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A Case Study in Abstract Interpretation Based Program Transformation: Blocking Command Elimination 1

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Abstract

We illustrate the design of correct semantics-based program transformations by abstract interpretation on blocking code elimination.

# Introduction

Static program analysis is closely related to program transformation since a preliminary program analysis is necessary in order to collect information about the program runtime behaviors which is then used to decide which offline trans- formations are applicable [12,14]. Abstract interpretation [4,6,8,9] has been used as a formal basis for static program analysis. Abstract interpretation can also be used to define a semantics-based program transformation framework. This is a new approach to the formal design of program transformations and a

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new application of the abstract interpretation theory. The idea is to formally design syntactic (that is source level) program transformations by abstrac- tion of transformations of the program semantics. Abstract interpretation is used to formalize the correspondence between semantic and syntactic transfor- mations. This yields the necessary formal basis for (hopefully mechanically) constructing correct program transformation tools and may be to systematize their design.

The framework is applied to *blocking code elimination*, which consists in eliminating blocking commands other than stop commands in imperative non- deterministic programs leaving non-terminating behaviors unchanged. The fi- nal algorithm is very simple and could have been designed empirically without error but this case study is simple enough to exemplify our approach.

We believe that this unified abstract interpretation based framework for reasoning on program transformation should be applicable to a wide variety of semantics-based program manipulations including constant propagation [15], transition compression [11], slicing [22], partial evaluation [13], continuation passing style transformation [18], call-by-name to call-by-value transformation [19], fold/unfold [3], deforestation [21], compilation [17], etc.

# A Few Elements of Abstract Interpretation

* 1. *Fixpoints*

≤

We write lfp

⊥

*F* for the ≤-least fixpoint of *F* ≤-greater than or equal to ⊥,

≤

when it exists. We write lfp *F* and lfp *F* when ⊥ and ≤ are understood from

≤

the context. Dually, gfp *F* is the ≤-greatest fixpoint of *F* ≤-less than or equal

T

to T, when it exists.

Theorem 2.1 (Least fixpoint) *Let* po⟨L; ≤⟩ *be a partially ordered set* L *with a binary relation* ≤ *which is a partial order (reflexive, antisymmetric and transitive). Assume that F is a monotone operator on* po⟨L; ≤⟩*. Assume that*

⊥ ∈ L *is such that* ⊥ ≤ *F* (⊥)*. Let L* ⊆ L *be a subset of* L *such that* ⊥ ∈ *L,*

∀*x* ∈ *L* : *x* ≤ *F* (*x*) ⇒ *F* (*x*) ∈ *L and if* ⟨*x*i*, i* ∈ ∆⟩ *is an* ≤*-increasing chain of elements of L then the least upper bound (lub)* ∨i∈∆*x*i *exists in* po⟨L; ≤⟩ *and*

≤

*satisﬁes* ∨i∈∆*x*i ∈ *L. Then* lfp *F exists, is unique and belongs to L.*

⊥

Proof. The proof easily derives from [7]. It is based on the iterative definition

of fixpoints in the tradition of Tarski [20] and Kleene [16] *F* 0 = ⊥, *F* δ+1 =

∆

∆

*F* (*F* δ) for successor ordinals *δ* +1 ∈ O, *F* λ =∆ ∨ *F* δ for limit ordinal *λ* ∈ O.✷

δ<λ

≤

Most often, we express the semantics in least fixpoint form lfp

*F* where the

semantic transformer *F* ∈ L −−m→ L is a monotone operator on the complete partial order (cpo) cpo⟨L; ≤*,* ⊥*,* ∨⟩. Dually, we can also use a greatest fixpoint gfp≤ *F* of a monotone semantic transformer *F* ∈ L −−m→ L on the dual complete partial order (co-cpo) ccpo⟨L; ≤*,* T*,* ∧⟩.

Theorem 2.2 (Least fixpoint iterates) *Under the hypotheses of Th. 2.1,*

∆

*for all x* ∈ L *such that* ⊥ ≤ *x* ≤ lfp≤ *F, the iterates F* 0 =

⊥

∆

*x, F* δ+1 =

*F* (*F* δ)

*for successor ordinals δ* +1 ∈ O *and F* λ =∆

∨δ<λ

*F* δ *for limit ordinal λ* ∈ O *are*

*ultimately stationary and converge to* lfp≤ *F* 4 *.*

⊥

Proof. By monotony and transfinite induction, the iterates of *F* starting from *x* are sandwiched between the iterates of *F* starting from ⊥ which are

≤

ultimately stationary and converge to lfp *F* and the iterates of *F* starting

⊥

≤

from lfp

⊥

≤

*F* which are all equal to lfp

⊥

*F* by the fixpoint property, proving the

ultimate convergence of all the iterates to that fixpoint. ✷

* 1. *Abstraction*

An abstraction *α*(*S*) of a concrete semantics *S* is defined by a Galois connec-

tion po⟨L; ≤⟩ ←−γ −

−−→

α

po⟨L; ≤⟩ between the concrete domain po⟨L; ≤⟩ and the

abstract domain po⟨L; ≤⟩ which are both posets 5 . By definition, we have

∀*X* ∈ L : ∀*Y* ∈ L : *α*(*X*) ≤ *Y* ⇔ *X* ≤ *γ*(*Y* ). It follows that *α* preserves exist- ing lubs, by duality *γ* preserves existing greatest lower bounds (glbs) and one adjoint uniquely determines the other. We write po⟨L; ≤⟩ ←−−γ −− po⟨L; ≤⟩ when *α* is surjective (or equivalently *γ* is injective) and po⟨L; ≤⟩ ←←−γ−− po⟨L;

α

−−−→→

α

−−−−→

≤⟩ when *α* is injective (or equivalently *γ* is surjective).

∆

Given *f* ∈ *A* −−→ *B*, a standard example is *α*(*X*) = {*f* (*x*) | *x* ∈ *X*} so

that

(1)

po⟨*℘*(*A*); ⊆⟩ ←−γ −

α

−−→

po⟨*℘*(*B*); ⊆⟩

where *γ*(*Y* ) = {*x* ∈ *A* | *f* (*x*) ∈ *Y* }.

∆

* 1. *Fixpoint Coabstraction*

We have the following sufficient condition for two fixpoints to have the same

≤ ≤

1 2

abstraction *α* (lfp *F* ) = *α* (lfp *F* ) which is based on the iterative definition

1 1 2 2

of fixpoints [7] :

*m*

Theorem 2.3 (Fixpoint coabstraction) *Let F*1 ∈ L1 −−→ L *and F*2 ∈

1

L −−*m*→ L *be respective monotone operators on the complete partial orders*

2

2

cpo⟨L1; ≤1*,* ⊥1*,* ∨1⟩ *and* cpo⟨L2; ≤2*,* ⊥2*,* ∨2⟩*. Let* cpo⟨L; ≤*,* ⊥*,* ∨⟩ *be a complete*

*partial order. Let α*1

⊥*,*↑

∈ L1 −−→

L *and α*2

⊥*,*↑

∈ L2 −−→

L *be* ⊥*-strict* 6 *Scott-*

4 Note that the iterates starting from *x* need not be an increasing chain.

5 Other equivalent formalizations (e.g. using closure operators) are given in [8] and weaker ones, not assuming the existence of a best approximation, are provided in [9].

