paths of length 2, all of them sharing a common vertex. Let *Fn* be the set of all firefly graphs with *n* vertices. Note that *Fn* contains the star *Sn Fs,*0*,*0, the stretched stars ( *Fs,*0*,t*), the friendship graphs ( *F*0*,r,*0) and the butterfly graphs ( *Fs,r,*0). The relevance of studying this family relates to the fact that many extremal graphs for functions depending on the eigenvalues of graph matrices belong to *Fn.* For unicyclic graphs, Hong [[4](#_bookmark13)] determined the unique graph, *Fn−*3*,*1*,*0, with maximum largest eigenvalue of *A*(*G*)*.* Fan *et al*. [[2](#_bookmark11)] determined the unique graph, *Fn−*3*,*1*,*0, with minimum least eigenvalue of *A*(*G*) among all unicyclic graphs of order *n* when *n ≥* 12. Petrovi´c *et al*. [[9](#_bookmark18)] determined the unique graph, *Fn−*3*,*1*,*0, with minimum least eigenvalue of *A*(*G*) among the cacti with *n* vertices (*n ≥* 12) and *k* cycles, where 0 *≤ k ≤ [ ♩*. Moreover, Li *et al* [[6](#_bookmark14)] characterized graphs, *Fn−[ n−*1 *♩−*1*,[ n−*1 *♩,*0, with the largest signless Laplacian spectral radius among all the

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*n−*1

cacti with *n* vertices.

Here, we address the problem of finding all the eigenvalues of *Aα*(*Fs,r,t*), which fills a literature gap and generalizes the eigenvalues of the adjacency and signless Laplacian matrix of a firefly graph for a convenient *α.*

The paper is organized such that preliminary results are presented in the next section and the main proofs are in Section [3.](#_bookmark6)

First, we present the equitable partition theorem of a matrix which can be found at Horn and Johnson [[5](#_bookmark15)] and the Propositions [2.2](#_bookmark2) and [2.3](#_bookmark3) that show the eigenvalues of *Aα*(*Kn*), *Aα*(*Sn*) and upper bounds for *λ*1(*Aα*)*,* respectively.

**Proposition 2.1 ( [**[**5**](#_bookmark15)**])** *Let M be a matrix of order n deﬁned by*

⎡ *M*1*,*1 *M*1*,*2 *··· M*1*,k* ⎤

*. .*

*M* = ⎢ *M*2*,*1 *M*2*,*2 *··· M*2*,k* ⎥ *,*

⎢ *. . . .* ⎥

⎣ *Mk,*1 *Mk,*2 *··· Mk,k* ⎦

*where Mi,j,* 1 *≤ i, j ≤ k, is a submatrix of order ni × nj such that the sum of each of its rows is equal to ci,j. If M* = [*ci,j*]*k×k, then the eigenvalues of M are also eigenvalues of M.*

**Proposition 2.2 ( [**[**8**](#_bookmark17)**])** *For α ∈* [0*,* 1]*, we have*

1. *Spec*(*Aα*(*Kn*)) = *{n −* 1*,* (*αn −* 1)[*n−*1]*};*
2. *Spec*(*Aα*(*Sn*)) = 1 (*αn* + *β*) *, α*[*n−*2]*,* 1 (*αn − β*)}*, where β* =

√

2 2

*α*2*n*2 + 4(*n −* 1)(1 *−* 2*α*)*.*

**Proposition 2.3 ( [**[**8**](#_bookmark17)**])** *If G is a graph of order n and has m edges, then*

, 1 Σ 2*m*

*λ* (*A* (*G*)) *≥ d*2(*u*) *and λ* (*A* (*G*)) *≥ .*

1 *α* , *n*

*u∈V* (*G*)

1 *α n*

*Equality holds in the second inequality if and only if G is regular. If α >* 0*, equality* *holds in the ﬁrst inequality if and only if G is regular.*

Proposition [2.4](#_bookmark4) states that the existence of twin vertices in *G* implies the presence of certain eigenvalues, *λ*, in the spectrum of *Aα*(*G*) and also provides a lower bound for the multiplicity, *m*(*λ*), of such eigenvalues.

