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A Survey on Mixed-Integer Programming Techniques in Bilevel Optimization

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Bilevel optimization is a field of mathematical programming in which some variables are constrained to be the solution of another optimization problem. As a consequence, bilevel optimization is able to model hierarchical decision processes. This is appealing for modeling real-world problems, but it also makes the resulting optimiza- tion models hard to solve in theory and practice. The scientific interest in computational bilevel optimization increased a lot over the last decade and is still growing. Independent of whether the bilevel problem itself con- tains integer variables or not, many state-of-the-art solution approaches for bilevel optimization make use of techniques that originate from mixed-integer programming. These techniques include branch-and-bound meth- ods, cutting planes and, thus, branch-and-cut approaches, or problem-specific decomposition methods. In this survey article, we review bilevel-tailored approaches that exploit these mixed-integer programming techniques to solve bilevel optimization problems. To this end, we first consider bilevel problems with convex or, in par- ticular, linear lower-level problems. The discussed solution methods in this field stem from original works from the 1980’s but, on the other hand, are still actively researched today. Second, we review modern algorithmic ap- proaches to solve mixed-integer bilevel problems that contain integrality constraints in the lower level. Moreover, we also briefly discuss the area of mixed-integer nonlinear bilevel problems. Third, we devote some attention to more specific fields such as pricing or interdiction models that genuinely contain bilinear and thus nonconvex aspects. Finally, we sketch a list of open questions from the areas of algorithmic and computational bilevel opti- mization, which may lead to interesting future research that will further propel this fascinating and active field of research.

# Introduction

min

*𝑦*∈*𝑌*

*𝑓* (*𝑥, 𝑦*) (2a)

In this paper, we consider bilevel optimization problems of the gen- eral form

s.t. *𝑔*(*𝑥, 𝑦*) ≥ 0*.* (2b)

Problem [(1)](#_bookmark7) is the so-called upper-level (or the leader’s) problem

min

*𝑥*∈*𝑋,𝑦*

*𝐹* (*𝑥, 𝑦*) (1a)

lem. Moreover, the variables *𝑥* ∈ ℝ*𝑛𝑥* are the upper-level variables (or and Problem [(2)](#_bookmark6) is the so-called lower-level (or the follower’s) prob-

s.t. *𝐺*(*𝑥, 𝑦*) ≥ 0*,* (1b)

*𝑦* ∈ *𝑆*(*𝑥*)*,* (1c)

where *𝑆*(*𝑥*) is the set of optimal solutions of the *𝑥*-parameterized prob-

lem

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leader’s decisions) and *𝑦* ∈ ℝ*𝑛𝑦* are lower-level variables (or follower’s decisions). The objective functions are given by *𝐹 , 𝑓* ∶ ℝ*𝑛𝑥* × ℝ*𝑛𝑦* → ℝ and the constraint functions by *𝐺* ∶ ℝ*𝑛𝑥* × ℝ*𝑛𝑦* → ℝ*𝑚* as well as *𝑔* ∶ ℝ*𝑛𝑥* × ℝ*𝑛𝑦* → ℝ𝓁. The sets *𝑋 ⊆* ℝ*𝑛𝑥* and *𝑌 ⊆* ℝ*𝑛𝑦* are typically used to denote in- tegrality constraints. For instance, *𝑌* = ℤ*𝑛𝑦* makes the lower-level prob-

lem an integer program. In what follows, we call upper-level constraints

*𝐺𝑖*(*𝑥, 𝑦*) ≥ 0, *𝑖* ∈ {1*,* … *, 𝑚*}, coupling constraints if they explicitly depend

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on the lower-level variable vector *𝑦*. Moreover, all upper-level variables

that appear in the lower-level constraints are called linking variables.

We use the nomenclature that the bilevel problem [(1)](#_bookmark7) is called an “UL-LL problem” where UL and LL can be LP, QP, MILP, MIQP, etc. if the upper-/lower-level problem is a linear, a quadratic, a mixed-integer linear, a mixed-integer quadratic, etc. program in both the variables of the leader and the follower. If the concrete specification of both levels is not required, we also use a shorter nomenclature and say, e.g., that the problem is a bilevel LP, if both levels are LPs.

Most of the time, we will consider the optimistic version of the bilevel

the lower-level outcome *𝑦* ∈ *𝑆*(*𝑥*) if the lower-level solution set *𝑆*(*𝑥*) is problem as it is given in [(1)](#_bookmark7). In this case, the leader also optimizes over

not a singleton. On the contrary, the pessimistic version is given by

min max *𝐹* (*𝑥, 𝑦*) s.t. *𝐺*(*𝑥, 𝑦*) ≥ 0 for all *𝑦* ∈ *𝑆*(*𝑥*)*.*

*𝑥*∈*𝑋 𝑦*∈*𝑆*(*𝑥*)

[For the general pessimistic setting, we refer to Wiesemann et al. (2013) and the recent surveys on pessimistic bilevel optimization in](#_bookmark253) [Liu et al. (2018)](#_bookmark184) and [Liu et al. (2020a)](#_bookmark185).

Instead of using the point-to-set mapping *𝑆* one can also use the so-

called optimal value function

*𝜑*(*𝑥*) ∶= min {*𝑓* (*𝑥, 𝑦*) ∶ *𝑔*(*𝑥, 𝑦*) ≥ 0} (3)

*𝑦*∈*𝑌*

and re-write Problem [(1)](#_bookmark7) as

[as (generalized) Benders decomposition (Benders, 1962; Geoffrion, 1972); see the books by](#_bookmark95) [Conforti](#_bookmark114) [et al. (2014);](#_bookmark95) [Jünger](#_bookmark179) [et al. (2010);](#_bookmark95) [Wolsey (1998)](#_bookmark257) for a comprehensive overview about mixed-integer lin- ear programming techniques. Moreover, also specific techniques from mixed-integer nonlinear programming such as outer approximation [(Bonami et al., 2008; Duran and Grossmann, 1986; Fletcher and Leyf- fer, 1994) or spatial branching (](#_bookmark107)[Horst](#_bookmark169) [and Tuy, 2013) are covered; see](#_bookmark107) [Belotti et al. (2013)](#_bookmark87); [Lee (2012)](#_bookmark224) for recent overviews on mixed-integer nonlinear optimization. For the more theoretical aspects of bilevel opti- mization we refer to [Dempe (2002)](#_bookmark124) and the references therein.

Obviously, the entire field of bilevel optimization is much broader and we thus are not able to cover everything. For instance, we do not [cover the fields of bilevel optimization under uncertainty (Besançon et al., 2019; 2020; Burtscheidt and Claus, 2020; Burtscheidt et al., 2020; Dempe et al., 2017; Ivanov, 2018; Jain et al., 2008; Pita et al., 2010;](#_bookmark100) [Yanikoglu](#_bookmark126) [and Kuhn, 2018), fractional bilevel optimization (Calvete and](#_bookmark100) [Galé, 1999; 2004), or purely continuous nonconvex bilevel problems](#_bookmark126) ([Dempe et al., 2019; Fliege et al., 2020](#_bookmark133)).

[Finally, let us mention already existing surveys (Colson et al., 2005; 2007) and books (](#_bookmark113)[Bard,](#_bookmark71) [1998; Dempe, 2002; Dempe et al., 2015) in the](#_bookmark113) field of bilevel optimization. Other very early survey articles include [Anandalingam and Friesz (1992)](#_bookmark54); [Ben-Ayed (1993)](#_bookmark89); [Kolstad (1985)](#_bookmark209); [Vicente and Calamai (1994)](#_bookmark241) as well as [Wen and Hsu (1991)](#_bookmark250) re- garding the field of linear bilevel optimization. Last but not least,

[Dempe (2020)](#_bookmark126) contains, to the best of our knowledge, the largest an-

min

*𝑥*∈*𝑋,𝑦*∈*𝑌*

*𝐹* (*𝑥, 𝑦*) (4a)

notated list of references in the field of bilevel optimization.

The remainder of this survey is structured as follows. In [Section 2](#_bookmark11),

s.t. *𝐺*(*𝑥, 𝑦*) ≥ 0*, 𝑔*(*𝑥, 𝑦*) ≥ 0*,* (4b)

*𝑓* (*𝑥, 𝑦*) ≤ *𝜑*(*𝑥*)*,* (4c)

to which we will refer as the value-function reformulation. This refor- mulation indicates that for the optimistic version of the problem, we can assume without loss of generality that all upper-level variables are linking variables; see [Bolusani and Ralphs (2020)](#_bookmark105).

Bilevel optimization problems date back to the seminal publications on leader-follower games of [von Stackelberg (1934, 1952)](#_bookmark243). The intro- duced formulation was first used in [Bracken and McGill (1973)](#_bookmark112) in the context of a military application regarding the cost-minimal mix of weapons. Another very early discussion of multilevel, or, in particular, two-level problems can be found in [Candler and Norton (1977)](#_bookmark84). Over the years, bilevel optimization has been recognized as an important model- ing tool since it allows to formalize hierarchical decision processes that often appear in application areas such as energy, security, or revenue management. We postpone the discussion of selected applied literature to the following sections.

The ability to model hierarchical decision processes also makes bilevel optimization problems notoriously hard to solve. For instance, already their easiest instantiation with a linear upper- and lower-level problem is strongly NP-hard; see [Section 3](#_bookmark12) for the details. Thus, eﬃcient,

i.e., polynomial-time, algorithms cannot be expected unless P = NP. This

also makes the development of solution algorithms a diﬃcult task on the

one hand—but on the other hand “allows” for enumeration-based algo- rithms such as branch-and-bound. During the last years and decades it turned out that the development of solution algorithms for bilevel op- timization problems strongly depends on the structure and properties of the lower-level problem as well as on the coupling between the up- per and the lower level. For instance, the solution techniques are very much different depending on whether the lower-level problem is con- tinuous and convex or whether it is nonconvex, e.g., due to the presence of integer variables.

In this survey, we focus on algorithmic techniques to actually solve bilevel problems. In particular, we discuss techniques from mixed-integer linear or nonlinear optimization that are applied in the field of bilevel optimization. These basic and well-studied tech- niques include branch-and-bound ([Land and Doig, 1960](#_bookmark223)) or cut- ting planes ([Kelley, 1960](#_bookmark186)) as well as decomposition techniques such

we collect selected applications from various different fields to moti- vate the study of bilevel problems. Afterward, in [Section 3](#_bookmark12), we discuss bilevel optimization problems with linear or, at least, convex lower-level problems. For this problem class, we study important general properties, derive classical single-level reformulations, and give a comprehensive overview of the algorithms used to solve these problems. The case of bilinear bilevel problems is discussed in [Section 4](#_bookmark29), where we focus on pricing problems and Stackelberg games. In [Section 5](#_bookmark35), we then turn to bilevel problems with mixed-integer (non)linear lower-level problems. Also for these problems, we first focus on general properties before we then turn to generic approaches for solving bilevel MILPs and bilevel MINLPs. [Section 6](#_bookmark40) is then devoted to interdiction problems. Here, we discuss both discrete as well as continuous interdiction problems, dif- ferent fields of applications, and different classes of algorithms to tackle these problems. The survey closes with a collection of possible directions for future research in [Section 7](#_bookmark50).

# Selected Applications

In this section, we present a selection of the vast literature on ap- plications of bilevel optimization. Due to the enormous number of publications, this review will be far from being comprehensive. Many other application-oriented papers can, e.g., be found in the survey by [Dempe (2020)](#_bookmark126) or by [Sinha et al. (2018b)](#_bookmark229).