6 A function *f* is ⊥*-strict*, written *f* : *D* −−⊥→ *E*, if and only if *f* (⊥) = ⊥.

*continuous* 7 *abstraction functions satisfying the following* local coabstraction condition 8 *:*

1. ∀*x* ∈ L1 : ∀*y* ∈ L2 : *α*1(*x*) = *α*2(*y*) ⇒ *α*1(*F*1(*x*)) = *α*2(*F*2(*y*)) *.*

*Then α* (lfp≤1 *F* ) *= α* (lfp≤2 *F* )*.*

1 1 2 2

Proof. Let *F* δ, *δ* ∈ O and *F* δ, *δ* ∈ O be the respective transfinite iterates for

1 2

*F*1 and *F*2 [7]. By monotony, they are increasing chains which are therefore well-defined in the respective complete partial orders cpo⟨L1; ≤1*,* ⊥1*,* ∨1⟩ and cpo⟨L2; ≤2*,* ⊥2*,* ∨2⟩.

*α*1 and *α*2 are ⊥-strict so that *α*1(⊥1) = ⊥ = *α*2(⊥2) hence *F* 0 = *F* 0.

1 2

Let *δ* + 1 be a successor ordinal such that *α*1(*F* δ) = *α*2(*F* δ) by induction

1 2

hypothesis. By the local coabstraction condition (2), we have *α*1(*F* δ+1) =

1

*α*1(*F*1(*F* δ)) = *α*2(*F*2(*F* δ)) = *α*2(*F* δ+1).

1 2 2

Let *λ* be a limit ordinal such that by induction hypothesis, ∀*δ < λ*: *α*1(*F* δ)

1

= *α*2(*F* δ). Then, by continuity of *α*1 and *α*2 and induction hypothesis, we have

2

λ δ δ

δ δ λ

*α*1(*F*1 ) = *α*1( 1 *F*1 ) = δ<λ *α*1(*F*1 ) =

δ<λ

δ<λ *α*2(*F*2 ) = *α*2( 2 *F*2 ) = *α*2(*F*2 ).

δ<λ

By transfinite induction, we conclude that ∀*δ* ∈ O: *α*1(*F* δ) = *α*2(*F* δ).

Let *ϵ*1

∈ O and *ϵ*2

∈ O be such that *ϵ*1

= lfp≤1 *F*

and *ϵ*2

1

= lfp≤2 *F*

1

2

2

[7]. We

have *α* (lfp≤1 *F* ) = *α* (*F* є1 ) = *α* (*F* max(є1,є2)) = *α* (*F* max(є1,є2)) = *α* (*F* є2 ) =

1 1 1 1 1 1 2 2 2 2

*α* (lfp≤2 *F* ). ✷

2 2

* 1. *Locally Complete Fixpoint Abstraction*

In particular when *α*1 = *α* and *α*2 is the identity, Th. 2.3 yields a suffi cient con-

≤ ≤ 

dition for complete (or exact) fixpoint abstractions *α*(lfp *F* ) = lfp *F* , which

≤ ≤

provides guidelines for designing lfp *F* from lfp *F* (or dually) in fixpoint form

[8, theorem 7.1.0.4(3)], [10, lemma 4.3], [2, fact 2.3] 9 :

Corollary 2.4 (Fixpoint transfer) *Let F* ∈ L −−*m*→ L *be a monotonic op-*

*erator on the* cpo⟨L; ≤*,* ⊥*,* ∨⟩*, let F* ∈ L −−*m*→ L *be a monotone operator*

⊥*,*↑ 

*on the* cpo⟨L; ≤*,* ⊥*,* ∨⟩ *and let α* ∈ L −−→ L *be a* ⊥*-strict Scott-continuous*

*abstraction function satisfying the* commutation condition *F* ◦ *α = α* ◦ *F* 10 *.*

7 A function *f* is *Scott-continuous*, written *f* :

↑

*D* −−→ *E*

, if and only if it preserves the lub

of any directed subset of *D* [1] (so that it is monotone).

8 As in Th. 2.1, it is sufficient to assume that *αi* is ⊥-strict, preserves the least upper bound of the iterates of *Fi* starting from ⊥*i*, *i* = 1*,* 2 and that the local coabstraction condition holds for these iterates or a given superset of the iterates.

9 The composition of relations *r*1 and *r*2 is *r*1 ◦ *r*2 =∆ {⟨*x, z*⟩| ∃*y* : ⟨*x, y*⟩∈ *r*1 ∧ ⟨*y, z*⟩∈ *r*2}

∆

whence the composition of functions is *f* ◦ *g*(*x*) = *f* (*g*(*x*)).

10 As in Th. 2.1, it is sufficient to assume that *α* is ⊥-strict, preserves the least upper bound of the iterates of *F* starting from ⊥ and that the commutation condition holds for these iterates.

≤ ≤

*Then α*(lfp *F* ) *=* lfp *F.*

* 1. *Fixpoint Approximation*

≤

Due to undecidability, it is often impossible to abstract a fixpoint *α*(lfp *F* )

≤ ≤

= lfp *F* exactly and to require simultaneously the abstract fixpoint lfp *F* to

be effectively computable. In that case, abstract interpretation theory offers

≤ ≤ 

fixpoint approximation methods so that *α*(lfp *F* ) ≤ lfp *F* [6,8,9]. Let us recall

these basic fixpoint approximation results in a generalized form:

Theorem 2.5 (Least fixpoint upper approximation) *Let F* ∈ L −−*m*→ L *be a monotonic operator on the complete partial order* cpo⟨L; ≤*,* ⊥*,* ∨⟩ *and let F* ∈ L −−*m*→ L *be a monotone operator on* cpo⟨L; ≤*,* ⊥*,* ∨⟩*.*

⊥*,*↑

*Assume that the* ⊥*-strict Scott-continuous abstraction function α* ∈ L −−→

L *is such that for all x* ∈ L *such that x* ≤ *F* (*x*) *there exists y* ≤ *x such that*

*α*(*F* (*x*)) ≤ *F* (*α*(*y*))*.*

≤ ≤

*Then α*(lfp *F* ) ≤ lfp *F.*

Proof. Let *F* δ and *F* δ, *δ* ∈ O be the respective ordinal-termed ≤ and ≤-in- creasing ultimately stationary chains of transfinite iterates of *F* and *F* [7]. We have *α*(*F* 0) = *α*(⊥) = ⊥ = *F* 0 by strictness of *α* and definition of the iterates. Assume *α*(*F* δ) ≤ *F* δ by induction hypothesis. We have *F* δ ≤ *F* (*F* δ) = *F* δ+1 so that, by hypothesis, ∃*y* ≤ *F* δ such that *α*(*F* δ+1) ≤ *F* (*α*(*y*)). By monotony of *F* and *α*, *F* (*α*(*y*)) ≤ *F* (*α*(*F* δ)) whence by transitivity, induction hypothesis, monotony of *F* and definition of the iterates, *α*(*F* δ+1) ≤ *F* (*α*(*F* δ)) ≤ *F* (*F* δ) = *F* δ+1. Given a limit ordinal *λ*, assume *α*(*F* δ) ≤ *F* δ for all *δ < λ*. Then by definition of the iterates, continuity of *α*, induction hypothesis and definition of lubs, *α*(*F* λ) = *α*( *F* δ) = *α*(*F* δ) ≤ *F* δ = *F* λ. By transfinite

δ<λ

δ<λ

δ<λ

induction, we conclude ∀*δ* ∈ O : *α*(*F* δ) ≤ *F* δ.