**Proposition 2.4** *Let G be a graph on n ≥* 2 *vertices, with vi and vjp,* 1 *≤ p ≤ r, twin vertices in G.*

1. *If vi /∼ vjp then αd*(*vi*) *∈ Spec*(*Aα*(*G*)) *and m*(*αd*(*vi*)) *≥ r.*
2. *If vi ∼ vjp then α*(*d*(*vi*)+ 1) *−* 1 *∈ Spec*(*Aα*(*G*)) *and m*(*α*(*d*(*vi*)+ 1) *−* 1) *≥ r .*

**Proof.** For a given *p ∈ {*1*,* 2*,..., r}*, let *vi* and *vjp* be twin vertices in *G*. Consider the vector **x**(*p*) *∈* R*n* with entries

⎧⎪ 1*,* if *k* = *i*; [**x**(*p*)]*k* = *−*1*,* if *k* = *jp*;

⎪⎨

⎪⎪⎩ 0*,* otherwise*.*

Since *Aα*(*G*)= *Aα* we have, for each *l ∈ {*1*,* 2*,,..., n}*,

[*Aα***x**(*p*)]*l* = Σ [*Aα*]*lk*[**x**(*p*)]*k* = [*Aα*]*li −* [*Aα*]*lj .* (1)

*n*

*p*

*k*=1

Now, consider the following three cases:

**Case 1** *l* = *i. In this case,*

*so,*

*p p*

[*Aα***x**(*p*)]*i* = [*Aα*]*ii −* [*Aα*]*ij* = *αd*(*vi*) *−* [*Aα*]*ij ,*

**Case 2** *l* = *jp. In this case,*

[*Aα*

**x**(*p*)]*i*

= ⎧⎨ *α*(*d*(*vi*)+ 1) *−* 1*, se vi ∼ vjp* ;

⎩ *αd*(*vi*)*, se vi /∼ vjp .*

*and,*

*p p p p p p*

[*Aα***x**(*p*)]*j* = [*Aα*]*j i −* [*Aα*]*j j* = [*Aα*]*j i − αd*(*vj* )*,*

[*Aα*

**x**(*p*)]*j*

= ⎧⎨ *−α*(*d*(*vjp* )+ 1) + 1*, if vi ∼ vjp* ;

⎩ *−αd*(*vjp* )*, if vi /∼ vjp .*

*p*

**Case 3** *l ∈/ {i, jp}.*

*Since vi and vjp are twin vertices, we have* [*Aα*]*li* = [*Aα*]*ljp . Then, for equa- tion* ([1](#_bookmark5)) [*Aα***x**(*p*)]*l* = 0*.*

Therefore, of the three previous cases we have

⎧⎨ (*αd*(*vi*)+ *α −* 1)**x***p,* if *vi ∼ vj* ;

*Aα***x**(*p*) =

*p*

⎩ *αd*(*vi*)**x**(*p*)*,* if *vi /∼ vj .*

*p*

It is easy to see that *{***x**(*p*)*}r* is linearly independent. Therefore *m*(*λ*) *≥ r*, for

*p*=1

*λ ∈ {αd*(*vi*)*, αd*(*vi*)+ *α −* 1*}*.