*Early Applications*

Among the first, bilevel optimization has been applied to military defense problems in [Bracken and McGill (1973)](#_bookmark112) and to agricultural plan- ning; see [Candler et al. (1981)](#_bookmark85); [Fortuny-Amat and McCarl (1981)](#_bookmark174). The latter topic is also picked up in [Bard et al. (2000)](#_bookmark77). Recent references [concerning the defense of critical infrastructure (Alguacil et al., 2014; Borrero et al., 2019; Caprara et al., 2016; DeNegre, 2011; Fioretto et al., 2019-07; Scaparra and Church, 2008; Wood, 2011) are related to the](#_bookmark52) mentioned early military applications. Many of these bilevel problems were originally considered in the field of game theory and are thus often called Stackelberg games. A particular attention has been given to those involving a finite number of strategies; see, e.g., [Sections 4.2](#_bookmark32) and [6](#_bookmark40).

Other early applications can be found in chemical process design [that involves thermodynamic equilibria; see, e.g., Clark and Wester-](#_bookmark109) [berg](#_bookmark155) [(1983),](#_bookmark109) [Clark](#_bookmark110) [and Westerberg (1990),](#_bookmark109) [Clark](#_bookmark108) [(1990), and Gümüş](#_bookmark109) [and Ciric (1997).](#_bookmark155)

*Traffic and Transportation*

Bilevel traﬃc and transportation planning problems are covered, [among others, in](#_bookmark92) [LeBlanc and Boyce (1986)](#_bookmark180)[,](#_bookmark92) [Marcotte (1986)](#_bookmark198)[, Ben- Ayed et al. (1988),](#_bookmark92) [Ben-Ayed](#_bookmark90) [et al. (1992), and](#_bookmark92) [Migdalas](#_bookmark205) [(1995), as well](#_bookmark92) as more recently in [Fontaine and Minner (2014)](#_bookmark173) or [Gairing et al. (2017)](#_bookmark137). Usually, the upper level models the decisions on the transportation net- work design, while the lower level models the individual behavior of the users of the network. Additionally, bilevel optimization is also used [for the detection and solution of aircraft conflicts (Cerulli et al., 2019; 2020), for which tailored cutting planes are proposed.](#_bookmark102)

*Management Science*

in a pool-based network-constrained electricity market are studied in [Ruiz et al. (2012)](#_bookmark248) and [Fampa et al. (2008)](#_bookmark156) analyze strategic pricing in competitive electricity markets. Other works consider demand-side management ([Aussel et al., 2020; Grimm et al., 2020](#_bookmark64)), the scheduling [of maintenance outages of a set of transmission lines (Pandzic et al., 2012), or how to economically exploit wind resources at a given lo-](#_bookmark225) cation from a transmission-cost perspective ([Morales et al., 2012](#_bookmark212)). For a recent survey on bilevel optimization in energy and electricity mar- kets see [Wogrin et al. (2020)](#_bookmark256). Besides electricity, gas markets are ad- dressed by bilevel optimization as well; see, e.g., [Böttger et al., 2021](#_bookmark80); [Grimm et al. (2019b)](#_bookmark153); [Schewe et al. (2021)](#_bookmark260) for models of the European entry-exit gas market.

# Continuous Linear and/or Convex Lower-Level Problems

The general form of an LP-LP bilevel problem, i.e., a bilevel problem in which all constraints and objective functions are linear, is as follows:

min *𝑐⊤𝑥* + *𝑐⊤𝑦* (5a)

In the context of management science, in [Bard (1983)](#_bookmark68), bilevel optimization is used to coordinate multi-divisional firms. Further, [Ryu et al. (2004)](#_bookmark251) address bilevel decision-making problems under un-

*𝑥,𝑦 𝑥 𝑦*

s.t. *𝐴𝑥* + *𝐵𝑦* ≥ *𝑎,* (5b)

certainty in the context of enterprise-wide supply chain optimization,

*𝑦* ∈ arg min { *⊤*

*𝑦̄* ∶ *𝐶𝑥* + *𝐷𝑦̄* ≥ *𝑏*}

(5c)

[Garcia-Herreros et al. (2016)](#_bookmark143) consider bilevel capacity expansion plan- ning problems, and [Reisi et al. (2019)](#_bookmark242); [Yue and You (2017)](#_bookmark270) consider sup- ply chain problems. In [Dan et al. (2020)](#_bookmark120); [Dan and Marcotte (2019)](#_bookmark121), the authors consider service firms deciding on the location and service lev- els of its facilities, taking into account the behavior of the user. This re- sults in mixed-integer nonlinear bilevel problems, for which tailored ap- proaches are provided. Finally, bilevel portfolio optimization problems are considered in, e.g., [González-Díaz et al. (2020)](#_bookmark146); [Leal et al. (2020)](#_bookmark179).

*𝑑*

*Machine Learning*

Bilevel problems are also discussed in the context of statistics and machine learning. In [Bennett et al. (2006)](#_bookmark96); [Bennett et al. (2008)](#_bookmark98), bilevel optimization is applied to hyper-parameter selection for statistical learn- ing methods. An evolutionary bilevel algorithm for the same purpose is given in [Sinha et al. (2014)](#_bookmark232). Very recently, [Franceschi et al. (2018)](#_bookmark176) intro- duce a framework based on bilevel programming that unifies gradient-

*𝑦̄*

with *𝑐𝑥* ∈ ℝ*𝑛𝑥* , *𝑐𝑦 , 𝑑* ∈ ℝ*𝑛𝑦* , *𝐴* ∈ ℝ*𝑚*×*𝑛𝑥* , *𝐵* ∈ ℝ*𝑚*×*𝑛𝑦* , and *𝑎* ∈ ℝ*𝑚* as well as

*𝐶* ∈ ℝ𝓁×*𝑛𝑥* , *𝐷* ∈ ℝ𝓁×*𝑛𝑦* , and *𝑏* ∈ ℝ𝓁 . Note that we already omitted a lin- ear term depending on the upper-level variables *𝑥* in the lower-level

objective function since this term would not have any influence on the optimal solutions of the lower level. Moreover, for the ease of presen- tation, we always use linear lower-level problems if this is suitable to describe the general ideas and only use nonlinear but convex lower-level problems if this is required.

* 1. *General Properties*

We introduce two concepts that are useful to derive solution algo-

lems. First, we consider the feasible region *𝐻* of the so-called high-point rithms since they lead to bounds on the optimal value of bilevel prob- relaxation (HPR), which is defined as the set of points (*𝑥, 𝑦*) satisfying

the leader and follower constraints, i.e., for Problem [(5)](#_bookmark13) it is given by

based hyper-parameter optimization and meta-learning. { }

*Energy Networks and Markets*

*𝐻* ∶= (*𝑥, 𝑦*) ∈ ℝ*𝑛𝑥* × ℝ*𝑛𝑦* ∶ *𝐴𝑥* + *𝐵𝑦* ≥ *𝑎, 𝐶𝑥* + *𝐷𝑦* ≥ *𝑏 .*

Clearly, the solution of the HPR

Arguably, energy networks and markets are two of the largest ar-

{ }

min *⊤ ⊤*

eas of application; see, e.g., the book of [Gabriel et al. (2012)](#_bookmark138) with

*𝑥,𝑦*

*𝑐𝑥 𝑥* + *𝑐𝑦 𝑦* ∶ (*𝑥, 𝑦*) ∈ *𝐻*

(6)

many applications and models. Some selected contributions that par- ticularly consider electricity networks and markets are given in the fol- lowing. [Arroyo (2010)](#_bookmark57) analyze the vulnerability of power systems and [Motto et al. (2005)](#_bookmark215) analyze the security of power grids under disrup- tive threats. Problems of generation and transmission expansion plan- [ning are studied in](#_bookmark183) [Garcés et al. (2009)](#_bookmark140)[,](#_bookmark183) [Jenabi et al. (2013)](#_bookmark181)[, or Jin and Ryan (2011). See also](#_bookmark183) [Bylling](#_bookmark124) [et al. (2020) for a stochastic bilevel model](#_bookmark183)

provides a lower bound on the optimal objective value of the bilevel problem, because it relaxes the optimality of the lower-level prob- lem [(5c)](#_bookmark14). Second, we consider the bilevel feasible region T, which is also denoted as the “inducible region” of the bilevel problem. This set particularly takes the optimal response of the follower into account and is given by

in this context. In [Grimm et al. (2016)](#_bookmark149), the authors propose a problem- tailored solution approach based on binary search to solve a similar

T ∶= {(*𝑥, 𝑦*) ∈ *𝐻* ∶ *𝑦*satisf ies

[(5c})](#_bookmark14)*.*

problem. Further, [Baringo and Conejo (2012)](#_bookmark78) deal with transmission

Having this notion at hand, we can write the bilevel LP [(5)](#_bookmark13) as

and wind power investment. Optimal placement of measurement de-

min

{ }

*𝑐⊤𝑥* + *𝑐⊤𝑦* ∶ (*𝑥, 𝑦*) ∈ T *.*

vices in an electrical network has been modeled as a bilevel MILP in [Poirion et al. (2020)](#_bookmark239). The authors develop a generic branch-and-cut pro- cedure that can be applied to problems with a similar type of bilevel constraints. [Grimm et al. (2019a)](#_bookmark147); [Kleinert and Schmidt (2019b)](#_bookmark199) de- velop a Benders-like decomposition approach to compute optimal price

zones of electricity markets. The approach is applied to the German elec- tricity market in [Ambrosius et al. (2020)](#_bookmark55). [Ruiz and Conejo (2009)](#_bookmark246) con-

*𝑥,𝑦 𝑥 𝑦*

This implies that any bilevel feasible point provides an upper bound on the optimal value of the bilevel LP.

To better understand the special features and properties of bilevel LPs, we illustrate them with some graphical examples involving one variable at each level. The problem

sider a strategic power producer that trades electric energy in an elec- { }

tricity pool. Similarly, the equilibria reached by strategic producers

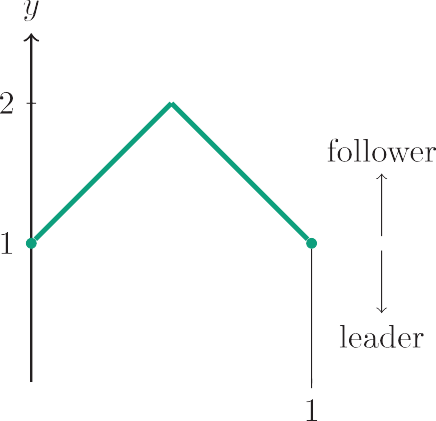
min

*𝑥,𝑦*

*𝑦* ∶ *𝑦* ∈ arg min {−*𝑦̄* ∶ (*𝑥, 𝑦̄*) ∈ P} *,*

*𝑦̄*

**Fig. 1.** Illustration of the example in [Section 3.1](#_bookmark15).



with the lower level’s feasible region given by

P = {(*𝑥, 𝑦*) ∶ *𝑦* ≥ 0*, 𝑦* ≤ 1 + *𝑥, 𝑦* ≤ 3 − *𝑥,* 0 ≤ *𝑥* ≤ 1}*,*

is depicted in [Figure 1](#_bookmark17) (left). The feasible points of the HPR coincide with the lower-level feasible region P since there are no upper-level constraint. The horizontal segment linking the origin and point (1,0) constitutes the set of solutions of the high-point relaxation, i.e., those

points in *𝐻* that minimize the upper-level objective function. Since the

corresponding upper-level objective function is 0 on this segment, this

leads to a lower bound of 0 for the entire bilevel LP. The bilevel feasible region T is given by the union of the two segments in thick green. In- terestingly, T is nonconvex although both levels are linear optimization problems. The problem has the two optimal solutions (0,1) and (1,1) with value 1.