Let *ϵ* and *ϵ*' be the respective ordinals such that *F* є = lfp≤ *F* and *F* є' =

lfp≤ *F* . In particular *α*(lfp≤ *F* ) = *α*(*F* є) = *α*(*F* max{є,є'}) ≤ *F* max{є,є'} = *F* є' =

≤

lfp *F* . ✷

The dual of the above Th. 2.5 leads to the *approximation of greatest ﬁx- points from below*. We also need to approximate greatest fixpoints *from above*, as follows:

Theorem 2.6 (Greatest fixpoint upper approximation 1) *Assume that F* ∈ L −−*m*→ L *is a monotonic operator on the co-cpo* ccpo⟨L; ≤*,* T*,* ∧⟩ *and that F* ∈ L −−*m*→ L *is a monotone operator on* ccpo⟨L; ≤*,* T*,* ∨⟩*.*

↓ 

*Let the Scott-co-continuous abstraction function α* ∈ L −−→ L *be such*

*that for all x* ∈ L *such that F* (*x*) ≤ *x there exists y* ≤ *x such that α*(*F* (*x*)) ≤

*F* (*α*(*y*))*.*

*Then α*(gfp

≤

*F* ) ≤ gfp

T

≤

*F.*

α(T)

Proof. Let *F* δ and *F* δ, *δ* ∈ O be the respective ordinal-termed ≤ and ≤-de- creasing ultimately stationary chains of transfinite iterates of *F* and *F* respec- tively starting from T and T [7].

We have *α*(*F* 0) = *α*(T) = *F* 0 by definition of the iterates.

Assume *α*(*F* δ) ≤ *F* δ by induction hypothesis. We have *F* δ+1 = *F* (*F* δ) ≤ *F* δ so that, by hypothesis, there exists *y* ≤ *F* δ such that *α*(*F* δ+1) ≤ *F* (*α*(*y*)). By monotony of *F* and *α*, *F* (*α*(*y*)) ≤ *F* (*α*(*F* δ)) whence by transitivity, in- duction hypothesis, monotony of *F* and definition of the iterates, *α*(*F* δ+1) ≤ *F* (*α*(*F* δ)) ≤ *F* (*F* δ) = *F* δ+1.

Given a limit ordinal *λ*, assume *α*(*F* δ) ≤ *F* δ for all *δ < λ*. Then by defini- tion of the iterates, co-continuity of *α*, induction hypothesis and definition of glbs, *α*(*F* λ) = *α*( *F* δ) = *α*(*F* δ) ≤ *F* δ = *F* λ.

δ<λ

δ<λ

δ<λ

By transfinite induction, we conclude that ∀*δ* ∈ O : *α*(*F* δ) ≤ *F* δ.

Let *ϵ* and *ϵ*' be the respective ordinals such that *F* є = gfp≤ *F* and *F* є' =

gfp≤ *F* . In particular *α*(gfp≤ *F* ) = *α*(*F* є) = *α*(*F* max{є,є'}) ≤ *F* max{є,є'} = *F* є' =

≤

gfp *F* . ✷

A useful variant is:

Theorem 2.7 (Greatest fixpoint upper approximation 2) *Assume that F* ∈ L −−*m*→ L *is a monotonic operator on the co-cpo* ccpo⟨L; ≤*,* T*,* ∧⟩ *and that F* ∈ L −−*m*→ L *is a monotone operator on* ccpo⟨L; ≤*,* T*,* ∧⟩*.*

−−−→

*Let* po⟨L; ≤⟩ ←−γ−− po⟨L; ≤⟩ *be such that for all x* ∈ L *such that F* (*x*) ≤ *x*

α

*there exists y* ≤ *x such that α*(*F* (*x*)) ≤ *F* (*α*(*y*))*.*

*Then α*(gfp

≤

*F* ) ≤ gfp

T

≤ 

*F.*

α(T)

Proof. The proof is similar to that of Th. 2.6 except for limit ordinals. Given a limit ordinal *λ* such that *α*(*F* δ) ≤ *F* δ for all *δ < λ*, we have *F* δ ≤ *γ*(*F* δ) for all *δ < λ*, by definition of Galois connections. Since ⟨*F* δ*, δ < λ*⟩ and

δ

⟨*F , δ < λ*⟩ are decreasing chains and *γ* is monotone, ⟨*γ*(*F*

δ

)*, δ < λ*⟩ is also

decreasing so that *F* δ and *γ*(*F* δ) on one hand and *F* δ on the other

δ<λ

δ<λ

δ<λ

hand do exist respectively in the co-cpos ccpo⟨L; ≤*,* T*,* ∧⟩ and ccpo⟨L; ≤*,*

T*,* ∧⟩. By definition of glbs and *γ* preserving existing glbs, we have *F* δ

δ<λ



≤ *γ*(*F* δ) = *γ*(  *F* δ ). By definition of Galois connections, it follows that

δ<λ δ<λ

*α*( *F* δ) ≤ *F* δ. By definition of the iterates, we conclude that *α*(*F* λ) =

δ<λ δ<λ

*α*( *F* δ) ≤ *F* δ = *F* λ. ✷

δ<λ δ<λ

# The Syntax and Semantics of Programs

Let us consider imperative iterative programs acting on global variables. Pro- grams are assumed to be compiled in an intermediate form as shown by the following example:

X := ?; a : X := ? → b;

while X > 0 do b : (X > 0) → c;

b: ¬ (X > 0) → d;

X := X + 1; c : X + 1 → b;

od; d : stop;

Programs are nondeterministic. The intuition is that if execution is at some label L then one of the transitions L : A → L'; labeled with L is executed, provided the action A is not blocking and the execution can go on by branching to the next label L'. Otherwise the execution is blocked at L, which is the case for the stop command L : stop; intended to correspond to normally expected termination while other blocking commands are supposed to be erroneous.

Nondeterminism is modeled by having several actions be referenced by the same label. For example, the random assignment {L1 : X := ? → L2;} which is a shorthand for {L1 : X := *z* → L2; | *z* ∈ Z}, where Z is the set of integers, can be used to model interactive integer inputs.

* 1. *Abstract Syntax of Programs*

X : X Program variables

E : E Arithmetic expressions

B : B Boolean expressions

A : A Program actions

A ::= X := E Assignment

| X := ? Random assignment

| B Test

| ¬ B Negated test

| skip Null action

| stop Stop action

Programs are collections of labelled nondeterministic commands:

L : L Program labels

C : C Commands

∆ ∆

C ::= L1 : A → L2; labelC*)* = L1*,* actionC*)* = A*,* Transition command

succC*)* = {L2}

∆

∆ ∆

| L1 : stop; labelC*)* = L1*,* actionC*)* = stop*,* Stop command

P : P =∆

succC*)* = ∅

*℘*(C) labelsP*)* =∆ {labelC*)* | C ∈ P} Programs

∆

* 1. *Semantics of Program Actions*

The commands of a program act on global variables X ∈ X which take their values in the semantic domain V.

An *environment ρ* ∈ E maps variables X to their value *ρ*(X) so E = X −−→

∆

V. *ρ*[X := *d*] is the environment *ρ* where the variable X is assigned the value *d*:

∆ ∆

*ρ*[X := *d*](X) = *d* and *ρ*[X := *d*](Y) = *ρ*(Y) when X /= Y.

The semantics of expressions is assumed to be given by AE*)* ∈ E −−→ V for arithmetic expressions E and by BB*)* ∈ E −−→ B where B = {tt*,* ff} for boolean expressions B.