*2*

# 3 Main results

In this section, we present the results involving the eigenvalues of graphs in the family *Fn*. There is a convenient vertex labelling of a graph *Fs,r,t ∈ Fn* such that *Aα* = *Aα*(*Fs,r,t*) can be written as

⎡ *α*(*t* + *s* + 2*r*) (1 *− α*)**J**1*×s* (1 *− α*)**J**1*×*2*r* (1 *− α*)**J**1*×t* **0**1*×t*

⎢ (1 *− α*)**J***s×*1 *α***I***s* **0***s×*2*r* **0***s×t* **0***s×t*

*Aα* = ⎢ (1 *− α*)**J**2*r×*1 **0**2*r×s B*2*r* **0**2*r×t* **0**2*r×t*

⎤

⎥

⎥ *,* (2)

⎢ (1 *− α*)**J***t×*1 **0***t×s* **0***t×*2*r* 2*α***I***t* (1 *− α*)**I***t* ⎥

⎣ **0***t×*1 **0***t×s* **0***t×*2*r* (1 *− α*)**I***t α***I***t* ⎦

where

⎡

*B*2*r* = ⎣

2*α***I***r* (1 *α*)**I***r*

*.* (3)

⎦

*—* ⎤

(1 *− α*)**I***r* 2*α***I***r*

The Figure [1](#_bookmark8) displays the firefly graph *F*3*,*2*,*2 with the adopted labelling.



*v*2 *v*3 *v*4

*v*11

*v*9

*v*8

*v*1

*v*12

*v*10

*v*6

*v*5 *v*7

Fig. 1. Firefly graph *F*3*,*2*,*2

**Remark 3.1** Let *G Fs,r,t*.The graph *G* has exactly one vertex of degree equal to 2*r* + *s* + *t*, 2*r* + *t* vertices of degree equal to 2 and *s* + *t* vertices of degree equal to

1. For *α* = 1 the eigenvalues of *A*1(*G*)= *D*(*G*) are *d*(*v*), *v ∈ V* (*G*), and for *α* =0 the eigenvalues of *A*0(*G*)= *A*(*G*) can be see in [[3](#_bookmark12)].

In Proposition [3.2](#_bookmark9), we were able to prove that the occurrence of some eigenvalues of a firefly graph depends on the existence of certain induced subgraphs.

**Proposition 3.2** *Given the nonnegative integers r, s and t, let G Fs,r,t and*

2

2

2

*α ∈* (0*,* 1)*. If t ≥* 2 *then θ*1

= 3*α*+*√*5*α*2*−*8*α*+4 *and θ*

= 3*α−√*5*α*2*−*8*α*+4 *are eigenvalues*

*of G, both with multiplicity at least t −* 1*. Moreover, if r ≥* 2 *then α* +1 *is an*

*eigenvalue of G with multiplicity at least r −* 1*.*

**Proof.** Given *λ ∈ {θ*1*, θ*2*}* suppose *t ≥* 2 and, for each *i ∈ {*1*,* 2*,...,t−* 1*}*, consider

the vector **x**(*i*) with 2*r* + *s* + 2*t* + 1 entries, where

⎧ *λ − α,* if *j* = *s* + 2*r* + 2;

⎪

⎪ 1 *− α*

⎪ *−λ − α,* if *j* = *s* + 2*r* +2+ *i*;

[**x**(*i*)]*j* =

⎪⎪⎨

⎪

⎪

⎪⎩

1 *− α*

1*,* if *j* = *s* + 2*r* + *t* + 2;

*−*1*,* if *j* = *s* + 2*r* + *t* +2+ *i*; 0*,* otherwise*.*

In this way, the entries of the vector *Aα*(*G*)**x**(*i*) *− λ***x**(*i*) are given by

⎧⎪

⎪⎪

*A* (*G*)**x**(*i*) *− λ***x**(*i*) = ⎨

*λ*2 3*αλ* + *α*2 + 2*α* 1

*,* if *j* = *s* + 2*r* + 2;

*− −*

*α −* 1

*λ*2 *−* 3*αλ* + *α*2 + 2*α −* 1

*α j* ⎪ *−*

⎪⎩

*,* if *j* = *s* + 2*r* +2+ *i*;

*α −* 1

0*,* otherwise*.*

Since *λ* is a root of the polynomial *x*2 *−* 3*αx* + *α*2 + 2*α −* 1*,* it follows that **x**(*i*) is an associated eigenvector to *λ*. Since *{***x**(*i*)*}t−*1 is a linearly independent set, the