Now, if we add the constraint *𝑦* ≤ *𝑎* with 1 *< 𝑎 <* 2 to the upper level,

the bilevel feasible region is reduced to two disjoint segments as de-

picted in [Figure 1](#_bookmark17) (right). Nonetheless, these segments constitute faces of the high-point relaxation. An even worse situation may happen if

to *𝑎* ∈ (0*,* 1). Then, the bilevel feasible region is empty, i.e., the bilevel the right-hand side of the constraint added to the upper level is set

LP has no feasible point, although the high-point relaxation is feasible. This last example is also useful to illustrate the effect of moving cou- pling constraints, i.e., upper-level constraints involving variables of the

lower level, between the two levels. If, e.g., the constraint *𝑦* ≤ 1∕2 is

points (*𝑥,* 1∕2) with 0 ≤ *𝑥* ≤ 1 are bilevel optimal. The two facts that (i) added to the lower level, then the problem becomes feasible and all

coupling constraints of a bilevel LP may lead to a disconnected bilevel feasible region and that (ii) they cannot be moved to the lower level without changing the set of optimal solutions have been discussed by [Audet et al. (2006)](#_bookmark58) and [Mersha and Dempe (2006)](#_bookmark203).

Another interesting property is that the unboundedness of the HPR [(6)](#_bookmark16) does not allow to conclude about the optimal solution of the bilevel problem. An illustrative example, borrowed from [Xu (2012)](#_bookmark261); [Xu and Wang (2014)](#_bookmark263) and slightly simplified here, demonstrates three different situations, in each of which the HPR solution is unbounded, but, depending on the objective function of the lower-level problem, the bilevel problem is either unbounded, infeasible, or admits an opti- mal solution. To this end, consider the bilevel problem

For *𝑑* = 0, the bilevel problem is unbounded as the lower-level prob- lem is feasible for all *𝑦*. For *𝑑* = 1, the bilevel problem is infeasible, as

*𝜑*(*𝑥*) = ∞. Finally, for *𝑑* = −1, the problem admits a unique optimal so- lution (*𝑥, 𝑦*) = (2*,* 2).

Despite the rather complicating properties of *𝐻* and T that

we described above, the two sets can be exploited algorithmically.

[Bard (1984)](#_bookmark72). For the ease of exposition, let us assume that *𝐻* is bounded The groundwork for this is laid in [Bialas and Karwan (1984)](#_bookmark103) and

and nonempty for what follows. In the following, we will explain that the bilevel feasible region is a union of faces of the high-point relax- ation and that a bilevel optimal solution is attained at one of the ver- tices of this union. This is already illustrated in the previous example.

A point (*𝑥, 𝑦*) belonging to the bilevel feasible region T must satisfy all

constraints defining the polyhedron *𝐻* and must be an optimal solution of the lower-level LP. Thus, (*𝑥, 𝑦*) must satisfy the Karush–Kuhn–Tucker

LP, which imply that each constraint is either active at (*𝑥, 𝑦*) or that the (KKT; see, e.g., [Nocedal and Wright (2006)](#_bookmark220)) conditions of the lower-level corresponding dual variable is equal to 0. Consider now the face *𝐹* of the polyhedron *𝐻* obtained by setting all constraints active at point (*𝑥, 𝑦*) at equality. All points on *𝐹* also satisfy the KKT conditions for a dual solu- tion corresponding to (*𝑥, 𝑦*) implying that *𝐹 ⊆* T. This property implies that a bilevel LP possesses an optimal solution that is a vertex of *𝐻* and

by [(5a)](#_bookmark13) over each (maximal) face of *𝐻* included in the bilevel feasible that it can be found by solving an LP whose objective function is given

region.

[The so-called *𝐾* th-Best algorithm proposed by Bialas and Kar- wan (1984) searches for a vertex of *𝐻* that is optimal for the bilevel](#_bookmark103)

LP by starting with a vertex that minimizes [(5a)](#_bookmark13) and then itera- tively generates adjacent vertices with nondecreasing value for [(5a)](#_bookmark13) un- til a vertex belonging to the bilevel feasible region is found. In the

worst case, the *𝐾* th-Best algorithm requests to visit an exponential

number of vertices of *𝐻* (remember that the bilevel feasible region may be empty even though *𝐻* is not). This is not surprising as

[Hansen et al. (1992)](#_bookmark157) have shown that bilevel LPs are strongly NP-hard (see also [Jeroslow (1985)](#_bookmark182) for NP-hardness) by reducing the graph prob- lem KERNEL and [Vicente et al. (1994)](#_bookmark242) have shown that even check-

[Audet et al. (1997)](#_bookmark59) remark that a binary constraint, say *𝑥* ∈ {0*,* 1}, ap- ing local optimality of a given point is NP-hard. In the same vein,

max

*𝑥,𝑦*

*𝑥* + *𝑦*

additional variable *𝑦* and the constraints *𝑦* = 0 and pearing in a single-level optimization problem can be modeled by an

s.t. 0 ≤ *𝑥* ≤ 2*,*

*𝑦* ∈ arg max {*𝑑𝑦*′ ∶ *𝑦*′ ≥ *𝑥*}

*𝑦*′

and its high-point relaxation

*𝑦* = arg max {*𝑦̄* ∶ *𝑦̄* ≤ *𝑥, 𝑦̄* ≤ 1 − *𝑥*}*.*

*𝑦̄*

As a consequence, linear optimization problems with binary variables

max

*𝑥,𝑦*

*𝑥* + *𝑦*

are a special case of bilevel LPs. Further hardness results are also stated in [Bard (1991)](#_bookmark73), where some general properties of bilevel LPs are dis-

s.t. 0 ≤ *𝑥* ≤ 2*,*

*𝑦* ≥ *𝑥.*

cussed as well. A survey about complexity results for bilevel LP problems

can be found in [Deng (1998)](#_bookmark141). The strongest complexity result was ob-

lems. Specifically, he showed that a *𝑘*-level LP problem belongs to the tained by [Jeroslow (1985)](#_bookmark182), who proved hardness of multilevel LP prob- complexity class Σ*𝑝* .

*𝑘*−1

Finally, given that the objective functions of both levels play a role in a bilevel problem, it would be tempting to conclude that the optimal solution of a bilevel LP is Pareto-optimal with respect to these objectives.

However, [Marcotte and Savard (1991)](#_bookmark200) have shown that this is not true

all) local solutions of the MPCC. We refer the reader to [Dempe (1987)](#_bookmark123); [Still (2002)](#_bookmark236), where this is used to solve the underlying bilevel problem to local optimality.

Besides this approach based on the lower level’s KKT conditions, one can also use a strong duality theorem for the lower-level problem. The dual problem to [(7)](#_bookmark20) is given by

unless *𝑐𝑦*

and *𝑑* are parallel.

max

*𝜆*

(*𝑏* − *𝐶𝑥*)*⊤𝜆* s.t. *𝐷⊤𝜆* = *𝑑, 𝜆* ≥ 0*.* (9)

* 1. *Single-Level Reformulations*

If the lower-level problem of the bilevel optimization model at hand is convex and satisfies a suitable constraint qualification (which, in the convex case, usually is Slater’s constraint qualification), then one can reformulate the bilevel problem into a single-level optimization prob- lem. To this end, one either uses the KKT conditions of the lower-level problem or a strong duality theorem applied to the lower-level prob- lem. In this section, we discuss both approaches and restrict ourselves, for the ease of presentation, to the case of LP-LP bilevel problems of the type given in [(5)](#_bookmark13). The lower-level problem [(5c)](#_bookmark14) can be seen as the

For a given decision *𝑥* of the leader, weak duality of linear optimization

states that

*𝑑⊤𝑦* ≥ (*𝑏* − *𝐶𝑥*)*⊤𝜆*

holds for every primal and dual feasible pair *𝑦* and *𝜆*. Thus, by strong du-

ality, we know that every such feasible pair is a pair of optimal solutions

if

*𝑑⊤𝑦* ≤ (*𝑏* − *𝐶𝑥*)*⊤𝜆*

holds. Consequently, we can reformulate the bilevel problem as

*𝑥*-parameterized linear problem

min *𝑐⊤𝑥* + *𝑐⊤𝑦* (10a)

*𝑥,𝑦,𝜆 𝑥 𝑦*

min

*𝑦*

*𝑑⊤𝑦* s.t. *𝐷𝑦* ≥ *𝑏* − *𝐶𝑥.* (7)

s.t. *𝐴𝑥* +

*𝐵𝑦* ≥ *𝑎, 𝐶𝑥* +

*𝐷𝑦* ≥ *𝑏,* (10b)

Its Lagrangian function is given by

G(*𝑦, 𝜆*) = *𝑑⊤𝑦* − *𝜆⊤*(*𝐶𝑥* + *𝐷𝑦* − *𝑏*)

and the KKT conditions are given by dual feasibility

*𝐷⊤𝜆* = *𝑑, 𝜆* ≥ 0*,*

primal feasibility

*𝐶𝑥* + *𝐷𝑦* ≥ *𝑏,*

and the KKT complementarity conditions

*𝜆𝑖* (*𝐶𝑖*⋅*𝑥* + *𝐷𝑖*⋅*𝑦* − *𝑏𝑖* ) = 0 for all *𝑖* = 1*,* … *,* 𝓁*.*

Here and in what follows, *𝐶𝑖*⋅ denotes the *𝑖*th row and *𝐶*⋅*𝑗* denotes the

*𝑗*th column of *𝐶*. Since the lower-level feasible region is polyhedral,

the Abadie constraint qualification holds and the KKT conditions are

both necessary and suﬃcient. Thus, the LP-LP bilevel problem can be reformulated as

*𝑐⊤𝑥 𝑐⊤𝑦* (8a)

min +

*𝑥,𝑦,𝜆 𝑥 𝑦*

s.t. *𝐴𝑥* + *𝐵𝑦* ≥ *𝑎, 𝐶𝑥* + *𝐷𝑦* ≥ *𝑏,* (8b)

*𝐷⊤𝜆* = *𝑑, 𝜆* ≥ 0*,* (8c)

*𝜆𝑖* (*𝐶𝑖*⋅*𝑥* + *𝐷𝑖*⋅*𝑦* − *𝑏𝑖* ) = 0 for all *𝑖* = 1*,* … *,* 𝓁*.* (8d)

we additionally have to include the lower-level dual variables *𝜆*. Since Note that we now optimize over an extended space of variables since we optimize over *𝑥*, *𝑦*, and *𝜆* simultaneously, any global solution of [(8)](#_bookmark22) is

an optimistic bilevel solution. Problem [(8)](#_bookmark22) is linear except for the KKT complementarity conditions that turn the problem into a nonconvex and nonlinear optimization problem (NLP). More precisely, Problem [(8)](#_bookmark22) is a mathematical program with complementarity constraints (MPCC); see, e.g., [Luo et al. (1996)](#_bookmark194). Unfortunately, standard NLP algorithms usually cannot be applied for such problems since classical constraint qualifica- tions like the Mangasarian–Fromowitz or the linear independence con- [straint qualification are violated at every feasible point; see, e.g., Ye and Zhu (1995). For a primer on constraint qualifications in nonlinear opti-](#_bookmark265)

*𝐷⊤𝜆* = *𝑑, 𝜆* ≥ 0*,* (10c)

*𝑑⊤𝑦* ≤ (*𝑏* − *𝐶𝑥*)*⊤𝜆.* (10d)

Here, the 𝓁 KKT complementarity constraints in [(8)](#_bookmark22) are replaced with the scalar inequality in [(10d)](#_bookmark21). Note that the general nonconvexity of LP- LP bilevel problems is reflected in this single-level reformulation due to

the bilinear products of the primal upper-level variables *𝑥* and the dual

lower-level variables *𝜆*.