∆

The *semantics* SA*)ρ* of an action A defines the effect of executing this action on the environment *ρ*. Nondeterministic statements such as the random assignment X := ? have more than one possible successor environment so we define S ∈ A −−→ (E −−→ *℘*(E)) as follows:

SB*)* = λ *ρ* • {*ρ* | BB*)ρ* = tt} SX := ?*)* = λ *ρ* • {*ρ*[X := *z*] | *z* ∈ Z}

∆

∆

∆ ∆

S¬ B*)* = λ *ρ* • {*ρ* | BB*)ρ* = ff} Sskip*)* = λ *ρ* • {*ρ*}

∆ ∆

SX := E*)* = λ *ρ* • {*ρ*[X := AE*)ρ*]} Sstop*)* = λ *ρ* • ∅

* 1. *States*

A *state s* ∈ S is a pair *s* = ⟨*ρ,* C⟩ where the environment *ρ* records the values of variables while C is the next command to be executed:

S = E × C

∆

The set of states SP*)* of a program P ∈ P is defined as:

1. SP*)* = E × P

∆

* 1. *Transitional Semantics*

The *transitional semantics* SP*)s* of a program P ∈ P specifies which successor states *s*' can follow state *s* during execution of program P:

SP*)* ∈ SP*)* −−→ *℘*(SP*)*)

∆

SP*)*⟨*ρ,* C⟩ =

{⟨*ρ*'*,* C'⟩| *ρ*' ∈ SactionC*))ρ* ∧ labelC'*)* ∈ succC*)*}

Observe that by Def. (3) of SP*)*, we have C ∈ P and C' ∈ P in (4). In particular

∀*ρ* ∈ E : SP*)*⟨*ρ,* L : stop;⟩ = ∅.

Example 3.1 The program:

1. P = {*a, a*'*, b, c*}

which commands are defined as follows:

*a* = a : Y *>* 0 → b; *a*' =∆

∆

a : ¬ (Y *>* 0) → c;

*b* = b : Y := Y − 1 → a; *c* = c : stop;

∆

∆

has the following transitional semantics:

SP*)*⟨*ρ, a*⟩ = {⟨*ρ, b*⟩| *ρ*(Y) *>* 0}*,* SP*)*⟨*ρ, a*'⟩ = {⟨*ρ, c*⟩| *ρ*(Y) ≤ 0}*,*

SP*)*⟨*ρ, b*⟩ = {⟨*ρ*[Y := *ρ*(Y) − 1]*, a*⟩}*,* SP*)*⟨*ρ, c*⟩ = ∅ *.*

✷

* 1. *Sequences of States*

Program executions are recorded in finite or infinite sequences of states over a given set C of commands. Formally, we define (*→ϵ* ∈ ∅ '−−→ SC*)* is the empty sequence of states, [*n, m*] = {*k* ∈ Z | *n* ≤ *k* ≤ *m*} so [*n, m*] = ∅ when *m < n*):

ΣnC*)* =∆

Σ∗C*)* =∆ Σ∞C*)* =∆

[0*,n* − 1] −−→ SC*),* Σ+C*)* =∆

Σ+C*)* ∪ {*→ϵ*}*,* ΣωC*)* =∆

Σ+C*)* ∪ ΣωC*),* Σ∝C*)* =∆

ΣnC*),*

n>0

N −−→ SC*),*

Σ∞C*)* ∪ {*→ϵ*} *.*

We define the length #*σ* of a sequence *σ* ∈ Σ∝C*)* as 0 when *σ* = *→ϵ*, *n >* 0 when *σ* ∈ ΣnC*)* and the first infinite limit ordinal *ω* when *σ* ∈ ΣωC*)*.

For short, we define (C is the set of commands defined in Sec. 3.1):

Σn =∆ Σω =∆

ΣnC*),* Σ+ =∆

ΣωC*),* Σ∞ =∆

Σ+C*),* Σ∗ =∆

Σ∞C*),* Σ∝ =∆

Σ∗C*),*

Σ∝C*) .*

* 1. *Complete Trace Semantics of Programs*

A *ﬁnite complete execution trace σ* ∈ SnP*)* of a program P ∈ P is a finite sequence *σ*0 *... σ*n−1 ∈ ΣnP*)* of states of length #*σ* = *n* such that:

* + - each state *σ*i*,i* = 1*,...,n* − 1 is the successor of the previous state *σ*i−1 so

*σ*i ∈ SP*)σ*i−1, and

* + - the last state *σ*n−1 is a blocking state so SP*)σ*n−1 = ∅.

The finite complete traces are not empty so S+P*)* =∆ S∗P*)* =∆ S+P*)* ∪ {*→ϵ*} where *→ϵ* is the empty trace.

n>0

SnP*)*. We define

An *inﬁnite execution trace σ* ∈ SωP*)* of a program P ∈ P is an infinite

sequence *σ*0 *... σ*i *...* ∈ ΣωP*)* of states of infinite length #*σ* = *ω* such that each state *σ*i+1 ∈ SP*)σ*i is the successor of the previous state *σ*i, 0 ≤ *i < ω*.

The *complete execution traces* of a program P ∈ P are S∞P*)* =∆ S+P*)*∪SωP*)*

and S∝P*)* =∆ S∞P*)* ∪ {*→ϵ*} = S∗P*)* ∪ SωP*)*.

Formally, the *trace semantics* of a program P ∈ P is defined as follows:

S∞P*)* =∆ S+P*)* =∆

SnP*)* =∆ SωP*)* =∆

S+P*)* ∪ SωP*),*

SnP*),*

n>0

{*σ* ∈ ΣnP*)* | ∀*i* ∈ [0*,n* − 2] : *σ*i+1 ∈ SP*)σ*i ∧

SP*)σ*n−1 = ∅} when *n >* 0,

{*σ* ∈ ΣωP*)* | ∀*i* ≥ 0 : *σ*i+1 ∈ SP*)σ*i} *.*

Example 3.2 The trace semantics of program P defined by (5) is the follow- ing:

S∞P*)* = {⟨*ρ*[Y := *n*]*, a*⟩⟨*ρ*[Y := *n*]*, b*⟩⟨*ρ*[Y := *n* − 1]*, a*⟩ *...*

*...* ⟨*ρ*[Y := 0]*, a*'⟩⟨*ρ*[Y := 0]*, c*⟩| *ρ* ∈ E ∧ *n >* 0}

∪ {⟨*ρ*[Y := *n*]*, a*'⟩⟨*ρ*[Y := *n*]*, c*⟩| *ρ* ∈ E ∧ *n* ≤ 0}

∪ {⟨*ρ*[Y := *n* + 1]*, b*⟩⟨*ρ*[Y := *n*]*, a*⟩⟨*ρ*[Y := *n*]*, b*⟩⟨*ρ*[Y := *n* − 1]*, a*⟩ *...*

*...* ⟨*ρ*[Y := 0]*, a*'⟩⟨*ρ*[Y := 0]*, c*⟩| *ρ* ∈ E ∧ *n >* 0}

∪ {⟨*ρ*[Y := *n* + 1]*, b*⟩⟨*ρ*[Y := *n*]*, a*'⟩⟨*ρ*[Y := *n*]*, c*⟩| *ρ* ∈ E ∧ *n* ≤ 0}

∪ {⟨*ρ, c*⟩| *ρ* ∈ E} *.*

✷

* 1. *Suﬃx-Closure*

The suffix *σ*+ of a trace *σ* ∈ Σ∞ is defined by *s*+ = *s* for traces of length 1 and *sσ*+ = *σ*. Intuitively, *σ*+ describes an execution starting one step after *σ*, if possible. When necessary we let *→ϵ*+ = *→ϵ*.

The suffix of a set T of traces is T + =∆ {*σ*+ | *σ* ∈T }.

a set T of traces is the least suffix-closed superset T = lfp⊆ λ X • T ∪ X + of A set T of traces is *suﬃx-closed* whenever T + ⊆ T . The *suﬃx-closure* of T .

Lemma 3.3 (Suffix-closed trace semantics) *The trace semantics (8) is suﬃx-closed.*

Proof. For finite traces *s* ∈ S∞P*)* of length 1, we have *s*+ = *s* ∈ S∞P*)*.

For finite traces *sσ* ∈ S∞P*)*, we have *sσ*+ = *σ* which belongs to S∞P*)* since each state *σ*i*,i* = 1*,...,n* − 1 is the successor of the previous state *σ*i−1 and the last state *σ*n−1 is a blocking state, by definition of *sσ* ∈ S∞P*)*.