*i*=1

multiplicity of *λ* is at least *t −* 1*.*

Now, suppose *r ≥* 2 and denote by *ek* the vector with *s* + 2*r* + 2*t* + 1 coordinates whose *k*-th entry is equal to 1 and the others entries are zero. For each *j*, *s* +2 *≤ j ≤ s* + *r*, it is easy to show that the vector *zj* = *ej − ej*+1 + *ej*+*r − ej*+*r*+1 is an eigenvector of *Aα*(*G*) associated with the eigenvalue *α* + 1. So, *α* + 1 is an eigenvalue of *Aα*(*G*) with multiplicity at least *r −* 1*.*

*2*

**Remark 3.3** As described in the Proposition [3.2](#_bookmark9), we use the notations *θ*1 and *θ*2

to represent the roots of the polynomial *x*2 *−* 3*αx* + *α*2 + 2*α −* 1.

For *s ≥* 1*, G Fs,*0*,*0 *Ss* and the eigenvalues of *Aα*(*Ss*) can be seen in the Proposition [2.2](#_bookmark2).

**Proposition 3.4** *Let G F*0*,r,*0*. If r ≥* 1 *and α ∈* (0*,* 1) *then*

*PAα*(*G*)(*x*)= (*x − α −* 1)*r−*1(*x −* 3*α* + 1)*r*(*x*2 *−* (2*αr* + *α* + 1)*x* + (6*α −* 2)*r*)*.*

*Moreover, if x*1 *and x*2 *denote the roots of the quadratic factor of PAα*(*G*)(*x*) *then*

⎨ 3

⎧⎪ *x*2 *≤* 3*α −* 1 *< α* +1 *< x*1*, if* 0 *< α ≤* 1 ;

⎪⎩ 3*α −* 1 *≤ x*2 *< α* +1 *< x*1*, if* 1 *< α <* 1*.*

3

**Proof.** For *G F*0*,r,*0, we have

⎡

*Aα*(*G*)= ⎣

2*αr* (1 *− α*)**J**1*×*2*r*⎤

(1 *− α*)**J**2*r×*1 *B*2*r*

⎦

*,*

where *B*2*r* is the matrix given in ([3](#_bookmark7)).

Applying the Propositions [2.4](#_bookmark4) for each triangles of *G*, we obtain that 3*α −* 1 is an eigenvalue of *Aα*(*G*) with multiplicity at least *r* and from Proposition [3.2](#_bookmark9), for *r ≥* 2, *α* + 1 is an eigenvalue of *Aα*(*G*) with multiplicity at least *r −* 1. If *r* = 1, the result follow by Proposition [2.2](#_bookmark2).

From Proposition [2.1](#_bookmark1), the spectrum of matrix

⎡ 2*αr* 2*r*(1 *− α*)⎤

⎣

⎦

*M* = *,*

1 *− α* 1+ *α*

whose characteristic polynomial is *g*(*x*)= *x*2*−*(2*αr*+*α*+1)*x*+(6*α−*2)*r*, is contained in the spectrum of *Aα*(*G*). Since 3*α −* 1 and *α* + 1 are not roots of *g*(*x*), we have

*PAα*

(*G*)(*x*)= (*x − α −* 1)*r−*1(*x −* 3*α* + 1)*rg*(*x*).

6*r* 3

As *G* has order *n* = 2*r*+1 and size *m* = 3*r*, we have *d* = 2*r* +1 = 3*−* 2*r* +1 *≥* 2.