Let us close this section with a remark on single-level reformulations

of problems more general than LP-LP bilevel problems. Both reformu- lations discussed can be applied as long as compact global optimality certificates for the lower level are available. This is, in general, the case if the lower-level problem is convex and if Slater’s constraint qualifi- cation holds. However, both the MPCC [(8)](#_bookmark22) and the nonconvex prob- lem [(10)](#_bookmark19) are only equivalent to the original bilevel problem if globally optimal solutions are considered and if Slater’s constraint qualification holds. In particular, locally optimal solutions of Problem [(8)](#_bookmark22) are not nec- [essarily locally optimal for the original bilevel problem; see Dempe and Dutta (2012) for the details.](#_bookmark127)

* 1. *Algorithms*

The most likely earliest published paper on mixed-integer program- [ming techniques for bilevel optimization is the one by Fortuny-Amat and McCarl (1981). The authors consider a bilevel optimization problem](#_bookmark174) with a quadratic programming problem (QP) in the upper and the lower level. For the ease of presentation, we explain the core ideas based on the LP-LP bilevel problem [(5)](#_bookmark13). The authors first derive the single-level reformulation [(8)](#_bookmark22) based on the lower-level’s KKT conditions and then linearize the KKT complementarity conditions [(8d)](#_bookmark26) by using additional binary variables. The key idea here is to consider the complementar-

ity conditions *𝜆𝑖* (*𝐶𝑖*⋅*𝑥* + *𝐷𝑖*⋅*𝑦* − *𝑏𝑖* ) = 0, *𝑖* = 1*,* … *,* 𝓁, as disjunctions stating

that either *𝜆𝑖* = 0 or *𝐶𝑖*⋅*𝑥* + *𝐷𝑖*⋅*𝑦* = *𝑏𝑖* needs to hold. These two cases can be modeled using binary variables *𝑧𝑖* ∈ {0*,* 1}, *𝑖* = 1*,* … *,* 𝓁, in the follow-

ing mixed-integer linear way,

*𝜆* ≤ *𝑀* d*𝑧 , 𝐶 𝑥* + *𝐷 𝑦* − *𝑏* ≤ *𝑀* p(1 − *𝑧* )*,*

mization, see, e.g., the seminal textbook by [Nocedal and Wright (2006)](#_bookmark220).

*𝑖 𝑖 𝑖*

*𝑖*⋅

*𝑖*⋅ *𝑖 𝑖 𝑖*

The inherent violation of suitable constraint qualifications for MPCCs

with suﬃciently large constants *𝑀* d and *𝑀* p for the dual variable and

*𝑖 𝑖*

lead to the development of both (i) tailored constraint qualifications and stationarity concepts ([Hoheisel et al., 2013](#_bookmark167)) as well as (ii) spe- cial solution techniques. However, the latter can achieve at most (if at

the primal constraint. Consequently, *𝑧𝑖* = 1 models the case that the

primal inequality is active, whereas *𝑧𝑖* = 0 models the inactive case in

which the dual variable is zero. The resulting MILP reformulation can

then be solved by general-purpose solvers. Unfortunately, this reformu- lation has a severe disadvantage because one needs to determine a big-

*𝑀* constant that both is valid for the primal constraint as well as for the

dual variable. The primal validity is usually ensured by the assumption

that the high-point relaxation is bounded, which is typically justified in practical applications. However, the dual feasible set is unbounded for bounded primal feasible sets; see [Clark (1961)](#_bookmark106) and [Williams (1970)](#_bookmark255). Thus, it is rather problematic to bound the dual variables of the follower.

In practice, often “standard” values such as 106 are used without any the-

oretical justification or heuristics are applied to compute a big-*𝑀* value, e.g., in [Pineda et al. (2018)](#_bookmark234), big-*𝑀* values are determined from local so-

lutions of the MPCC [(8)](#_bookmark22). In [Pineda and Morales (2019)](#_bookmark235) it is shown by an illustrative counter-example that such heuristics may deliver invalid

values. Moreover, validating the correctness of a given big-*𝑀* is shown

to be NP-hard in general in [Kleinert et al. (2020c)](#_bookmark195).

All the mentioned methods so far solve a certain reformulation of the bilevel problem with general-purpose solvers. In addition, one can also develop bilevel-tailored solution techniques. Already in their pa- per from 1981, [Fortuny-Amat and McCarl](#_bookmark174) briefly discuss the possibility to set up a bilevel-specific branch-and-bound scheme. In this scheme, Problem [(8)](#_bookmark22) without the KKT complementarity conditions [(8d)](#_bookmark26) is solved at the root node. Afterward, it is checked whether all KKT complemen- tarity conditions are satisfied. If not, the most violated one is chosen and

two subproblems are constructed with either *𝜆𝑗* = 0 or *𝐶𝑗*⋅*𝑥* + *𝐷𝑗*⋅*𝑦* = *𝑏𝑗*

added as a constraint if *𝑗* ∈ {1*,* … *,* 𝓁} is the most violated condition. In

this manner, the method proceeds as a usual branch-and-bound method.

This method is also used in [Bard and Moore (1990)](#_bookmark74), where it is compu- tationally evaluated for bilevel problems with LP upper-level problems and lower-level problems that are convex QPs. Note that for convex QPs in the lower level, all problems to be solved in the nodes of the branch-

In [Audet et al. (2007b)](#_bookmark61), three further cuts are presented that can again be derived from the solution of the root node problem. The first one is a Gomory-like cut. For each violated complementarity constraint of the lower level, two inequalities can be derived. One of them is act- ing on the primal upper- and lower-level variables and the other one on the dual lower-level variables. The presentation of these inequalities is rather technical and we thus refer to the paper for the details. At least one of the two inequalities must be valid and is actually a cut. Since the valid one is not known, both inequalities are added to the problem and a binary switching variable is used to select the valid inequality. In this light, the two inequalities add a rather implicit coupling of the con- straints [(8b)](#_bookmark23) and [(8c)](#_bookmark24). Another variant are so-called extended cuts that, similar to the Gomory-like cuts, also involve binary switching variables. However, it is noted that these cuts are deeper than the Gomory-like cuts. One can also derive two cuts that do not involve a switching vari- able. These cuts are called simple cuts in [Audet et al. (2007b)](#_bookmark61). Again, the combination of both cuts implicitly couples the primal upper and lower level with the dual lower level. In a small numerical study it is shown that applying a cut generation phase at the root node that adds cuts of either one of the three types, outperforms pure branch-and-bound. Fi- nally, [Wu et al. (1998)](#_bookmark262) propose Tuy’s cut for LP-LP problems but did not test it in a numerical study.

Very recently, a new valid inequality for LP-LP bilevel optimization

based on strong duality of the lower-level problem has been presented in [Kleinert et al. (2020b)](#_bookmark191), which couples primal bilevel variables as well as dual variables of the lower-level problem:

*𝜆⊤𝑏* − *𝜆⊤𝐶* + − *𝑑⊤𝑦* ≤ 0*,*

with *𝐶* + being an upper bound on *𝐶𝑖*⋅*𝑥*. For instance, the bounds *𝐶* + can be computed with the auxiliary LPs

*𝑖*

and-bound tree are convex, which would not be the case anymore if bi-

+ {

{ } }

linear terms as products of upper- and lower-level variables are present in the lower level. A very similar branch-and-bound algorithm for con-

*𝐶𝑖*

∶= max

*𝑥,𝑦,𝜆*

*𝐶𝑖*⋅*𝑥* ∶ (*𝑥, 𝑦, 𝜆*) ∈ *𝐻* ×

*𝜆* ∶ *𝐷⊤𝜆* = *𝑑, 𝜆* ≥ 0 *,* (*𝑥, 𝑦, 𝜆*) ∈ C *,*

tinuous bilevel problems is presented in [Bard (1988)](#_bookmark75). Here, bilevel problems with strictly convex upper-level objective function, convex quadratic lower-level objective function, polyhedral feasible set of the upper level, and convex feasible region of the lower level are considered. Moreover the lower-level problem needs to satisfy a suitable constraint qualification. Another extension of [Bard and Moore (1990)](#_bookmark74) for nonlinear but convex problems is given in [Edmunds and Bard (1991)](#_bookmark151). A branch- ing rule different from most-violated complementarity is discussed in [Hansen et al. (1992)](#_bookmark157). At this point in time, problems with 250 leader variables, 150 follower variables, and 150 follower constraints were the largest instances that have been solved. Finally, we note that it is already stated in [Fortuny-Amat and McCarl (1981)](#_bookmark174) that the complementarity conditions can also be modeled as special ordered sets (SOS) of type 1; see [Beale and Tomlin (1970)](#_bookmark86). Modern mixed-integer solvers can handle SOS1 conditions out-of-the-box such that it is not necessary to imple- ment the branching on complementarity conditions. The branching rule [is then left to the solver. This approach is also proposed by Siddiqui and Gabriel (2013) in an MPEC context and by](#_bookmark226) [Pineda](#_bookmark234) [et al. (2018) in a](#_bookmark226) bilevel context.

In the history of integer programming, the basic branch-and-bound

method has been extended to the so-called branch-and-cut (B&C) method. This means that, besides branching, additional valid inequal- ities or cuts are introduced at the nodes of the branch-and-bound tree to tighten the formulation. Whereas the literature on cutting planes in integer programming is huge, there are only a few papers dealing with valid inequalities in the bilevel case.

In [Audet et al. (2007a)](#_bookmark60), the complementarity conditions [(8d)](#_bookmark26) have been used to obtain so called disjunctive cuts that are applied at the root node of the branch-and-bound tree. For each violated complementarity constraint, solving a linear optimization problem yields such a cut. In a small example, the usefulness of the cut is demonstrated. It is also shown that sometimes this cut couples primal feasibility [(8b)](#_bookmark23) and dual feasibility [(8c)](#_bookmark24) and sometimes it does not.

where C is a constraint set containing already added valid inequalities of any type as well as branching decisions or might be empty. While the inequality can be applied throughout the entire branch-and-bound [tree, it is shown that it is most effective at the root node. In Kleinert and Schmidt (2020), it is shown that when equipping both approaches, the](#_bookmark200)

classical big-*𝑀* approach and an SOS1-approach for the KKT comple-

mentarity conditions, with the root node inequality, then the two ap-

suffer from the possible theoretical issues of invalid big-*𝑀* values. The proaches perform very competitive—but the SOS1-approach does not

computational study in [Kleinert and Schmidt (2020)](#_bookmark200) is based on a LP-LP test set containing 1077 instances with up to several thousands of upper- and lower-level variables and constraints. We note that the approaches tested in [Kleinert and Schmidt (2020)](#_bookmark200) are capable of solving 1051 out of the 1077 instances within a time limit of 1h.

So far, most approaches discussed exploit the (structure of the) KKT reformulation [(8)](#_bookmark22) of the bilevel problem. On the other hand, there also exist approaches that are based on reformulation [(10)](#_bookmark19). The issues with this reformulation are the nonconvex bilinear terms involving primal upper- and dual lower-level variables. In principle, such nonconvex problems can be solved using classical convex envelopes—like those obtained using McCormick inequalities; see [McCormick (1976)](#_bookmark201). These convex envelopes can be refined by spatial branching to reduce the do- main of the considered part of the nonconvex function. We refer the interested reader to [Horst and Tuy (2013)](#_bookmark169) for details and a conver- gence analysis of spatial branching methods in specific as well as for an overview of global optimization in general. Today, also general-purpose mixed-integer solvers such as Gurobi ([Achterberg, 2019](#_bookmark51)) and CPLEX ([Klotz, 2017](#_bookmark206)) can solve problems including these bilinear nonconvex- ities.

