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A Note On the Turing Degrees

of Divergence Bounded Computable Reals

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Abstract

The Turing degree of a real number is defined as the Turing degree of its binary expansion. In this note we apply the double witnesses technique recently developed by Downey, Wu and Zheng [[2](#_bookmark2)] and show that there exists a ∆0-Turing degree which contains no divergence bounded computable real numbers. This extends the result of [[2](#_bookmark2)] that not every ∆0-Turing degree contains a d-c.e. real.

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*Keywords:* Turing degree of reals, divergence bounded computable reals.

# Introduction

The computability of real numbers was introduced by Turing [[11](#_bookmark11)] by means of decimal expansions. Namely, a real number *x* is *computable* if it has a computable decimal expansion, i.e., *x* = *n*∈N *f* (*n*) · 10−*n* for a computable function *f* : N → {0*,* 1*,* ··· *,* 9}. This notion is independent of the represen- tations of the reals as shown by Robinson [[9](#_bookmark9)] and others. Concretely, a real

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*x* ∈ [0; 1] is computable iff the Dedekind cut *Lx* := {*r* ∈ Q : *r < x*} is a computable set; iff *x* has a computable binary expansion, i.e., there is a com- putable set *A* such that *x* = *xA* := *n*∈*A* 2−(*n*+1); and iff there is a computable

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sequence (*xs*) of rational numbers which converges to *x* effectively in the sense that |*xs* − *xs*+1| ≤ 2−*s* for all *s*, and so on. Similarly, the Turing degree of a real number and the Turing reducibility between real numbers can be nat- urally defined based on binary expansion, Dedekind cut or Cauchy sequence representations of the reals, respectively and it is not difficult to see that they are equivalent (see e.g., [[5](#_bookmark5)]). For example, *xA* is Turing reducible to *xB* (de- noted by *xA* ≤*T xB*) if *A* ≤*T B* and *xA* ≡*T xB* if *xA* ≤*T xB* & *xB* ≤*T xA*. In addition, the Turing degree of a real *x* is defined as the class of all reals which are Turing equivalent to *x*, namely, deg*T* (*x*) := {*y* ∈ R : *y* ≡*T x*}.

Especially, the degree of a computable real number consists of all computable

reals. On the other hand, Ho [[6](#_bookmark6)] shows that a real number is Turing reducible to 0', the Turing degree of the halting problem, if and only if it is computably approximable, where the computably approximable reals are simply the lim- its of computable sequences of rational numbers (see [[1](#_bookmark1)]). The Turing degree of a computably approximable real is also called ∆0-Turing degree or simply

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∆0-degree.

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Between the classes of computable and computably approximable real numbers there are a lot of interesting classes of real numbers introduced in literatures ([[1](#_bookmark1),[7](#_bookmark7),[12](#_bookmark12)]). For example, *x* is left (right) computable if it is the limit of an increasing (decreasing) computable sequence of rational numbers. The left computable reals are also called computably enumerable (c.e., for short) (see [[3](#_bookmark3),[4](#_bookmark4)]) because their left Dedekind cuts are c.e. sets of rational numbers. *x* is d-c.e. if it is the difference of two c.e. reals. The class of d-c.e. real numbers (denoted by WC) is a very interesting class. Ambos-Spies, Weihrauch and the first author showed in [[1](#_bookmark1)] that the class WC is the arithmetical closure

of the c.e. reals and hence is a field and that *x* is d-c.e. if and only if there is a computable sequence (*xs*) of rational numbers which converges to *x weakly effectively* in the sense that the sum *s*∈N |*xs* − *xs*+1| is bounded.

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Since any c.e. real has a c.e. left Dedekind cut, the Turing degree of any

c.e. real is c.e., although not every c.e. real has a c.e. binary expansion as observed by Jockusch (see [[10](#_bookmark10)]). On the other hand, every non-computable

c.e. degree contains a non-c.e. real. For the d-c.e. reals, the situation is a little bit complicated. The case for the d-c.e. reals is different. The first author shows in [[13](#_bookmark13)] that there is a d-c.e. real whose Turing degree is not even *ω*-c.e. Here a set *A* ⊆ N is *ω*-c.e. if there are a computable function *h* and a computable sequence (*As*) of finite sets which converges to *A* such that

|{*s* ∈ N : *n* ∈ *As*∆*As*+1}| ≤ *h*(*n*) for all *n*. A Turing degree is called *ω*-c.e. if

it contains an *ω*-c.e. set. Recently, Downey, Wu and the first author showed in [[2](#_bookmark2)] that any *ω*-c.e. Turing degree contains at least a d-c.e. real, but there exists a ∆0-Turing degree which does not contain any d-c.e. real.