From Proposition [2.3](#_bookmark3), *λ*1(*Aα*) *≥* 2. Let *x*1 and *x*2 be the roots of *g*(*x*), with *x*2 *< x*1. Since 3*α −* 1 *< α* +1 *<* 2, for *α ∈* (0*,* 1), we conclude that *x*1 = *λ*1(*Aα*). Now, we have *g*(*α* + 1) = *−*2*r*(*α −* 1)2 *<* 0 for all *r ≥* 1 and *α ∈* (0*,* 1). On the other hand, *g*(3*α −* 1) = *−*2(*r −* 1)(*α −* 1)(3*α −* 1), thus *g*(3*α −* 1) *≤* 0 if *α ∈* (0*,* 1 ] and

3

*g*(3*α −* 1) *>* 0 if *α ∈* ( 1 *,* 1). So it easy to see that

3

⎧⎪ *x*2 *≤* 3*α −* 1 *< α* +1 *< x*1*,* if 0 *< α ≤* 1 ;

⎨ 3

⎪⎩ 3*α −* 1 *≤ x*2 *< α* +1 *< x*1*,* if 1 *< α <* 1*,*

3

and the equalities only hold if *r* =1 or *α* = 1 .

3

*2*

**Proposition 3.5** *Let G F*0*,*0*,t. If t ≥* 1 *and α ∈* (0*,* 1) *then*

*P* (*x*)= (*x*2 *−* 3*αx* + *α*2 + 2*α −* 1)*t−*1*h*(*x*)*,*

*Aα*(*G*)

*where h*(*x*)= *x*3 *− α*(*t* + 3)*x*2 + [*α*2*t* + (*α*2 + 2*α −* 1)(*t* + 1)]*x −* 2*α*(2*α −* 1)*t. If x*1*, x*2 *and x*3 *are the roots of h*(*x*) *then x*3 *< θ*2 *< x*2 *< θ*1 *< x*1*.*

**Proof.** For *G F*0*,*0*,t*, we have

⎡ *αt* (1 *− α*)**J**1*×t* **0**1*×t* ⎤

⎣ **0***t×*1 (1 *− α*)**I***t α***I***t* ⎦

*Aα*(*G*)= ⎢(1 *− α*)**J***t×*1 2*α***I***t* (1 *− α*)**I***t*⎥ *.*

From Proposition [3.2](#_bookmark9), if *t ≥* 2, *θ*1 and *θ*2 are eigenvalues of *Aα*(*G*), both with multiplicity at least *t −* 1. From Proposition [2.1](#_bookmark1), the spectrum of matrix

⎡ *αt t*(1 *− α*) 0 ⎤

⎣ 0 1 *− α α* ⎦

*M* = ⎢1 *− α* 2*α* 1 *− α*⎥ *,*

whose characteristic polynomial is *h*(*x*)= *x*3 *− α*(*t* + 3)*x*2 + [*α*2*t* + (*α*2 + 2*α −* 1)(*t* + 1)]*x −* 2*α*(2*α −* 1)*t,* is contained in the spectrum of *Aα*(*G*). As *θ*1 and *θ*2 = 3*α − θ*1

are not roots of *h*(*x*), we have *P*

*Aα*(*G*)

(*x*) = (*x*2 *−* 3*αx* + *α*2 + 2*α −* 1)*t−*1*h*(*x*). Since

*h*(*θ*1)= *−*(*α −* 1)2(*θ*1 *− α*)*t <* 0 and *h*(*θ*2)= (*α −* 1)2(*θ*1 *−* 2*α*)*t >* 0 for all *t ≥* 1 and

*α ∈* (0*,* 1), there is a root of *h*(*x*) in (*θ*2*, θ*1). As lim *h*(*x*)= *∞* and lim *h*(*x*)=

*x→∞ x→−∞*

*−∞*, the previous inequalities imply *x*3 *< θ*2 *< x*2 *< θ*1 *< x*1. If *t* = 1, *G P*3 and

the result follows.