In bilevel optimization, very often the assumption is made that the

linking variables, i.e., those upper-level variables that also appear in

linear terms *𝜆⊤𝐶𝑥* can be linearized if upper bounds on *𝜆* are avail- the lower-level constraints, are bounded integers. In this case, the bi-

able. Note, however, that finding these upper bounds is the same task

for these activities *𝑦*1 that are essential, one may assume that the set

as finding big-*𝑀* values for the KKT reformulation. Nevertheless, if such

{ }

*𝑦*2 ∶ *𝐷*2 *𝑦*2 ≥ *𝑏*

is nonempty. Indeed, in this case, there exists a feasible

a big-*𝑀* is at hand, in [Zare et al. (2019)](#_bookmark271) it is shown that in case of

large lower-level problems (measured in terms of the number of con-

straints), the strong-duality based reformulation [(10)](#_bookmark19) outperforms the KKT-based approach. The same assumption and linearization technique is used in [Kleinert et al. (2021)](#_bookmark192), where an outer approximation algo- rithm for MIQP-QP bilevel problems with convex-quadratic lower levels is presented.

Let us close this section with some brief pointers to local methods. Recently, classical MPCC regularization techniques such as the famous regularization proposed by [Scholtes (2001)](#_bookmark263) have been used to com- [pute C-stationary solutions of the KKT reformulation in Dempe and Franke (2019). In](#_bookmark128) [Dempe](#_bookmark125) [(2019), even locally optimal solutions of the](#_bookmark128) linear bilevel problem are obtained based on the KKT reformulation. Sta- tionary points of [(10)](#_bookmark19) are computed in [Kleinert and Schmidt (2019a)](#_bookmark197) by

point for the lower level that does not use any activity influenced by the

upper level.

We now discuss some interesting geometrical properties of the bilevel pricing problem. First, remark that the feasible region of the

lower level [(11c)](#_bookmark31) is independent of the upper-level variables *𝑥*, which is

in contrast to the lower level [(7)](#_bookmark20) of the LP-LP problem. Assuming that

the feasible region of the lower level is bounded, i.e., a polytope, allows us to conclude that for every upper-level decision the optimal solution of the lower level is attained at a vertex of the feasible polytope of the lower level. In addition, strong duality holds for every parametric lower level problem [(11c)](#_bookmark31). Second, we look at the single-level reformulation of Problem [(11)](#_bookmark30) obtained by using the KKT conditions of the lower-level problem [(11c)](#_bookmark31):

using a penalty alternating direction method. The quality of this method as a primal heuristic for the bilevel problem at hand is evaluated in

max

*𝑥,𝑦*=(*𝑦*1 *,𝑦*2 )*,𝜆*

*𝑥⊤𝑦*1 (12a)

an extensive computational study. It demonstrates that the approach is capable of computing feasible points for large instances with thou-

sands of variables and constraints, often in a fraction of a second. Re-

s.t. *𝐴𝑥* ≤ *𝑎, 𝐷*1 *𝑦*1 + *𝐷*2 *𝑦*2 ≥ *𝑏,* (12b)

*𝐷⊤𝜆* = *𝑥* + *𝑑*1 *, 𝐷⊤𝜆* = *𝑑*2 *, 𝜆* ≥ 0*,* (12c)

1 2

lated penalty methods for the linear bilevel problem are discussed in

[Anandalingam and White (1990)](#_bookmark56); [Campelo et al. (2000)](#_bookmark82); [Lv et al. (2007)](#_bookmark196). Last but not last, let us refer to the recent survey chapter by

[Calvete and Galé (2020)](#_bookmark129) on algorithms for linear bilevel problems.

# Bilinear Lower Levels

which is a linear problem when the upper-level variables *𝑥* are fixed can A bilevel problem for which the lower level contains bilinearities but

also be reformulated as a single-level optimization problem by using any of the two techniques described in [Section 3.2](#_bookmark18). Pricing problems and bimatrix Stackelberg games constitute two classes of bilevel problems that present this feature.

* 1. *Pricing Problems*

A first bilevel pricing problem with linear constraints, linear upper- level objective and bilinear lower-level objective has been proposed by [Bialas](#_bookmark213) [and Karwan (1984)](#_bookmark103)[. The following problem considered in Labbé et al. (1998) provides a general framework for pricing:](#_bookmark213)

*𝜆⊤*(*𝐷*1 *𝑦*1 + *𝐷*2 *𝑦*2 − *𝑏*) = 0*.* (12d)

Let (*𝑦̄*1*, 𝑦̄*2) be a fixed vertex of the feasible polytope of the lower level. Then, the constraints of [(12)](#_bookmark27) are linear in *𝑥* and *𝜆*, i.e., they con- stitute a polyhedral set for fixed (*𝑦̄*1*, 𝑦̄*2). By considering all vertices of

the lower level, we determine a partition of the feasible set of Prob-

the property that all price vectors *𝑥* belonging to a cell share the same lem [(12)](#_bookmark27) into a (possibly exponential) number of polyhedral cells with

lower-level optimal solution. Some of these cells may be empty. As a

neither convex nor continuous in *𝑥* but is linear in each cell. consequence, the objective function of the bilevel pricing problem is

Formulation [(12)](#_bookmark27) contains nonlinear terms both in its objective func- tion [(12a)](#_bookmark27) and in constraints [(12d)](#_bookmark28). To circumvent the nonlinearity of [the latter one might use the approach proposed by Fortuny-Amat and McCarl (1981) that is described in](#_bookmark174) [Section](#_bookmark25) [3.3 but to do so, again one](#_bookmark174) needs to bound the dual variables, which is NP-hard in general as men- tioned earlier. Another approach consists of replacing the complemen- tarity constraints by the strong duality condition

(*𝑥* + *𝑑*1 )*⊤𝑦*1 + *𝑑⊤𝑦*2 ≤ *𝑏⊤𝜆.*

2

max

*𝑥,𝑦*=(*𝑦*1 *,𝑦*2 )

*𝑥⊤𝑦*1 (11a)

that involve the same bilinear term as the objective function [(12a)](#_bookmark27). [Grimm et al. (2020)](#_bookmark152) use the latter kind of reformulation for the lower-

s.t. *𝐴𝑥* ≤ *𝑎,* (11b)

level problem for particular cases of the above bilevel pricing prob- lem [(11)](#_bookmark30) that correspond to different electricity retailer pricing schemes.

*𝑦* { *⊤ ⊤*

} [Zugno et al. (2013)](#_bookmark274), on the other hand, consider a similar electricity pric-

∈ arg min

*𝑦̄*

(*𝑥* + *𝑑*1 ) *𝑦̄*1 + *𝑑*2 *𝑦̄*2 ∶ *𝐷*1 *𝑦̄*1 + *𝐷*2 *𝑦̄*2 ≥ *𝑏 .* (11c)

ing problem but use the KKT optimality conditions and the single-level

The vector *𝑦* of lower-level variables is partitioned into two sub- vectors *𝑦*1 and *𝑦*2, called plans, that specify the levels of some activities

plan *𝑦*1 through a price vector *𝑥* it charges to the lower level and max- such as goods or services. The upper level influences the activities from imizes its revenue given by *𝑥⊤𝑦*1. The price vector *𝑥* is subject to linear

on the prices. Vectors *𝑑*1 and *𝑑*2 represent linear disutilities faced by constraints that may, among others, impose lower and upper bounds the lower level when executing the activity plans *𝑦*1 as well as *𝑦*2. Note that *𝑑*2 may also encompass the price for executing the activities not

influenced by the upper level. These activities may, e.g., be substitutes

level determines its activity plans *𝑦*1 and *𝑦*2 to minimize the sum of to- offered by competitors for which prices are known and fixed. The lower tal disutility and the price paid for plan *𝑦*1 subject to linear constraints.

Remark that if the model allows negative prices then it implicitly per- mits subsidies, which may be appropriate, e.g., in the context of a cen- tral agency determining taxes. In order to avoid the situation in which the upper level would maximize its profit by setting prices to infinity

reformulation á la [Fortuny-Amat and McCarl (1981)](#_bookmark174).

If all vertices of the feasible polytope of the lower level are binary, bi- linear terms can be linearized more eﬃciently when using the approach proposed by [McCormick (1976)](#_bookmark201). This particularly applies to lower-level problems that are polynomial graph problems. [Van Hoesel (2008)](#_bookmark240) and [Labbé and Violin (2013)](#_bookmark217) present surveys about such so-called network pricing problems that we briefly sketch in the following. Consider a graph whose arc weights represent travel costs. In the toll setting prob- lem, the upper level determines the prices (or tolls) of a subset of arcs of a network in order to maximize its revenue obtained by collecting tolls paid by the lower level that consists in a given number of users, each one being an independent follower. Each user selects a path from her origin to her destination that minimizes her disutility given by the sum of the prices of the arcs in the path that are controlled by the upper level plus the total travel costs.

[Labbé et al. (1998)](#_bookmark213) show that the toll setting problem with (pos- sibly negative) lower bounds on the prices is NP-hard even for a

single user and that it is polynomial in the special case that one single arc is to be priced. [Roch et al. (2005)](#_bookmark244) strengthen the com- plexity result by showing that the single-user toll setting problem is already strongly NP-hard if all lower bounds on the prices are equal to 0. [Joret (2011)](#_bookmark184) shows that the problem is also APX-hard. [Labbé et al. (1998)](#_bookmark213) propose an MILP reformulation of the toll set-

ting problem that involves big-*𝑀* values. [Dewez et al. (2008)](#_bookmark142) show

how to derive eﬃcient big-*𝑀* s and propose valid inequalities that

strengthen the MILP model. [Brotcorne et al. (2001)](#_bookmark117) propose heuristics

and [Bouhtou et al. (2007)](#_bookmark111) present a preprocessing method to reduce the graph size. [Didi-Biha et al. (2006)](#_bookmark144) and [Brotcorne et al. (2011)](#_bookmark115) ex- ploit the fact that revenue maximizing prices that are compatible with a given lower-level solution can be easily determined. They propose exact algorithms as well as heuristics based on multi-path generation.

[Heilporn et al. (2010b)](#_bookmark162) and [Heilporn et al. (2011)](#_bookmark165) study the particu- lar case in which each follower uses at most one arc priced by the leader. [Heilporn et al. (2010b)](#_bookmark162) show that the problem is strongly NP-hard. Fur- ther, exploiting the fact that there exists a limited number of feasible solutions for each follower, they provide an MILP formulation based on the optimal value function, a polyhedral study of this formulation, and provide a complete description of the convex hull of feasible points

[Cardinal et al. (2011)](#_bookmark93) show that this problem is APX-hard, whereas [Morais et al. (2016)](#_bookmark214) and [Labbé et al. (2021)](#_bookmark216) propose different MILP formulations.