The same way for infinite traces *sσ* ∈ S∞P*)*, we have *sσ*+ = *σ* which belongs to S∞P*)* since each state *σ*i+1 ∈ SP*)σ*i is the successor of the previous state *σ*i, 0 ≤ *i < ω*, by definition of *sσ* ∈ S∞P*)*. ✷

* 1. *Complete Trace Semantics of Programs in Fixpoint Form (1)*

The trace transformer F∞P*)* of a program P ∈ P is defined as follows:

F∞P*)* ∈ *℘*(Σ∞P*)*) −−→ *℘*(Σ∞P*)*)

∆

F∞P*)*T =

{*s* | SP*)s* = ∅} ∪ {*sσ* | *σ*0 ∈ SP*)s* ∧ *σ* ∈T }

Example 3.4 The trace transformer of the program P defined by (5) is the following:

F∞P*)*T = {⟨*ρ, c*⟩| *ρ* ∈ E}

∪ {⟨*ρ, a*⟩⟨*ρ, b*⟩*σ* | *ρ*(Y) *>* 0 ∧ ⟨*ρ, b*⟩*σ* ∈T }

∪ {⟨*ρ, a*'⟩⟨*ρ, c*⟩*σ* | *ρ*(Y) ≤ 0 ∧ ⟨*ρ, c*⟩*σ* ∈T }

∪ {⟨*ρ*[Y := *ρ*(Y)+ 1]*, b*⟩⟨*ρ, a*⟩*σ* | ⟨*ρ, a*⟩*σ* ∈T }

∪ {⟨*ρ*[Y := *ρ*(Y)+ 1]*, b*⟩⟨*ρ, a*'⟩*σ* | ⟨*ρ, a*'⟩*σ* ∈T }

✷

We have the following fixpoint characterizations of the program trace se- mantics [5]:

S∞P*)* = gfp⊆

Σ

∞

F∞P*)*

F∞P*)* is ⊆-monotone which ensures the existence of the fixpoints [20].

* 1. *Feasible Traces*

Some finite or infinite sequences of states such as ⟨*ρ,* L : stop;⟩ω do not corre- spond to any execution of any program. In order to eliminate such infeasible sequences of states, we restrict *traces* to the finite or infinite sequences of states corresponding to potential program executions:

Tn =∆ Tω =∆

SnC*),* T+ =∆

SωC*),* T∞ =∆

S+C*),* T∗ =∆

S∞C*),* T∝ =∆

S∗C*),*

S∝C*) .*

* 1. *Complete Trace Semantics of Programs in Fixpoint Form (2)*

The trace transformer F∞P*)* of a program P ∈ P can also be defined using feasible traces only or arbitrary state sequences containing only commands of P only, as follows:

Theorem 3.5 (Fixpoint complete trace semantics of programs) *For all*

T ∈ *℘*(Σ∞) *such that* S∞P*)* ⊆ T *, we have*

⊆

1. S∞P*)* = gfp F∞P*) .*

T

Proof. By (8) and the dual of Th. 2.2 since Σ∞ ⊇ T ⊇ S∞P*)*. ✷

Corollary 3.6

S∞P*)* = gfp⊆

Σ P)

∞

F∞P*)* = gfp⊆

T

∞

F∞P*)*

Proof. Obviously Σ∞P*)* ⊆ gfp⊆

Σ∞

F∞P*)* by (8) and Def. (6) of S∞P*)* which

implies that all commands appearing along a trace of S∞P*)* belongs to P

proving that this trace belongs to Σ∞P*)*.

⊆

By (8), gfp

∞

Σ

F∞P*)* contains only feasible traces so gfp⊆

Σ

∞

F∞P*)* ⊆ T∞.

Applying Th. 3.5, we conclude that S∞P*)* = gfp⊆

∞

Σ P)

F∞P*)* = gfp⊆

T

∞

F∞P*)*.✷

# Correspondence between Syntax and Trace Seman- tics of Programs

The trace semantics maps programs to sets of traces. Inversely, we map sets of traces to programs by collecting commands appearing along traces.

* 1. *Trace-wide Command Collection*

The abstraction ∞ collects all commands on all traces, as follows:

∞ ∈ *℘*(T∞) −−→ P ∈ T∞ −−→ P ∈ S −−→ C

∆

∆

∞[T ] =∆

σ∈T

[*σ*] [*σ*] =

{ [*σ*i] | 0 ≤ *i <* #*σ*} [⟨*ρ,* C⟩] = C

This correspondence is formalized by the following Galois connection:

Lemma 4.1

po⟨*℘*(T∞ S∞

); ⊆⟩ −←−−−−−−→−→− po⟨P; ⊆⟩

∞

Proof. ∞ and S∞ are obviously ⊆-monotone.

For all programs P ∈ P, we have S∞P*)* ∈ (E×P)+ ∪(E×P)ω so ∞[S∞P*)*] ⊆

P.

Inversely, for all C ∈ P, there may exist an environment *ρ* ∈ E such that

SP*)*⟨*ρ,* C⟩ = ∅ in which case the trace ⟨*ρ,* C⟩ belongs to S∞P*)* et so C belongs to ∞[S∞P*)*]. Otherwise, ∀*ρ* ∈ E : SP*)*⟨*ρ,* C⟩ /= ∅. Let ⟨*ρ*0*,* C0⟩ = ⟨*ρ,* C⟩ and

⟨*ρ*1*,* C1⟩∈ SP*)*⟨*ρ*0*,* C0⟩. We have built a sequence *σ*n = ⟨*ρ*0*,* C0⟩ *...* ⟨*ρ*n*,* Cn⟩ of states, up to *n* = 1, such that ∀*i < n* : ⟨*ρ*i+1*,* Ci+1⟩∈ SP*)*⟨*ρ*i*,* Ci⟩. Having built *σ*n, we may have SP*)*⟨*ρ*n*,* Cn⟩ = ∅ in which case *σ*n ∈ S∞P*)* and consequently C ∈ ∞[S∞P*)*] by definition of ∞. Otherwise, we have ∃⟨*ρ*n+1*,* Cn+1⟩ ∈

SP*)*⟨*ρ*n*,* Cn⟩, so that *σ*n can be extended to *σ*n+1. If we can go on in this way for ever, we obtain a limit trace *σ* which nonempty prefixes are the *σ*n*,n* ≥ 0. We have *σ* ∈ S∞P*)* and *σ* starts with ⟨*ρ,* C⟩ so that C ∈ ∞[S∞P*)*] by definition of ∞. We conclude that P ⊆ ∞[S∞P*)*].

By antisymmetry, we conclude that ∞[S∞P*)*] = P.

Let T ⊆ T∞ and *σ* ∈T . For all 0 ≤ *i <* #*σ*, let *σ*i = ⟨*ρ*i*,* Ci⟩. By definition of ∞, we have {Ci | 0 ≤ *i <* #*σ*} ⊆ ∞[T ]. Moreover, if *σ* is finite so that *n* = #*σ >* 0, we have SC*)σ*n−1 = ∅ = S∞[T ]*)σ*n−1 since Cn−1 ∈ ∞[T ].