*2*

**Proposition 3.6** *Let G Fs,r,*0*. If s ≥* 1*, r ≥* 1 *and α ∈* (0*,* 1) *then*

*PA* (*G*)(*x*)= (*x − α*)*s−*1(*x −* 3*α* + 1)*r*(*x − α −* 1)*r−*1*h*(*x*)*,*

*α*

*where h*(*x*) = *x*3 *−* (*αs* + 2*αr* + 2*α* + 1)*x*2 + ((*α*2 + 3*α −* 1)(*s* + 2*r*)+ *α*2 + *α*)*x −*

(2*α*2 + *α−* 1)*s* + 2*αr*(1 *−* 3*α*)*. If x*1*, x*2 *and x*3 *are the roots of polynomial h*(*x*) *then*

3

⎧⎪ *x*3 *<* 3*α −* 1 *< α < x*2 *< α* +1 *< x*1*, if* 0 *< α ≤* 1 ;

⎪

⎪⎨ min*{x*3*,* 3*α −* 1*} <* max*{x*3*,* 3*α −* 1*} < α < x*2 *< α* +1 *< x*1*, if* 1 *< α <* 1 ;

3 2

⎪

⎪⎩ *x*3 *< α <* 3*α −* 1 *< x*2 *< α* +1 *< x*1*, if* 1 *≤ α <* 1*.*

2

**Proof.** For *G Fs,r,*0, we have

⎡ *α*(*s* + 2*r*) (1 *− α*)**J**1*×s* (1 *− α*)**J**1*×*2*r*⎤

*Aα*(*G*)= ⎢ (1 *− α*)**J***s×*1 *α***I***s* **0***s×*2*r*

⎣(1 *α*)**J 0** *B*

*—* 2*r×*1 2*r×s* 2*r*

⎥⎦ *,*

where *B*2*r* is the matrix given in ([3](#_bookmark7)).

From the vector obtained in the proof of Proposition [2.4](#_bookmark4) it is possible to obtain *r* linearly independent eigenvectors related to the eigenvalue 3*α −* 1 and *s−* 1 linearly independent eigenvectors associated to the eigenvalue *α*. By Proposition [3.2](#_bookmark9), *α* +1 is an eigenvalue of *Aα*(*G*) with multiplicity at least *r −* 1 and, from Proposition [2.1](#_bookmark1), the eigenvalues of the reduced matrix

⎡*α*(*s* + 2*r*) (1 *− α*)*s* (1 *− α*)2*r*⎤

*M* = ⎢⎣

1 *− α α* 0

1 *− α* 0 1 + *α*

⎥⎦ *,*

whose characteristic polynomial is *h*(*x*)= *x*3 *−* (*αs* + 2*αr* + 2*α* + 1)*x*2 + ((*α*2 + 3*α −* 1)(*s* + 2*r*)+ *α*2 + *α*)*x−* (2*α*2 + *α−* 1)*s* + 2*αr*(1 *−* 3*α*)*,* is contained in the spectrum of *Aα*(*G*). Note that *h*(*α* + 1) = *−*2(*α −* 1)2*r <* 0 and *h*(*α*)= (*α −* 1)2*s >* 0 for all *α ∈* (0*,* 1), *s* 1 and *r* 1. So, *h*(*x*) has a root, *x*2, in (*α, α* + 1). As lim *h*(*x*)= and

*≥ ≥ ∞*

*x→∞*

*h*(*α* + 1) *<* 0, we concluded that the largest root of *h*(*x*), *x*1, satisfies *α* +1 *< x*1. We have *h*(3*α−*1) = 2(1*−α*)[(3*α*2 *−*3*α*+1)*s*+(6*α*2 *−*5*α*+1)(*r−*1)]. As 3*α*2 *−*3*α*+1 *>* 0

2 1 1

for all *α ∈* (0*,* 1) and 6*α −* 5*α* +1 *≥* 0 for *α ∈* 0*, ∪ ,* 1 we have *h*(3*α −* 1) *>* 0

*∈* *∪*

for all *α* 0*,* 1 1 *,* 1 . As lim

3 2 *x→−∞*

3 2

*h*(*x*) = *−∞* we have *x*3 *<* min

*{*3*α −* 1*, α}*.