* 1. *Stackelberg Bimatrix Games*

The determination of optimal mixed strategies in a Stackelberg bima- trix game under normal form constitutes another typical bilevel problem in which both objectives are bilinear (in both the upper- and lower-level variables) and all constraints are linear. In such a game, two players,

say A and B are endowed with a set of pure strategies *𝐼* and *𝐽* with

|*𝐼* | = *𝑛,* |*𝐼* | = *𝑚*. The matrices *𝑅* = [*𝑅𝑖𝑗* ] and *𝐶* = [*𝐶𝑖𝑗* ] encode the respec- tive utilities when A plays strategy *𝑖* and B plays strategy *𝑗*. A mixed strategy for player A (B) is a probability distribution *𝑥* (*𝑦*) over her pure strategy set *𝐼* (*𝐽* ). Both players want to maximize their respective ex- pected utility given by *𝑥⊤𝑅𝑦* and *𝑥⊤𝐶𝑦*. Now assume that the players

choose their mixed strategy sequentially: A is the leader and plays first, then B, informed of A’s decision, reacts optimally with respect to her own objective. The solution to this Stackelberg bimatrix game, called a Strong Stackelberg Equilibrium (SSE), is given by an optimal solution of the bilevel problem

for the special case of one single follower. In [Heilporn et al. (2011)](#_bookmark165), a branch-and-cut procedure is proposed.

max

*𝑥,𝑦*

*𝑥⊤𝑅𝑦* (13a)

[Heilporn et al. (2010a)](#_bookmark163) show the equivalence of this problem with the so-called product line pricing problem. In the upper level

of this problem, prices of products must be determined to maxi-

s.t. 1*⊤𝑥* = 1*, 𝑥* ≥ 0*,* (13b)

*𝑦* ∈ arg max { *⊤ ⊤* }

mize total revenue. In the lower level, customers choose the prod-

*𝑦̄*

*𝑥 𝐶 𝑦̄* ∶ 1 *𝑦̄* = 1*, 𝑦̄* ≥ 0 *,* (13c)

uct that maximizes their welfare given by the difference of their reservation price (also called willingness to pay) for the product and its price. The product line design and pricing was originally intro- duced by [Dobson and Kalish (1988)](#_bookmark148). [Guruswami et al. (2005)](#_bookmark158) show that it is APX-hard. MILP formulations different than the one used in [Heilporn et al. (2010b)](#_bookmark162) are presented in [Shioda et al. (2011)](#_bookmark268), [Myklebust et al. (2016)](#_bookmark218), and [Fernandes et al. (2016)](#_bookmark160). Moreover, heuris- tics are proposed in [Dobson and Kalish (1993)](#_bookmark150), [Shioda et al. (2011)](#_bookmark268) as well as [Myklebust et al. (2016)](#_bookmark218). Instance generators that are publicly available are described in [Fernandes et al. (2016)](#_bookmark160).

[Castelli et al. (2017)](#_bookmark101) show that the special case in which the price of all arcs controlled by the leader must be equal is polynomial. Further- more, they also show that the problem is pseudo-polynomial when arc prices must be proportional to their length and they also consider a ro- bust variant of these problems. [Castelli et al. (2013)](#_bookmark100) apply the model with proportional prices in the context of air traﬃc management to determine how much Air Navigation Service Providers (ANSPs) should charge airlines to use their airspace.

[Marcotte et al. (2009)](#_bookmark199) use the toll setting problem to determine road tolls to regulate the use of roads for hazardous shipments and show that an optimal toll policy is more eﬃcient then a network de- sign approach that determines road segments to be closed to dangerous materials.

[Brotcorne et al. (2008)](#_bookmark118) consider the more general problem in which the leader faces a joint design and pricing problem. Here, in the upper- level objective, a fixed cost is incurred for each arc that is installed (and priced) by the leader. The lower level is the same as in the toll setting problems. They show that the coupling constraints linking the design variables and the user arc choice variables appearing in the lower level can be moved to the upper level. These constraints forbid the followers to use arcs that are not installed. Moving them to the upper level is allowed because the leader can prevent the followers to use them by setting their price very high. Finally, they suggest a single-level MILP formulation as well as heuristics.

Network pricing problems with different lower-level problems have also been studied. [Brotcorne et al. (2000)](#_bookmark116) consider a lower level given by an uncapacitated transshipment problem and provide an MILP formulation as well as some heuristics. Another variant is obtained by assuming that the lower level selects a minimum spanning tree.

in which 1 denotes the vector of all ones in appropriate dimension. The term “strong” stands for the fact that the optimistic version of the prob- lem is considered. [Conitzer and Sandholm (2006)](#_bookmark116) describe and discuss different representations of such leader-follower games as well as the appropriateness and the utility of using pure or mixed strategies. Fur- thermore, SSE’s may not coincide with Nash equilibria, as shown by the example provided in [Korzhyk et al. (2011)](#_bookmark211).

that for a given leader’s solution *𝑥*, the lower level is an LP on the Problem [(13)](#_bookmark33) can be solved using linear programming. First notice

for the follower that is one of the *𝑛* vertices of the unit simplex. Sec- unit simplex. In other words, there always exists an optimal solution ond, a solution *𝑥* that maximizes the leader’s utility and for which some solution *𝑦̄*, with *𝑦̄𝑗* = 1 for some *𝑗* ∈ *𝐽* , is optimal for the fol-

function is *𝑥⊤𝑅*⋅*𝑗* and in which the lower-level problem [(13c)](#_bookmark34) is lower can be found by solving problem [(13)](#_bookmark33) whose objective

replaced with

*𝑥⊤𝐶.*⋅*𝑗*′ ≤ *𝑥⊤𝐶*⋅*𝑗* for all *𝑗*′ ∈ *𝐽.*

Hence, solving this LP for every possible pure strategy of the follower and retaining the one that yields the highest utility for the leader pro- vides an SSE; see [Conitzer and Sandholm (2006)](#_bookmark116).

Problem [(13)](#_bookmark33) can be adapted to the case in which the leader does not know the follower’s preferences over the outcomes of the game with

certainty. This is done by considering different types *𝑘* ∈ *𝐾* of followers.

In this case, the game is called Bayesian. Utility matrices *𝑅𝑘* and *𝐶 𝑘* are then given for each follower type *𝑘* as well as a probability *𝜋𝑘* that the type of the follower is indeed *𝑘*. The leader’s expected utility is then equal to ∑ *𝜋𝑘𝑥⊤𝑅𝑘𝑦𝑘* and a lower-level problem

*𝑘*∈*𝐾*

*𝑦𝑘* ∈ arg max { *⊤ 𝑘 𝑘* ∶ 1*⊤𝑦̄𝑘* = 1*, 𝑦̄𝑘* ≥ 0}

*𝑥 𝐶 𝑦̄*

*𝑦̄𝑘*

is introduced for each follower type. A Bayesian Stackelberg bimatrix game can be seen as a regular Stackelberg bimatrix game in which the

set of pure strategies of the follower is composed of all *𝑛*|*𝐾*| possible

combined choices of pure strategies of the different follower types; see

[Harsanyi and Selten (1972)](#_bookmark161). As a consequence, an SSE in a Bayesian Stackelberg bimatrix game can be determined in polynomial time when the number of types is fixed. If not, the problem is NP-hard; see again [Conitzer and Sandholm (2006)](#_bookmark116).

The bilevel optimization problem that determines an SSE of a gen- eral Bayesian Stackelberg bimatrix game can be reformulated as a single-level MILP. In fact any of the three approaches consisting in using KKT conditions, strong duality, or the optimal value function leads to an equivalent single-level reformulation. Then, to circum- vent the bilinearities in the objective functions of both levels, one may exploit the fact that there always exists an optimal follower’s re- sponse that is binary, i.e., it is a pure strategy. [Paruchuri et al. (2008)](#_bookmark227), [Kiekintveld et al. (2009)](#_bookmark188), and [Yin and Tambe (2012)](#_bookmark267) propose models based on these principles. The LP relaxation of the formulation proposed by [Yin and Tambe (2012)](#_bookmark267) is the strongest and provides a complete de-

Another common feature of Stackelberg security games is that the utility of both the defender and the attacker depend only on whether the target that is attacked is protected or not. There are two cases, de- pending on whether or not the target is covered by the defender. The

defenders utility for an uncovered attack of type *𝑘* on target *𝑗* is denoted

*𝐷𝑘*(*𝑗*|*𝑢*) and for a covered attack of type *𝑘* it is denoted as *𝐷𝑘*(*𝑗*|*𝑐*). Sim- ilarly, *𝐴𝑘*(*𝑗*|*𝑢*) and *𝐴𝑘*(*𝑗*|*𝑐*) represent the type *𝑘* attackers utilities. With

these new notations at hand, one can formulate the following bilevel problem that determines an SSE in a Bayesian Stackelberg security game:

∑ ∑

max *𝜋𝑘* (*𝑥 𝐷𝑘*(*𝑗*|*𝑐*) + (1 − *𝑥* )*𝐷𝑘*(*𝑗*|*𝑢*))*𝑦𝑘*

scription of the convex hull of feasible points in the case of a single

follower. See [Casorrán et al. (2019)](#_bookmark99) for comparison of the three above

*𝑥,𝑦*

*𝑘*∈*𝐾*

*𝑗*

*𝑗*∈*𝐽*

*𝑗 𝑗*

mentioned formulations from both theoretical and computational point of views. On the other hand, decomposition methods scale better when the problem involves many resources and/or follower types. In this per- spective, [Paruchuri et al. (2008)](#_bookmark227) propose a solution approach involving Benders decomposition and [Jain et al. (2010)](#_bookmark177) and [Lagos et al. (2017)](#_bookmark219) use column generation.

Stackelberg bimatrix games have been shown to be useful for many real-world applications in security domains. In these so-called Stack- elberg security games, the leader (defender) places security resources (e.g., guards) at various potential targets (possibly in a randomized man- ner), and then the follower (attacker) chooses a target to attack; see e.g. [Jain et al. (2013)](#_bookmark175). Examples of such applications include disrupting drug traﬃcking networks ([Washburn and Wood, 1995](#_bookmark252)), assigning Federal Air Marshals to transatlantic flights ([Pita et al., 2008](#_bookmark236)), determining random- [ized port and waterways patrols for the U.S. Coast Guard (Shieh et al.,](#_bookmark267) [2012](#_bookmark266)[), preventing fare evasion in public transport systems (Yin et al.,](#_bookmark267) [2012), protecting endangered wildlife (](#_bookmark266)[Yang](#_bookmark264) [et al., 2014), or coordinat-](#_bookmark266) ing resources to organize patrols of the Chilean national police force ([Bucarey et al., 2019](#_bookmark121)). See also the book edited by [Tambe (2011)](#_bookmark237) that describes many applications and the survey by [Sinha et al. (2018a)](#_bookmark228) that presents recent advances in Stackelberg security games. In these security games, playing a mixed strategy of the defender is particularly appro- priate because even if the attacker is aware of this mixed strategy, she does not know which pure strategy will actually be put in action when she attacks. This is especially relevant when the game is played in a repeated way, e.g., every day.

A common feature of Stackelberg security games is that pure strate-

gies of the leader consist in allocating several resources to protect tar- gets, leading to an exponential number of such pure strategies. In the

simplest case, *𝐽* represents a set of targets that may be attacked and each

sume that the defender has a set of *𝑚 < 𝑛* (identical) resources available target attack corresponds to a pure strategy of the attacker. Further, as-

to cover these targets. The possible pure strategies of the defender con-

s.t.1*⊤𝑥* ≤ *𝑚,* 0 ≤ *𝑥* ≤ 1*,*

{ }

∑

*𝑦𝑘* ∈ arg max (*𝑥𝑗 𝐴𝑘*(*𝑗*|*𝑐*) + (1 − *𝑥𝑗* )*𝐴𝑘*(*𝑗*|*𝑢*))*𝑦̄𝑘* ∶ 1*⊤𝑦̄* = 1*, 𝑦̄* ≥ 0 *.*

*𝑗*

*𝑦̄𝑘 𝑗*∈*𝐽*

Three single-level MILP reformulations similar to the ones proposed for general Stackelberg games can be derived for this problem; see [Casorrán et al. (2019)](#_bookmark99). The authors also compare them with extended formulations that involve all possible mixed strategies, i.e., formulations of the general Stackelberg game version of such security games.