Whether *σ* is finite or not, we have *σ*i ∈ SC*)σ*i−1 for all 0 ≤ *i <* #*σ*. But

Ci−1 ∈ ∞[T ] so *σ*i ∈ S∞[T ]*)σ*i−1. It follows that *σ* ∈ S∞∞[T ]*)* proving that T ⊆ S∞∞[T ]*)*. ✷

* 1. *Trace First Command Collection*

Let us define the *ﬁrst command of a trace* as:

∞ ∈ *℘*(T∞) −−→ P 0 ∈ T∞ −−→ C

0

∆

∞[T ] =∆ {

0[*σ*] | *σ* ∈T }

0[*σ*] =

[*σ*0]

Observe that if T is suffix-closed then ∞[T ] = ∞[T ]. It immediately follows

0

0

from (12) and (1) that:

where *γ*∞[Q] =∆ {*σ* ∈ T∞ |

0

γ∞

po⟨*℘*(T∞); ⊆⟩ ←−−0 −− po⟨P; ⊆⟩

−−−∞−→

0

0[*σ*] ∈ Q}.

Moreover, for transformations which eliminate commands from the subject programs, we can use the following correspondence between suffix-closed sets of traces and programs:

Corollary 4.2 *For all programs* P ∈ P*, we have:*

po⟨{T ⊆ Σ∞

P*)* |T +

∞

= T }; ⊆⟩ −←−−−−−−→−→− po⟨*℘*(P); ⊆⟩

S

∞

0

Proof. For all T ⊆ Σ∞P*)* such that T + = T and Q ⊆ P, we have:

∞[T ] ⊆ Q

0

⇔ ∞[T ] = ∞[T ] since T is suffix-closed*3*

0

∞[T ] ⊆ Q

⇔ by Lem. 4.1*3*

T ⊆ S∞Q*) .*

✷

# Blocking Command Elimination

In the following, we consider the *blocking code elimination*, which consists in eliminating blocking commands other than stop commands. The final iterative algorithm is trivial but this case study is simple enough to exemplify the design of correct program transformations by abstract interpretation. In particular the iterative nature of the blocking code elimination algorithm follows from the fixpoint definition (10) of the trace semantics.

* 1. *Introduction to Blocking Command Elimination*

A command C of the form L1 : A → L2; of a program P is *semantically blocking* if and only if it has no possible successor for all evaluation environments (for- mally SP*)*⟨*ρ,* C⟩ = ∅ for all environments *ρ* ∈ E that can be encountered when executing command C in program P). We have singled out a stop command L : stop; corresponding to a normally expected termination. Other blocking commands are considered as undesirable (for example they might correspond to some abnormal termination such as e.g. a runtime error freezing the com-

puter screen). The use of such undesirable semantically blocking commands may be considered as bad program design, and a removal function (prefer- ably an algorithm) tb[P] would be useful to eliminate blocking commands or to check that a program P = tb[P] is well designed according to this criterion. Non-terminating program behaviors should be left unchanged. Because of

tests and iteration, the problem is obviously undecidable so that any effective algorithm b is necessarily an approximation of function tb. For example:

 1 : false → 1; 

2 : skip → 3;





1 : false → 1;

=





b 3 : skip → 5;

4 : stop;

4 : stop;

since the command 3 : skip → 5; and therefore 2 : skip → 3; are blocking. The command 1 : false → 1; is also blocking but is not removed by the syntactic blocking command elimination algorithm b. This is because it is in general not decidable whether B is false in the command 1 : B → 1;. So the syntactic elimination algorithm b only gets rid of syntactically blocking

commands where a command C of the form L1 : A → L2; of a program P is *syntactically blocking* if L2 /∈ labelsP*)*. The command 1 : false → 1; would have been eliminated by the incomputable semantic elimination function tb. In that sense, b is an abstraction of tb.

Obviously a preliminary static program analysis could also be used to determine a larger subset of the semantically blocking actions by taking values of variables into account (e.g. by using the constant propagation static analysis [15]). We do not consider this more refined offline transformation because infinitely many such variants of b can be designed and we choose the simplest one to illustrate our purpose.

* 1. *Semantic Blocking Trace Elimination*

The semantic blocking trace elimination is:

t ∈ *℘*(Σ∞) −−→ *℘*(Σ∞)

b

tb[T ] =

*C*b[*σ*] =

∆

∆

(T ∩ Σm) ∪ {*σ* ∈T | *C* [*σ*]}

*σ* ∈ Σ+ ∧ actionlast[*σ*]*)* = stop

b

where last[*σ*] denotes the command C = last[*σ*] in the last state ⟨*ρ,* C⟩ = *σ*#σ−1

of the finite trace *σ* ∈ Σ+ of length #*σ*.

We define:

*γ*t ∈ *℘*(Σ∞) −−→ *℘*(Σ∞)

b

∆

b

so that:

Lemma 5.1

*γ*tb [Y] =

Y ∪ {*σ* ∈ Σ+ | ¬*C* [*σ*]}

po⟨*℘*(Σ∞

γtb

); ⊆⟩ −←−−−−→− po⟨*℘*(Σ∞

tb

); ⊆⟩

Proof.

tb[X ] ⊆Y

⇔ def. (14) of tb*3*

(X ∩ Σm) ∪ {*σ* ∈X | *C*b[*σ*]}⊆Y

⇔ def. lubs, def. intersection and X ⊆ Σ∞*3* (X ∩ Σm) ⊆Y ∧ (X ∩ {*σ* ∈ Σ∞ | *C*b[*σ*]}) ⊆Y

⇔  *C*b[*σ*] ⇒ *σ* ∈ Σ+ by def. (15) of *C*b*3*

(X ∩ Σm) ⊆Y ∧ ((X ∩ Σ+) ∩ {*σ* ∈ Σ∞ | *C*b[*σ*]}) ⊆Y

⇔  *A* ∩ *B* ⊆ *C* if and only if *A* ⊆ (¬*B* ∪ *C*)*3*

(X ∩ Σm) ⊆Y ∧ (X ∩ Σ+) ⊆ (¬{*σ* ∈ Σ∞ | *C*b[*σ*]}∪ Y)

⇔ (*X* ∩ *B*) ⊆ *Y* if and only if (*X* ∩ *B*) ⊆ (*Y* ∩ *B*)*3*

(X ∩ Σm) ⊆ (Y ∩ Σm) ∧ (X ∩ Σ+) ⊆ ((¬{*σ* ∈ Σ∞ | *C* [*σ*]}∪ Y) ∩ Σ+)

b

⇔ def. complement*3*

(X ∩ Σm) ⊆ (Y ∩ Σm) ∧ (X ∩ Σ+) ⊆ (({*σ* | *σ* /∈ Σ∞ ∨ ¬*C* [*σ*]}∪ Y) ∩ Σ+)

b

⇔ Σ+ ⊆ Σ∞ and def. intersection*3*

(X ∩ Σm) ⊆ (Y ∩ Σm) ∧ (X ∩ Σ+) ⊆ (({*σ* ∈ Σ+ | ¬*C* [*σ*]}∪ Y) ∩ Σ+)

b

⇔ Σ+ ∩ Σm = ∅*3*

(X ∩ Σm) ⊆ (({*σ* ∈ Σ+ | ¬*C* [*σ*]}∪ Y) ∩ Σm) ∧ (X ∩ Σ+) ⊆ (({*σ* ∈ Σ+ |

b

¬*C* [*σ*]}∪ Y) ∩ Σ+)

b

⇔ (*X* ∩ *B*) ⊆ *Y* if and only if (*X* ∩ *B*) ⊆ (*Y* ∩ *B*)*3*

(X ∩ Σm) ⊆ ({*σ* ∈ Σ+ | ¬*C* [*σ*]}∪Y) ∧ (X ∩ Σ+) ⊆ ({*σ* ∈ Σ+ | ¬*C* [*σ*]}∪Y)

b b

⇔ def. lubs*3*

(X ∩ Σm) ∪ (X ∩ Σ+) ⊆ ({*σ* ∈ Σ+ | ¬*C* [*σ*]}∪ Y)

b

⇔ X ⊆ Σ∞ = Σ+ ∪ Σm *3*

X ⊆ ({*σ* ∈ Σ+ | ¬*C* [*σ*]}∪ Y)

b

⇔ def. (16) of *γ*tb *3*

X ⊆ *γ*tb [Y]

✷

Intuitively Lem. 5.1 states that *the transformed semantics is an abstrac- tion of the subject semantics*. This corresponds to the idea that the program transformation looses some information on the original program. For exam- ple the elimination of blocking commands looses all behavior about blocking program behaviors, constant propagation looses all information about how constants are computed, partial evaluation looses all information on program computations for input values other than the ones for which the program is

specialized, etc.