Similarly, when *α ∈* 1 *,* 1 , it is shown that max*{x*3*,* 3*α −* 1*} < α*. It is easy to sort

3

2

the eigenvalues of *Aα*(*G*) and the result follows.

*2*

**Proposition 3.7** *Let G Fs,*0*,t. If s ≥* 1 *and t ≥* 1 *then*

*PAα*(*G*)(*x*)= (*x − α*)*s−*1(*x*2 *−* 3*αx* + *α*2 + 2*α −* 1)*t−*1*h*(*x*)*,*

*where h*(*x*) = *x*4 *− α*(*s* + *t* + 4)*x*3 + [(3*α −* 1)(*α* + 1)(*s* + *t* + 1)+ *α*2]*x*2 *− α*[(*α*2 + 2*α −* 1)(*s* + 2*t* + 1)+ (2*α −* 1)(3*s* + *t*)]*x* + (2*α −* 1)[*α*2(*s* + 2*t*)+ (2*α −* 1)*s*]*. Moreover, the polynomial h*(*x*) *has four distinct roots, x*1*, x*2*, x*3 *and x*4*, such that x*4 *< θ*2 *< x*3 *< α < x*2 *< θ*1 *< x*1*.*

**Proof.** For *G Fs,*0*,t*, we have

⎡ *α*(*s* + *t*) (1 *− α*)**J**1*×s* (1 *− α*)**J**1*×t* **0**1*×t* ⎤

*A* (*G*)= ⎢(1 *− α*)**J***s×*1 *α***I***s* **0***s×t* **0***s×t* ⎥ *.*

*α*

⎢⎣(1 *− α*)**J**

*t×*1

**0***t×*1

2*α***I***t*

(1 *− α*)**I***t*⎥⎦

**0***t×*1 **0***t×s* (1 *− α*)**I***t α***I***t*

From Proposition [2.4](#_bookmark4), *α* is eigenvalue of *Aα*(*G*) with multiplicity at least *s −* 1. If *t* = 1, *θ*1 and *θ*2 are not eigenvalues of *Aα*(*G*). From Proposition [3.2](#_bookmark9), if *t ≥* 2, *θ*1 and *θ*2 are eigenvalues of *Aα*(*G*), both with multiplicity at least *t −* 1. From Proposition [2.1](#_bookmark1), the eigenvalues of the reduced matrix

⎡*α*(*s* + *t*) (1 *− α*)*s* (1 *− α*)*t* 0 ⎤

*M* = ⎢

⎢

1 *− α α* 0 0 ⎥ *,*

1 *− α* 0 2*α* 1 *− α*⎥

⎣ 0 0 1 *− α α* ⎦

whose characteristic polynomial is *h*(*x*), is contained in the spectrum of *Aα*(*G*). For

*θ*1 and *θ*2 = 3*α − θ*1, defined in Proposition [3.2](#_bookmark9), we have

*√*

*h*(*α*)= (*α −* 1)4*s >* 0, *h*(*θ*1)= *−*

(*α −* 1)2*t*

2

(3*α*2

*—* 4*α* +2+ *α*

5*α*2 *−* 8*α* + 4) *<* 0

2(*α* 1)6*t*

*—*

and *h*(*θ*2)= *−* 3*α*2 4*α* +2+ *α√*5*α*2 8*α* +4 *<* 0, for all *α ∈* (0*,* 1), *s ≥* 1 and

*− −*

*t ≥* 1*.*

*s−*1 2

Since *α*, *θ*1 and *θ*2 are not roots of *h*(*x*)*,* we have *PAα*(*G*)(*x*)= (*x − α*) (*x −*

3*αx* + *α*2 + 2*α −* 1)*t−*1*h*(*x*). The above inequalities imply that the polynomial *h*(*x*)

has a root in the interval (*θ*2*, α*) and other root in the interval (*α, θ*1). Moreover, as *h*(*θ*1) *<* 0 and lim *h*(*x*) = , there is a root of *h*(*x*) in the interval (*θ*1*,* ).