Other variants of Stackelberg security games involve more so- phisticated pure strategies of the leader. Resources can be heteroge- neous meaning that each resource can only cover a subset of tar- gets. Resources can cover at once a subset of targets, called sched- ule. [Korzhyk et al. (2010)](#_bookmark210) investigate the complexity of such variants with one type of follower. They show that a Stackelberg security game with homogeneous resources is polynomial if the schedules have size at most 2 and is NP-hard otherwise. When resources are heterogeneous, they show that the problem is polynomial when schedules have size 1 and NP-hard otherwise. [Jain et al. (2010)](#_bookmark177) propose a branch-and- price approach for such variants by iteratively generating columns [representing pure strategies of the leader. Finally, Letchford and Conitzer (2013) study the complexity of the case of Stackelberg secu-](#_bookmark181) rity games in which the targets are vertices of a graph and schedules are subgraphs with a particular structure such as path or tree.

# Mixed-Integer (Non)Linear Lower Levels

In this section, we focus on a general bilevel MILPs, which are de- fined as

*𝑐⊤𝑥 𝑐⊤𝑦* (14a)

min +

*𝑥*∈*𝑋,𝑦 𝑥 𝑦*

s.t. *𝐴𝑥* + *𝐵𝑦* ≥ *𝑎,* (14b)

sist in all subsets of *𝐽* of cardinality at most *𝑚*. As a consequence, any

of the formulations proposed for finding an SSE in a general bimatrix

*𝑦* ∈ arg min { *⊤*

*𝑦̄*∈*𝑌*

*𝑑*

}

*𝑦̄* ∶ *𝐶𝑥* + *𝐷𝑦̄* ≥ *𝑏 ,* (14c)

Stackelberg game becomes rapidly intractable when the number of tar- gets and/or resources increase.

code a leader’s mixed strategy by a vector *𝑥* whose entries *𝑥𝑗* represent To alleviate this situation, [Kiekintveld et al. (2009)](#_bookmark188) propose to en- the marginal probabilities of covering each target *𝑗* in this mixed strat-

egy. The marginal probability of a target is equal to the sum of the prob-

vector *𝑥* of marginal probabilities is a point belonging to the convex hull abilities of the pure strategies covering the said target. In other words, a

all binary vectors with at most *𝑚* entries equal to 1. It can be readily seen of the binary vectors corresponding to all possible pure strategies, i.e.,

{ }

that this convex hull is *𝑥* ∶ 1*⊤𝑥* ≤ *𝑚,* 0 ≤ *𝑥* ≤ 1 . Indeed, the constraint

matrix is totally unimodular so that all the vertices of this polytope are

binary vectors. Further, as explained in [Kiekintveld et al. (2009)](#_bookmark188), the mixed strategy corresponding to a given vector of marginal probabil- ities can be retrieved in polynomial time since it amounts to solve a linear system with a polynomial number of constraints. In the context

where the vectors *𝑐𝑥 , 𝑐𝑦 , 𝑑, 𝑎, 𝑏* and matrices *𝐴, 𝐵, 𝐶, 𝐷* are defined as in [Section 3](#_bookmark12). The sets *𝑋* and *𝑌* specify integrality constraints on a subset of *𝑥*- and *𝑦*-variables, respectively.

as the set of points (*𝑥, 𝑦*) ∈ *𝑋* × *𝑌* satisfying all constraints of the upper The HPR’s feasible region of this bilevel MILP is, as usual, defined

and lower level, i.e.,

*𝐻* ∶= {(*𝑥, 𝑦*) ∈ *𝑋* × *𝑌* ∶ *𝐴𝑥* + *𝐵𝑦* ≥ *𝑎, 𝐶𝑥* + *𝐷𝑦* ≥ *𝑏*}*.*

points, i.e., all points (*𝑥, 𝑦*) ∈ *𝐻* for which for a given *𝑥*, the vector *𝑦* is The inducible region of a bilevel MILP consists of all bilevel feasible

an optimal solution of the lower-level problem. This means,

*𝑑⊤𝑦* ≤ *𝜑*(*𝑥*)*,*

holds. Here, *𝜑*(*𝑥*) again is the optimal value of the lower-level problem,

which is defined as

of a scheduling problem, [McNaughton (1959)](#_bookmark204) proposes an alternative

and faster polynomial procedure.

*𝜑*(*𝑥*) = min { *⊤𝑦*

*𝑦*∈*𝑌*

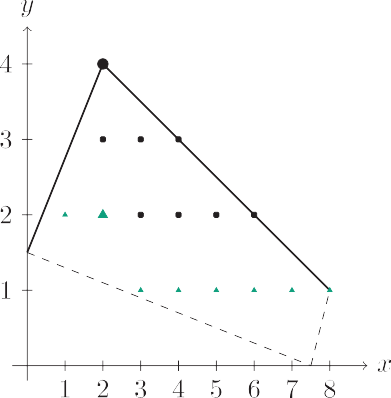
*𝑑*

∶ *𝐷𝑦* ≥ *𝑏* −

}

*𝐶𝑥 .*

(15)

**Fig. 2.** Example of a bilevel MILP: Discrete points are feasible for the high-point relaxation. The point (2,4) is the optimal solution of the high-point relaxation and (2,2) is the optimal solution of the bilevel MILP. Triangles represent bilevel feasible solutions and dashed lines represent the feasible region of the bilevel LP in which the integrality constraints on the upper- and lower-level variables are relaxed.

The value function *𝜑*(*𝑥*) thus corresponds to a parametric MILP, and hence it is nonconvex, not continuous, and in general very diﬃcult to describe. Moreover, in contrast to bilevel LPs, it is NP-hard to check

whether a given point (*𝑥, 𝑦*) is a feasible solution of the bilevel MILP.

[Jeroslow (1985)](#_bookmark182) showed that *𝑘*-level discrete optimization problems are

Σ*𝑝* -hard, even when the variables are binary and all constraints are lin-

*𝑘*

ear. This means that, e.g., a discrete bilevel optimization problem can be

solved in nondeterministic polynomial time, provided that there exists an oracle that solves problems that are in NP in constant time.

The inducible region of the bilevel MILP is contained in the set *𝐻* ,

the set *𝐻* (which represents another MILP) provides a valid lower bound and therefore, minimizing the objective function of the upper level over

for the bilevel MILP. Consequently, solving the LP-relaxation of the HPR

**Fig. 3.** The attainability counterexample by [Köppe et al. (2010)](#_bookmark185)

variables are relaxed. In general, such obtained set does not even have to contain a single bilevel feasible point.

For a general study of representability of sets by extended formula- tions using mixed-integer bilevel programs we refer to the recent paper by [Basu et al. (2021)](#_bookmark79).

*Attainability of Optimal Solutions*

In [Vicente et al. (1996)](#_bookmark245), the authors consider three cases of bilevel MILPs and study the following different assumptions:

1. only upper-level variables are discrete,
2. all upper- and lower-level variables are discrete, and
3. only lower-level variables can take discrete values.

Assuming that all discrete variables are bounded and that the in- ducible region is nonempty, they show that for Case (i) and (ii), an optimal solution always exists and that (i) can be reduced to a linear bilevel program (cf. [Section 3](#_bookmark12)), whereas (ii) can be reduced to a linear trilevel problem. However, for Case (iii), [Moore and Bard (1990)](#_bookmark211) and also [Vicente et al. (1996)](#_bookmark245) provided examples that demonstrate that the bilevel feasible region may not be closed, and hence, the optimal solu- tion may not be attainable. The following simpler example (see [Figure 3](#_bookmark38)) is due to [Köppe et al. (2010)](#_bookmark185):

provides another (and usually much weaker) lower bound of the bilevel { }

MILP.

[Moore and Bard (1990)](#_bookmark211) initiated the studies of bilevel optimization

inf

0≤*𝑥*≤1*,𝑦*

*𝑥* − *𝑦* ∶ *𝑦* ∈ arg min {*𝑦̄* ∶ *𝑦̄* ≥ *𝑥,* 0 ≤ *𝑦̄* ≤ 1} *,*

*𝑦* ∈ℤ

′

problems involving discrete variables. Their illustrative example (cf. [Figure 2](#_bookmark38)) is frequently used in the literature to highlight the major dif- ferences and pitfalls arising in discrete bilevel optimization. Since then, studies have been carried out considering only special cases, e.g., by as- suming binary variables at both levels or by considering purely linear problems at the lower level. Exact MILP-based procedures for the gen- eral case in which both the upper and the lower level are MILPs have been mainly studied in the last decade.

* 1. *General Properties*

The following example is provided by [Moore and Bard (1990)](#_bookmark211):

which is equivalent to

inf {*𝑥* − ⌈*𝑥*⌉ ∶ 0 ≤ *𝑥* ≤ 1}*.*

*𝑥*

In this problem, the infimum is -1, which is never attained. In the ex- isting literature on bilevel MILPs, it is therefore frequently assumed that the linking variables are discrete. We recall that nonlinking upper [level variables can be moved to the lower level (Bolusani and Ralphs, 2020; Tahernejad et al., 2020), which effectively translates the latter as-](#_bookmark105) sumption into “all upper-level variables are discrete”. Alternatively, for bilevel MILPs with continuous linking variables, methods that achieve

*𝜀*-optimal solutions are considered if the optimal solution cannot be at-

tained; see, e.g., [Zeng and An (2014)](#_bookmark273). [Fanghänel and Dempe (2009)](#_bookmark159) an-

{ } alyzed the structure of bilevel MILPs with continuous upper-level and

min

*𝑥*∈ℤ*,𝑦*∈ℤ

−*𝑥* − 10*𝑦* ∶ *𝑦* ∈ arg min {*𝑦̄* ∶ (*𝑥, 𝑦̄*) ∈ P} *,*

*𝑦̄*∈ℤ

discrete lower-level variables. They also discussed optimality conditions for local and global optimality.

where P is a polytope defined by

−25*𝑥* + 20*𝑦̄* ≤ 30*, 𝑥* + 2*𝑦̄* ≤ 10*,* 2*𝑥* − *𝑦̄* ≤ 15*,* 2*𝑥* + 10*𝑦̄* ≥ 15*.*

The HPR of this problem is an integer linear problem, whose feasible region is depicted in [Figure 2](#_bookmark38). The unique optimal solution for this ex- ample is the point (2,2), which is in the interior of the convex hull of the HPR. This is in contrast to bilevel LPs, whose optimal solution is always a vertex of the HPR; see [Section 3](#_bookmark12). The example also shows that relaxing the integrality constraints for the lower-level problem does not provide neither lower nor upper bounds for the bilevel MILP. Dashed lines in [Figure 2](#_bookmark38) correspond to the inducible region of the problem in which the integrality constraints for both the upper-level and the lower-level

*Unboundedness of the Lower-Level Problem*

A common assumption for algorithms dealing with bilevel MILPs is that the feasible region of the HPR is compact. Sometimes, this condition is relaxed and it is only assumed that discrete variables are bounded. For the latter case, [Xu and Wang (2014)](#_bookmark263) demonstrate that the unboudedness of the optimal HPR value does not reveal the nature of the underlying bilevel problem. It can happen that the underlying bilevel MILP is infea- sible, unbounded, or admits an optimal solution; see also [Section 3](#_bookmark12) for

that if the lower level MILP [(15)](#_bookmark37) is unbounded (i.e., *𝜑*(*𝑥*) = −∞ for a an illustrative example. [Xu and Wang (2014)](#_bookmark263) (cf. Lemma 2) also show

certain *𝑥* from the HPR’s feasible region), then the bilevel MILP [(14)](#_bookmark36) is

infeasible. Later, [Fischetti et al. (2018)](#_bookmark168) showed that for any bilevel MILP

whose HPR value is unbounded, one can detect upfront whether the lower-level problem is unbounded or not. To this end, it is suﬃcient to

solve a single LP (not depending on *𝑥*) in a presolve phase. The solution

of this LP, cf. Theorem 1 of [Fischetti et al. (2018)](#_bookmark168), provides a direction

bounded—no matter the choice of the vector *𝑥* from the HPR’s feasible (if such exists) in which the lower-level problem defined by [(15)](#_bookmark37) is un-

region.