Let 1S = λ *x* ∈ *S* • *x* be the identity operator on a set *S* and tb[T ] =

∆

∆

{tb[*σ*] | *σ* ∈T } be the right image of T by tb. We have:

Lemma 5.2 *If* T ⊆ Σ∞ *then:*

po⟨*℘*(t [T ]); ⊆⟩ **←**−−−tb−−−− po⟨*℘*(T ); ⊆⟩

b −−−−−−−−→

1℘(tb[T ])

Proof. Observe that tb is a lower closure operator that is reductive (∀X ⊆ T : tb[X ] ⊆ X ), idempotent (tb ◦ tb = tb) and monotone (∀X *,* Y ⊆ T : (X ⊆ Y) ⇒ (tb[X ] ⊆ tb[Y])). It follows that for all X ⊆ tb[T ] and Y ⊆T , we have:

1℘(tb[T ])(X ) ⊆Y

⇒def. identity*3*

X ⊆ Y

⇒ X ∈ *℘*(tb[T ]) so that there exists Z ∈ *℘*(T ) such that X = tb[Z]*3* tb[Z] ⊆Y

⇒ tb is monotone*3*

tb[tb[Z]] ⊆ tb[Y]

⇒ tb is idempotent*3*

tb[Z] ⊆ tb[Y]

⇒ def. X = tb[Z]*3*

X ⊆ tb[Y]

⇒ tb is reductive and ⊆ is transitive*3*

X ⊆ Y

⇒ def. identity*3* 1℘(t [T ])(X ) ⊆Y

b

proving that 1℘(t [T ])(X ) ⊆Y if and only if X ⊆ tb[Y]. Moreover 1℘(t [T ]) ∈

b b

*℘*(tb[T ]) −−→ *℘*(T ) is injective. ✷

It immediately follows from Lem. 5.2 with T = Σ∞P*)* that:

po⟨*℘*(t [Σ∞P*)*]); ⊆⟩ ←←−−−−t−b −−−−− po⟨*℘*(Σ∞P*)*); ⊆⟩*,*

b

so that by duality:

−−−−−−−−−−→

1℘(tb[Σ∞ P)])

1℘(tb[Σ∞ P)])

1. po⟨*℘*(Σ∞P*)*); ⊇⟩ ←−−−−−−−−−− po⟨*℘*(t [Σ∞P*)*]); ⊇⟩ *.*

−−−−−−−−−−→→ b

t

b

The intuition is that tb is a dual abstraction which can be used to approximate greatest fixpoints from above.

* 1. *Observational Abstraction*

For a program transformation to be correct, the semantics of the subject and transformed programs should be equivalent at some level of observation. This observational equivalence can be formalized in the abstract interpretation framework by requiring that the abstraction of the semantics of the subject and of the transformed programs into an abstract observation domain should to be identical:

∀P ∈ P : α (S∞P*)*) = α (S∞P*))*) *.*

O O

The specification of the observational abstraction αO must be considered as part of the problematics (in that it explicitly defines the chosen correctness criterion).

* 1. *Observational Abstraction for Blocking Code Elimination*

In the particular case of blocking code elimination, the observational abstrac- tion αO(T ) of traces T is tb[T ], in that:

* + - all infinite behaviors of T are observed in tb[T ];
    - all complete finite behaviors of T terminating with a stop command are observed in tb[T ];
    - no other trace of T is observed in tb[T ] so none of the complete finite behaviors terminating of T with a non-stop blocking command is observed in tb[T ].
  1. *Transformation Design Strategy*

Our objective is to constructively derive a blocking code elimination algorithm b transforming a subject program P into a transformed program bP*)* such that P and bP*)* have equivalent semantics for the tb observational abstrac- tion:

α (t [S∞P*)*]) = α (S∞P*))*)

O b O b

this is

t [t [S∞P*)*]] = t [S∞P*))*] since αO = tb for blocking command elimination, hence

b b b b

t [S∞P*)*] = t [S∞P*))*] *.*

b b b

since tb is idempotent.

Our design strategy is to first derive the non-blocking trace semantics of programs t [S∞P*)*] by abstraction of the trace semantics S∞P*)* and then to design the blocking command elimination algorithm bP*)* as an abstraction of t [S∞P*)*].

b

b

* 1. *Non-Blocking Trace Semantics of Programs*

We define the non-blocking trace semantics of a program P as:

S∞P*)* =∆

t [S∞P*)*] *.*

We observe that S∞P*)* is suffix-closed since, by (18) and (14), it contains all infinite execution traces of S∞P*)* (which suffix is also an infinite execution trace of S∞P*)*), the traces *s* of length 1 reduced to a stop command (such that *s*+ = *s*) and finite traces of the form *sσ* which are execution traces of S∞P*)* which, by (15), end with a stop command so that their suffix *sσ*+ = *σ* is also a finite execution trace of S∞P*)* ending with a stop command.

b

b

b

In order to express S∞P*)* algorithmically as a fixpoint iteration, we can start from the fixpoint form (10) of the program execution trace semantics,

b

such that S∞ = t [gfp⊆ F∞P*)*] where F∞P*)*T = {*s* | SP*)s* = ∅} ∪{*sσ* | *σ* ∈

∆

b b Σ∞ P) 0

SP*)s* ∧ *σ* ∈ T }. Then (17) leads to the idea of using the dual of Cor. 2.4 to

express S∞P*)* in greatest fixpoint form

b

T-strictness

⊆

gfp

b

F∞P*)*. We have:

t [Σ∞P*)*] is the ⊆-supremum of *℘*(t [Σ∞P*)*]); Scott co-continuity

b b

By (17), tb is a complete ∩-morphism hence Scott co-continuous; For the commutation condition, we have:

t [F∞P*)*T ]

b

= By def. (14) of tb*3*

(F∞P*)*T ∩ Σm) ∪ {*σ* ∈ F∞P*)*T | *C* [*σ*]}

b

= By def. (7) of F∞P*)3*

({*s* | SP*)s* = ∅} ∪ {*sσ* | *σ*0 ∈ SP*)s* ∧ *σ* ∈ T } ∩ Σm) ∪ {*σ* ∈ {*s* | SP*)s* =

∅} ∪ {*sσ* | *σ*0 ∈ SP*)s* ∧ *σ* ∈T }| *C*b[*σ*]}

= def. Σm and ∪*3*

{*s* | SP*)s* = ∅∧ *C* [*s*]}∪ {*sσ* | *σ* ∈ SP*)s* ∧ *σ* ∈ T ∩ Σm}∪ {*sσ* | *σ* ∈

b 0 0

SP*)s* ∧ *σ* ∈T ∧ *C*b[*σ*]}

= def. (15) of *C*b[*σ*]*3*

{*s* | SP*)s* = ∅ ∧ ∃*ρ,* L : *s* = ⟨*ρ,* L : stop;⟩ ∪ {*sσ* | *σ*0 ∈ SP*)s* ∧ *σ* ∈