*∞ ∞*

*x→∞*

Similarly, we conclude that *h*(*x*) has the smallest root in the interval (*−∞, θ*2).

*2*

The next propositions, whose proofs are similar to the previous results, complete all cases in the family *Fn*.

**Proposition 3.8** *Let G F*0*,r,t. If r ≥* 1*, t ≥* 1 *and α ∈* (0*,* 1) *then*

*PAα*(*G*)(*x*)= (*x −* 3*α* + 1)*r*(*x − α −* 1)*r−*1(*x*2 *−* 3*αx* + *α*2 + 2*α −* 1)*t−*1*h*(*x*)*,*

*where h*(*x*)= *x*4 *−* (*α*(*t* + 2*r* + 4)+ 1)*x*3 + ((3*α*2 + 3*α −* 1)(*t* + 2*r* + 1)+ *α*2 + 2*α*)*x*2 *−*

((*α*3 + 2*α*2 + *α −* 1)(2*t* + 2*r* + 1)+ (2*α*2 *−* 3*α* + 1)(*t* + 8*r*)+ (14*α −* 6)*r*)*x* + 2*αt*(2*α*2 +

*α −* 1) + 2*r*(3*α* + 5*α −* 5*α* + 1)*. If x*1*, x*2*, x*3 *and x*4 *are the roots of polynomial*

3 2

*h*(*x*) *then*

⎨ 3

⎧⎪*x*4 *< θ*2 *<* min*{x*3*,* 3*α −* 1*} <* max*{x*3*,* 3*α −* 1*} < θ*1 *< x*2 *< α* +1 *< x*1*, if* 0 *< α <* 1 ;

⎪⎩*x*4 *< θ*2 *<* 3*α −* 1 *< x*3 *< θ*1 *< x*2 *< α* +1 *< x*1*, if* 1 *≤α <* 1*.*

3

**Proposition 3.9** *Let G Fs,r,t. For s ≥* 1*, r ≥* 1*, t ≥* 1 *and α ∈* (0*,* 1)*,*

*P* (*x*)= (*x − α*)*s−*1(*x −* (3*α −* 1))*r*(*x −* (*α* + 1))*r−*1(*x*2 *−* 3*αx* + *α*2 + 2*α −* 1)*t−*1*g*(*x*)*,*

*Aα*(*G*)

*where g*(*x*)= *x*5 + (*−αt −* 2*αr − αs −* 5*α −* 1)*x*4 + ((4*α*2 + 3*α −* 1)*t* + (8*α*2 + 6*α −*

2)*r* + (4*α*2 + 3*α −* 1)*s* + 8*α*2 + 6*α −* 1)*x*3 +((*−*5*α*3 *−* 11*α*2 + 2*α* + 1)*t* + (*−*8*α*3 *−* 28*α*2 +

10*α*)*r* + (*−*4*α −* 13*α* + 3*α* + 1)*s −* 5*α −* 8*α* + 1)*x* +((2*α* + 12*α* + *α −* 3*α*)*t* + (2*α*4 + 28*α*3 + 2*α*2 *−* 10*α* + 2)*r* + (*α*4 + 11*α*3 + 7*α*2 *−* 8*α* + 1)*s* + *α*4 + 3*α*3 + *α*2 *− α*)*x*

3 2 3 2 2 4 3 2

+(*−*4*α −* 2*α* + 2*α* )*t* + (*−*6*α −* 10*α* + 10*α −* 2*α*)*r* + (*−*2*α −* 5*α* + *α* + 3*α−* 1)*s.*

4 3 2 4 3 2 4 3 2

*Moreover, g*(*x*) *has ﬁve roots x*1*, x*2*, x*3*, x*4*, x*5 *which are arranged as*

*x*5 *< θ*2 *< x*4 *<* min*{*3*α −* 1*, α} <* max*{*3*α −* 1*, α} < x*3 *< θ*1 *< x*2 *< α* +1 *< x*1

*and the result follows.*

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