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In this paper, we explore the Turing degrees of another class of reals, namely, the divergence bounded computable reals which were introduced by the authors together with Gengler and von Braunmu¨hl in [[7](#_bookmark7)]. A real number

*x* is called *divergence bounded computable* (dbc, for short) if there are a total

computable function *h* and a computable sequence (*xs*) of rational numbers which converges to *x h-bounded effectively* in the sense that there are at most *h*(*n*) non-overlapping 2−*n*-jumps (*i, j*) for all *n*. Here an index pair (*i, j*) is a 2−*n*-jump of the sequence (*xs*) if |*xi* − *xj*| ≥ 2−*n*. It is shown that, every

d-c.e. real is dbc but there is computably approximable real which is not dbc, i.e., the class of all dbc reals (denoted by DBC) is strictly between the classes of d-c.e. and computably approximable reals. Like WC, the class DBC is also a field. Furthermore, the class DBC is also closed under total computable real functions while WC is not so. Actually, DBC is the closure of WC under computable total real functions (see [[8](#_bookmark8)]). We will show in this paper that, even the Turing degrees of dbc reals does not exhaust all the ∆2-Turing degrees.

# Main Result

This section gives a construction which shows that not every ∆0-degree con- tains a divergence bounded computable real. The construction is another interesting example of the “double witnesses” technique introduced in [[2](#_bookmark2)].

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In the following, the use function of computable functionals Φ and Ψ are denoted by the lower case *ϕ* and *ψ*, respectively. W.l.o.g. we assume that *n* ≤ *ϕe,s*(*n*) ≤ *s* for all *n* and *s*, where *ϕe,s*(*n*) denotes the use function of Φ*e* for the input *n* up to stage *s*. We identify a set *A* with its characteristic function. That is, *A*(*n*) = 1 ⇐⇒ *n* ∈ *A* and *A*(*n*) = 0 ⇐⇒ *n* ∈*/ A* for all *n*. For convenience, we write *A*(*n*)(*m*) := *A*(*n*)*A*(*m*).

Theorem 2.1 *There exists a* ∆0*-degree which contains no divergence bounded*

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*computable reals.*

Proof. We will construct a computable sequence (*As*) of finite subsets of natural numbers which converges to *A* such that *A* is not Turing equivalent to any divergence bounded computable real. To this end, let (*be, he,* Φ*e,* Ψ*e*) be an effective enumeration of all tuples of computable functions *be* :⊆ N → D, *he* :⊆ N → N, and computable functionals Φ*e,* Ψ*e*. For any *e, s* ∈ N, if *be*(*s*) is defined, then let *Be,s* be a finite set of natural numbers such that *be*(*s*) = *xB*e,s . Thus, the set *A* has to satisfy all the following requirements.

*R* : *be* and *he* are total and (*be*(*s*)) con- =⇒ *A* /= Φ*B*e ∨ *B*

*e*

/= Ψ*A.*

verges *he*-bounded effectively to *xB*e

*e e e*

The sequence (*As*) is constructed in stages such that *As* is the approximation of *A* at the end of stage *s* and *A* = lim*s*→∞ *As*. We define a length function *l* as follows:

*l*(*e, s*) := max{*x* : *As* T *x* = Φ*B*e,s T *x* & *Be,s* T *ϕe,s*(*x*) = Ψ*A*s T *ϕe,s*(*x*)}*,*

*e,s e,s*

where *ϕe* is the use function of the functional Φ*e*. Thus, to satisfy a require- ment *Re*, it suffices to guarantee that *l*(*e, s*) is bounded from above, if the premisses of *Re* hold.

Now let’s describe the strategy to satisfy a single requirement *Re*. We first choose a witness *ne* large enough. At the beginning, let *A*(*ne* − 1)(*ne*) = 00, i.e., both *ne* − 1 and *ne* are not in *A*. Then we wait for a stage *s* such that *l*(*e, s*) *> ne*. If there does not exist such *s* at all, then *l*(*e, s*) ≤ *ne* for all *s* and we are done. Otherwise, suppose that *s*1 is the first stage such that

*l*(*e, s* ) *> n* . In this case, both computations Φ*B*e,s1 (*n* ) and Φ*B*e,s1 (*n*

— 1)

1 *e e,s*1 *e e,s*1 *e*

halt and hence *ϕe,s*1 (*ne* − 1) and *ϕe,s*1 (*ne*) are defined. Furthermore, the initial

segment Ψ*A*s1

*e,s*1

T *ϕe,s*1

(*ne*) is defined too. Let *me*

:= *ψe,s*1

(*ϕe,s*1

(*ne*)). Assume

w.l.o.g. that *ne < me*. If *he,s*1 (*me*) is also defined, then we put *ne* − 1 into