(T ∩ Σm) ∪ {*σ*' ∈T | *C* [*σ*']}}

b

= def. (4) of SP*)* and (14) of tb*3*

{⟨*ρ,* L : stop;⟩| L : stop; ∈ P ∧ *ρ* ∈ E}∪ {*sσ* | *σ*0 ∈ SP*)s* ∧ *σ* ∈ tb[T ]}

= F∞P*)* ◦ t [T ]

b b

by defining:

b

F∞P*)* =∆

λ T • {⟨*ρ,* L : stop;⟩| L : stop; ∈ P ∧ *ρ* ∈ E}∪

{*sσ* | *σ*0 ∈ SP*)s* ∧ *σ* ∈T } *.*

We conclude, by the dual of Cor. 2.4, that:

⊆

S∞P*)* =∆ t [S∞P*)*] = t [gfp

F∞P*)*] = gfp⊆ F∞P*) .*

b b b

Σ∞ P)

tb[Σ∞ P)] b

* 1. *Blocking Command Elimination Algorithm*

We can now design the syntactic blocking command elimination algorithm bP*)* as an upper approximation of the non-blocking trace semantics of pro- grams:

P*)* ⊇ ∞[S∞P*)*] = ∞[S∞P*)*] = ∞[gfp⊆

F∞P*)*]

b b 0 b

0 tb[Σ∞ P)] b

since S∞P*)* is suffix-closed and by (20). Then (13) leads to the idea of using Th. 2.7 to constructively derive the algorithmP*)*. For all T ⊆ Σ∞P*)*, we have:

b

b

∞[F∞P*)*T ]

0 b

= def. (12) of ∞*3*

0

{ 0[*σ*] | *σ* ∈ F∞P*)*T}

b

= def. (19) of F∞P*)3*

b

{ 0[⟨*ρ,* L : stop;⟩] | L : stop; ∈ P ∧ *ρ* ∈ E}∪{ 0[*sσ*] | *σ*0 ∈ SP*)s* ∧ *σ* ∈T }*,*

= def. (12) of 0 and (11) of , *s, σ*0 ∈ SP*)* and def. (3) of SP*)* so that *s* = ⟨*ρ,* C⟩, *σ*0 = ⟨*ρ*'*,* C'⟩ and *σ* = *σ*0*σ*'*3*

{L : stop; | L : stop; ∈ P}∪{ [⟨*ρ,* C⟩] | ∃*ρ*' ∈ E : ∃*σ*' ∈ Σ∞P*)* : ∃C' ∈ C :

⟨*ρ*'*,* C'⟩∈ SP*)*⟨*ρ,* C⟩∧ ⟨*ρ*'*,* C'⟩*σ*' ∈T }

= def. (11) of and (4) of SP*)3*

{L : stop; | L : stop; ∈ P}∪ {C ∈ P | ∃*ρ*' ∈ E : ∃*σ*' ∈ Σ∞P*)* : ∃C' ∈ P :

*ρ*' ∈ SactionC*))ρ* ∧ labelC'*)* ∈ succC*)* ∧ ⟨*ρ*'*,* C'⟩*σ*' ∈T }

⊆ Ignoring the (maybe undecidable) condition *ρ*' ∈ SactionC*))ρ3*

{L : stop; | L : stop; ∈ P}∪ {C ∈ P | ∃*ρ*' ∈ E : ∃*σ*' ∈ Σ∞P*)* : ∃C' ∈ P :

labelC'*)* ∈ succC*)* ∧ ⟨*ρ*'*,* C'⟩*σ*' ∈T }

= def. (12) of 0*3*

{L : stop; | L : stop; ∈ P}∪

{C ∈ P | ∃C' ∈ P : labelC'*)* ∈ succC*)* ∧ C' ∈ 0[T ]}

= def. ∩*3*

{L : stop; | L : stop; ∈ P}∪

{C ∈ P | {labelC'*)* | C' ∈ 0[T ] ∩ P}∩ succC*)* /= ∅}

= by def. of labels in Sec. 3.1*3*

{L : stop; | L : stop; ∈ P}∪ {C ∈ P | succC*)* ∩ labels0[T ] ∩ P*)* /= ∅}

= bP*)* ◦ 0[T ]

by defining:

1. bP*)* = λ Q • {L : stop; | L : stop; ∈ P}∪

∆

{C ∈ P | succC*)* ∩ labelsQ∩ P*)* /= ∅}*,*

Moreover 0[t [Σ∞P*)*]] = P so by (13) and Th. 2.7, we conclude that:

b

P*)* =

⊆

gfp

∆

P*)* ⊇ ∞[gfp⊆

F∞P*)*] = ∞[S∞P*)*] *.*

b P b

0 tb[Σ∞ P)] b b

⊆

All iterates of gfp

P

bP*)* are included in P so that we have

⊆

bP*)* = gfp

P

'P*)*

with

b

bP*)* = λ Q • {L : stop; | L : stop; ∈ Q} ∪

∆

{C ∈Q| succC*)* ∩ labelsQ*)* /= ∅}*,*

Observe that po⟨P; ⊆⟩ satisfies the descending chain condition so that the above fixpoint form of bP*)* immediately leads to an effective iteration algo- rithm, that we can describe informally as follows:

* Start from Q := P;
* Repeat

Suppress the commands C from Q such that C /= L : stop; and succC*)* ∩

labelsQ*)* = ∅;

Until Q is left unchanged;

* Return Q.
  1. *Correctness of the Blocking Command Elimination Algorithm*

The correctness of the transformation is stated by the fact that the observation of the semantics of the subject and transformed programs by the observational abstraction α = t is the same. Formally, α (S∞P*)*) = α (S∞P*))*), that is

O b O O b

t [S∞P*)*] = t [S∞P*))*] *.*

b b b

Proof. By Lem. 4.1, S∞ is monotone. By (17), t is monotone. By (22),

b

⊆

we have bP*)* = gfp bP*)* so bP*)* ⊆ P. By monotony we conclude that

P

t [S∞P*))*] ⊆ t [S∞P*)*].

b

b

b

By Lem. 4.1, S∞ ◦ ∞ is extensive so that tb[S∞P*)*] ⊆ S∞∞[tb[S∞P*)*]]*)*.

By (22), ∞[S∞P*)*] ⊆ P*)*. By Lem. 4.1, S∞ is monotone. So by monotony,

b

∆

b

b

b

S∞∞[S∞P*)*]*)* ⊆ S∞

b

P*))*. By (18), S∞P*)* =

t [S∞P*)*] so that we have

S∞∞[t [S∞P*)*]]*)* ⊆ S∞P*))*.

b

b b

By transitivity, t [S∞P*)*] ⊆ S∞P*))*. By Lem. 5.2, t is monotone and idempotent so t [S∞P*)*] = t [t [S∞P*)*]] ⊆ t [S∞P*))*].

b b b

b b b b b

By antisymmetry, we conclude that t [S∞P*)*] = t [S∞P*))*]. ✷

b b b

# Conclusion

The general idea to formalize program transformation by abstract interpreta- tion is to define a semantic transformation as an abstraction of the subject program semantics. This transformation is an abstraction in that the trans- formed semantics has lost some information on the subject semantics (e.g. the existence of blocking traces). The correctness of the semantic transformation is proved using an observational abstraction specifying which details about the subject and transformed semantics should be abstracted away to consider them as equivalent. Then the syntactic – source to source – program trans- formation is constructively derived by abstraction of transformed semantics into a transformed program. This new approach has been illustrated on the simple case of blocking command elimination.

Many more complex examples such as transition compression, constant propagation, partial evaluation, slicing, etc. have to be treated similarly in order to convince that this point of view is quite general. This will probably require the generalization of the present program transformation framework, for example using weaker hypotheses on abstraction in absence of a best ap- proximation [9].

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