*A* (the number *ne* remains out of *A*) to destroy the agreement. That is, we define *As*1+1(*ne* − 1)(*ne*) = 10. Then we wait for a new stage *s*2 *> s*1 such that *l*(*e, s*2) *> ne* holds again. If no such a stage exists, then we are done again. Otherwise, we put *ne* into *A*, i.e., let *As*2+1(*ne* − 1)(*ne*) := 11. If there exists another stage *s*3 *> s*2 such that *l*(*e, s*3) *> ne*, then we take both *ne* − 1 and *ne* out of *A*, i.e., let *As*3+1(*ne* − 1)(*ne*) := 00. In this case, the set *As*3 +1 is recovered to that of stage *s*1, i.e., *As*3+1 = *As*1 . This closes a cycle in which the values *A*(*ne* − 1)(*n*2) change in the order of 00 → 10 → 11 → 00. This process will continue as long as the number of 2−*m*e -jumps of the sequence (*xB*e,s ) (i.e., the sequence (*be*(*s*)) ) does not exceed *he*(*me*) yet.

Thus, we achieve a temporary disagreement between *A* and Φ*B*e by chang- ing the values *A*(*ne* −1)(*ne*) whenever the length of agreement goes beyond the witness *ne*. After that, if the agreement becomes bigger than *ne* again, then the corresponding value Φ*B*e (*ne* − 1)(*ne*) has to be changed too and this forces the

*e*

*e*

initial segment *Be* T *ϕe*(*ne*) to be changed, say, *Be,s* T *ϕe,s*(*ne*) /= *Be,t* T *ϕe,t*(*ne*).

There are two possibilities now.

Case 1. This corresponds to a 2−*m*e -jump, i.e., |*xB*e,s − *xB*e,t | ≥ 2−*m*e . If the sequence (*be*(*s*)) converges *he*-bounded effectively, then (*be*(*s*)) has at most *he*(*me*) non-overlapping 2−*m*e -jumps. Thus, this can happen at most *he*(*me*) times

Case 2. The change of the initial segment *Be* T *ϕe*(*ne*) does not lead to a 2−*m*e -jump. That is, |*xB*e,s − *xB*e,t | = 2−*m <* 2−*m*e for a natural number *m > me*. In this case, there exists a (least) natural number *n < me* such that *Be,s*(*n*) /= *Be,t*(*n*) because *Be,s* T *me* /= *Be,t* T *me* (remember that *me* ≥ *ϕe*(*ne*)).

In this case, as binary word, *Be,s* has one of the following forms

form 1 := 0 *. w* 1 0 ··· 0 ··· 0 *v*

form 2 := 0 *. w* 0 1 ··· 1 ··· 1 *v*

↑ ↑ ↑

(positions : *n me m* )

for some *w, v* ∈ {0*,* 1}∗ and *Be,t* takes another one. Here *n, me* and *m* indicate the corresponding positions. This implies that, if the sequence (*xB*e(*s*)) does not have 2−*m*e -jumps after some stage *s* any more, then the initial segment *Be,s* T *me* can have only two possible forms: 0*.w*10 ··· 0 or 0*.w*01 ··· 1. Corre-

spondingly, the combination Φ*B*e,s (*n* − 1)(*n* ) can have at most two possibil-

*e,s e e*

ities too. However, in every circle described above, *A*(*ne* − 1)(*ne*) takes three different forms, i.e., 00*,* 10 and 11. In other words, we can always achieve a disagreement *A* /= Φ*B* at some stage and hence the requirement *Re* is satisfied

*e*

eventually.

To satisfy all requirements simultaneously, we apply a finite injury priority construction. In this case, *Re* has higher priority than *Ri* if *e < i*. To preserve the requirement *Re* from the disturbance by lower priority *Ri*, the initial seg- ment *A* T *ψeϕe*(*ne*) should be preserved. To this end, only the elements which

are larger than *ψeϕe*(*ne*) are allowed to be appointed as witnesses of *Ri* for

*i > e* afterward. *me* := *ψe*(*ϕe*(*ne*)) is called a restriction of *Re*. A requirement

*Re* being *initialized* means that *ne* and *me* (if any) are set to be undefined.

The following is a formal construction of the sequence (*As*). Stage 0: Let *A*0 := ∅ and all requirements *Re* are initialized.

Stage *s* + 1: A requirement *Re requires attention* if the following conditions

hold.

1. the witness *ne,s* is defined such that *ne,s < l*(*e, s*) holds (in this case the restriction *me,s* := *ψe,s*(*ϕe,s*(*ne,s*)) is defined too);
2. *he,s*(*me,s*) is defined and the sequence (*be*(*t*))*t*≤*s* does not make more than

*he,s*(*me,s*) non-overlapping 2−*m*e,s -jumps so far.

If no requirement requires attention at this stage, then choose a least *e* such that *ne* is currently not defined and let *ne,s*+1 := *s* + 2 (remember the convention that *ϕe,s*(*n*) ≤ *s* for all *n, e*). Otherwise, suppose that *Re* is the re- quirement of highest priority (i.e., of the least index) which requires attention. Then we define

 01*,* if *As*(*ne* − 1)(*ne*) = 00;



*As*+1(*ne* − 1)(*ne*) := 11*,* if *As*(*ne* − 1)(*ne*) = 01;

 00*,* if *As*(*ne* − 1)(*ne*) = 11.



In addition, all requirements *Ri* of lower priority (i.e., *i > e*) are *initialized*. In this case, we say that *Re receives attention*.

This completes the construction.

To show that the constructed sequence (*As*) converges to a set *A* which satisfies all requirements *Re*, it suffices to prove the following claim.

Claim 2.2 *For any e* ∈ N*, the requirement Re receives attention ﬁnitely many times and is eventually satisﬁed.*

*Proof of Claim*: The claim can be proved by an induction on *e*. Assume that, for any *i < e*, the requirements *Ri* receive attention only finitely many times and are satisfied eventually. Let *s*0 be the first stage such that no requirement *Ri* for *i < e* requires and receives attention after stage *s*0 any more. By the minimality of *s*0, *Re* is initialized at stage *s*0. By construction, there is a stage *s*1 *> s*0 at which the witness *ne,s*1 is defined. This witness will not be changed any more, i.e., *ne,s*1 = *ne,s* for all *s* ≥ *s*1, because *Re* will never be initialized after stage *s*1 again. For convenience, let *ne* := *ne,s*1 .

If *Re* does not require attention after stage *s*1, then either the length *l*(*e, s*)

does not go beyond *ne*, or *l*(*e, s*) *> ne* for some *s* but either *he*(*me*) (for *me* := *ψe,s*(*ϕe,s*(*ne*))) is not defined or the sequence (*be*(*s*))*s* has already more 2−*m*e -jumps than *he*(*me*). In any of these cases, *Re* is satisfied.

Otherwise, suppose that *Re* requires and receives attention after stage *s*1 at stages *t*0 + 1*, t*1 + 1*, t*2 + 1*,* ·· ·. Thus, both *me* := *ψe,t*0 (*ϕe,t*0 (*ne*)) and *he*(*me*) := *he,t*0 (*me*) are defined. At stage *t*0 + 1, all requirements *Ri* for *i > e* are initialized. If a witness *ni* for *Ri* is appointed later, then we have *ni > t*0 + 1 *> me*. This implies that the initial segment *As* T *me* will not be changed after stage *t*0 + 1 except the elements *ne* − 1 and *ne*. Furthermore, since *A*(*ne* − 1)(*ne*) changes always in the order 00 → 10 → 11 → 00, we can prove by a simple induction on *n* that *At*i T *me* = *At*3n+i T *me* for all *n* ∈ N and *i* = 0*,* 1*,* 2. This implies obviously that

Ψ*A*ti T *ϕ*

(*n* ) = Ψ*A*t3n+i T *ϕ*

(*n* )

*e,t*i

*e,t*i *e*

*e,t*3n+i

*e,t*3n+i *e*

and, because of *ne < l*(*e, t*3*n*+*i*), hence

*Be,t*i T *ϕe,t*i (*ne*) = *Be,t*3n+i T *ϕe,t*3n+i (*ne*)*.*

Thus, we have *ϕe,t*0 (*ne*) = *ϕe,t*3n (*ne*) and *ψe,t*0 (*ϕe,t*0 (*ne*)) = *ψe,t*3n (*ϕe,t*3n (*ne*)) for

all *n*. Since *A*(*ne* − 1)(*ne*) changes whenever *Re* receives attention, Φ*B*e (*ne* −

*e*

1)(*ne*) has to be changed before *Re* receives a new attention. That is, *Be,t*n T

*me* /= *Be,t*n+1

n+1

T *me* for all *n*. By construction, the case |*xB*e,tn

− *xB*e,t

|≥ 2−*m*e

can happen at most *he*(*me*) times. If |*xB*e,tn

n+1

— *xB*e,t

| *<* 2−*m*e , then we have

*B**e,t*n T *me* = 0*.w*10 ··· 0 or *Be,t*n T *me* = 0*.w*01 ··· 1 for some binary word *w*.

That is, *Be,t*n T *me* switches between two different forms. On the other hand,

*A*(*ne* − 1)(*ne*) takes three different values consecutively. This means that after

some stage, an agreement between *A* and Φ*B*e of a length bigger than *ne* is

*e*

impossible and *Re* stops requiring further attention. Therefore, *Re* receives attention only finitely often and it is satisfied eventually.

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