* 1. *Generic Approaches for Bilevel MILPs*

Most of the exact methods studied in the literature start with solv-

ing the high-point relaxation, i.e., min{*𝑐⊤𝑥* + *𝑐⊤𝑦* ∶ (*𝑥, 𝑦*) ∈ *𝐻* }, and con-

in polynomial time for a fixed dimension *𝑛𝑥* + *𝑛𝑦* . The algorithm applies

binary search by targeting the optimal value of the bilevel MILP.

*Multi-Way Branching*

[Xu and Wang (2014)](#_bookmark263), see also the PhD thesis by [Xu (2012)](#_bookmark261), apply a multi-way branching method to solve bilevel MILPs in which all leader variables are required to be integer and bounded. The algorithm solves a series of MILPs obtained by restricting the values of slack variables of the lower-level constraints. Another enhanced version of this method, which provides a heuristic solution in the case that the lower-level problem has multiple optimal solutions, is given by [Liu et al. (2020b)](#_bookmark187).

In their “watermelon algorithm”, [Wang and Xu (2017)](#_bookmark247) exploit multi-

way branching to “carve out” bilevel infeasible points from the feasible

*𝑥 𝑦*

cutting planes, by approximating the value function *𝜑*(*𝑥*) given in [(15)](#_bookmark37), tinue by discarding bilevel infeasible solutions by branching, by adding

or by a combination of all of them. In the following, we review these methods and point out to their differences.

*Branch-and-Bound Methods*

In their seminal paper, [Moore and Bard (1990)](#_bookmark211) develop the first branch-and-bound method for discrete bilevel optimization. Their al- gorithm terminates after a finite number of iterations if all upper-level variables are integer or all lower-level variables are continuous (assum- ing an optimum exists). In addition, the authors assume that the HPR’s feasible region is compact and that there are no coupling constraints at the upper level. The authors point out that two of the three standard B&B fathoming rules for mixed-integer optimization are not valid in the bilevel context and discuss further computational challenges of solving discrete bilevel problems. [Bard and Moore (1992)](#_bookmark76) then propose another

exact algorithm for bilevel MILPs assuming that all variables (*𝑥, 𝑦*) are

binary.

[Fischetti et al. (2018)](#_bookmark168) developed another branch-and-bound method that works for mixed-integer upper- and lower-level problems and al- lows coupling constraints at the upper level. The major assumption is that the discrete variables are bounded and that the linking variables are discrete. Necessary modifications of a standard B&B-based MILP solver are introduced to properly handle branching, node evaluation, and fathoming rules. The method checks unboundedness of the lower- [level problem in a presolve phase; see](#_bookmark263) [Section 5.1](#_bookmark39)[. Together with Xu and Wang (2014), see below, the proposed B&B algorithm is one of the few](#_bookmark263) methods that return a provably optimal solution (if such exists) within a finite number of iterations without assuming that the HPR’s feasible re- gion is compact. Instead, only the discrete variables need to be bounded.

*Parametric Integer Programming Methods*

[Faísca et al. (2007)](#_bookmark154) assume that discrete variables of the bilevel

act method that works in two phases. In the first phase, all *𝐾* lower- MILP are binary and use parametric programming to develop an ex-

Then, each solution is plugged into the upper-level problem, yielding *𝐾* level solutions are enumerated using parametric integer programming.

single-level MILP problem reformulations, from which the best one rep- resents the global optimum. The approach is picked up and extended to bilevel MIQPs in [Avraamidou and Pistikopoulos (2019b)](#_bookmark65). The au- thors also provide a computational study for bilevel MILPs and bilevel MIQPs. A more detailed description of the implementation can be found in [Avraamidou and Pistikopoulos (2019a)](#_bookmark66).

[Köppe et al. (2010)](#_bookmark185) also approach bilevel MILPs from the paramet- ric programming perspective. They view the lower-level problem as a parametric (integer) program whose right-hand side is parameterized

by *𝑥*. The authors propose an algorithm that runs in polynomial time

for a fixed dimension *𝑛𝑦* of the lower-level problem and for the case

that the linking variables are continuous. In case the linking variables

are discrete, the authors show that there exists an algorithm that runs

region of the HPR. Whenever a bilevel infeasible point (together with a polyhedron around it that contains no bilevel feasible points) is dis- covered, it is discarded by decomposing the search space into a fam- ily of smaller polyhedra, which are then solved in a recursive fashion. Two different ways to determine the bilevel-free polyhedron around a given infeasible point are proposed along with MILP-based procedures for their determination.

*Branch-and-Cut Methods*

[By extending the ideas from](#_bookmark139) [Moore and Bard (1990)](#_bookmark211)[, DeNegre and Ralphs (2009), see also the dissertation by](#_bookmark139) [DeNegre](#_bookmark135) [(2011), develop an](#_bookmark139) MILP-based branch-and-cut approach. Their method does not allow for any continuous variables and coupling constraints at the upper level. It is also assumed that all coeﬃcients in the upper- and lower-level con- straints are integer. Bilevel infeasible solutions are cut off on the fly by adding “integer no-good cuts” that exploit the integrality property of the upper- and lower-level variables. These cuts are guaranteed to sepa- rate bilevel infeasible points from the convex hull of the bilevel feasible region.

An extension of the former method that allows for a mixed-integer setting at both levels is given by [Tahernejad et al. (2020)](#_bookmark238). In this setting, “generalized no-good cuts” are used to remove all solutions for which the linking variables have a certain fixed value, and thus no integrality of the coeﬃcients in the constraint matrices is required. The authors provide a comprehensive implementation that integrates many compu- tational and algorithmic features proposed in the recent literature on bilevel MILPs.

A cutting plane method for bilevel MILPs in which all variables are discrete (and all coeﬃcients at the upper- and lower-level are integer) is given by [Caramia and Mari (2015)](#_bookmark94). The authors solve the HPR and

utilize a variant of “no-good” constraints (involving big-*𝑀* s and 𝓁∞-

norms) to cut off nonoptimal responses from the follower on the fly.

They also propose a B&C method with a specific branching rule derived from rounding the value of the optimal follower’s response.

[Dempe and Kue (2017)](#_bookmark131) consider two special cases of bilevel MILPs:

(i) both levels contain discrete variables only and the leader influences the objective of the follower (i.e., the objective function is bilinear), and (ii) only the lower level contains discrete variables and the leader influences the right-hand-side of the follower. For the former case, the authors propose a B&C algorithm based on covering-type valid inequal- ities. For the latter case, the authors exploit the structural properties of the value function and derive an iterative MILP-based procedure in which the value function is refined. The methods have been illustrated on two small examples.

To enhance the performance of their basic B&B method, [Fischetti et al. (2018)](#_bookmark168) introduce intersection cuts to separate inte- ger bilevel infeasible points, thus obtaining a B&C approach for bilevel MILPs. These cuts, which are traditionally used for mixed-integer programming (see, e.g., [Balas (1971)](#_bookmark69)) are used here for the first time to solve bilevel MILPs: LP-optimal solutions (being integer but bilevel infeasible) are cut off by deriving a cut in which the LP-cone of this

feasible points. These cuts can be derived under the assumption that *𝑑* solution is intersected with a convex set that contains no bilevel and *𝐶𝑥* + *𝐷𝑦* − *𝑏* are integer for any (*𝑥, 𝑦*) ∈ *𝐻* . In a follow-up article,

[Fischetti et al. (2017b)](#_bookmark166) provide additional computational techniques to further improve their B&C method. These techniques include new ways to derive intersection cuts, follower upper-bound cuts and variable fixing based on the properties of the lower-level problem. The results also include hypercube intersection cuts, which can deal with lower levels with continuous variables (and thus do not require any additional assumptions regarding the coeﬃcients of the lower-level problem). The authors conducted a computational study on a set of 874 benchmark instances and reported optimal solutions for 822 of them. The code of [Fischetti et al. (2017b)](#_bookmark166) is publicly available ([Fischetti et al., 2017a](#_bookmark164)), and represents the current state-of-the-art exact method for general bilevel MILPs. The code is integrated within the commercial solver CPLEX. An alternative open-source implementation that includes features of [Fischetti et al. (2017b)](#_bookmark166), but also many additional ones, has been devel- oped by [Tahernejad et al. (2020)](#_bookmark238) and is available online ([Ralphs, 2018](#_bookmark240)). Unsurprisingly, specialized approaches for solving particular interdic- tion problems, like those of [Fischetti et al. (2019)](#_bookmark170); [Furini et al. (2020b)](#_bookmark136), are outperforming the generic approaches by [Fischetti et al. (2017b)](#_bookmark166); [Tahernejad et al. (2020)](#_bookmark238) on interdiction instances.

*Benders-like Decomposition*

A Benders-like decomposition scheme for general bilevel MILPs is given in [Saharidis and Ierapetritou (2009)](#_bookmark254), assuming that the HPR’s fea- sible region is compact. Valid Benders-like cuts are derived by fixing the value of integer variables at the master level and using the active-set strategy together with the KKT reformulation of the resulting continu-

ous lower-level problem. The algorithm terminates when an *𝜀*-optimal

solution is achieved.

In a recent article by [Bolusani et al. (2020)](#_bookmark104), the authors make a parallel between bilevel MILPs and two-stage stochastic MILPs with re- course. By exploiting their common mathematical structure given by the value-function reformulation and using the MILP-duality theory, a uni- fied algorithmic framework is provided. In [Bolusani and Ralphs (2020)](#_bookmark105), a Benders-like decomposition to approximate the value function and a cutting-plane method are discussed as two possible solution strategies.

*Other Approaches*

[Zeng and An (2014)](#_bookmark273) proposed a single-level reformulation and a decomposition algorithm based on a column-and-constraint generation scheme for general bilevel MILPs. The authors even allow the linking variables of the leader to be continuous. Under the assumption that the optimal solution is attainable, the algorithm finds an optimal so-

lution. Otherwise, it finds an *𝜀*-solution. Their idea is picked up by

[Yue et al. (2019)](#_bookmark269) who propose to project out integer variables of the

lower-level problem and work with KKT conditions of the remaining continuous lower-level problem.

Another alternative approach for binary lower-level problems is re- cently proposed by [Shi et al. (2020)](#_bookmark266). The authors consider bilevel MILPs in which the lower-level variables are all binary. The method is based

on the *𝑘*-optimality of the lower-level solution: It is a relaxation of the

leader as long as it is within the *𝑘*-Hamming distance neighborhood of lower-level problem in which the follower’s response is accepted by the

any bilevel feasible solution. This way, it is possible to model not com- pletely rational decisions of the follower. The authors provide a hierar-

chy of decisions linked with the value of *𝑘*, along with a hierarchy of

sponds to *𝑘* = 0. upper and lower bounds of the original bilevel problem, which corre-

* 1. *Bilevel MINLPs*

one can only expect to compute *𝜀*-optimal solutions. Thus, the same also For single-level nonconvex mixed-integer optimization problems,

tion 3 in [Mitsos et al. (2008)](#_bookmark210) for a formal definition of *𝜀*-optimality in the holds for nonconvex mixed-integer bilevel problems. We refer to Defini-

bilevel context and discuss some approaches for bilevel problems with general nonconvex mixed-integer lower-level problems in the following. In [Mitsos (2010)](#_bookmark207), general bilevel MINLPs with continuity assump- tions on all functions are considered. In addition, all variables are as- sumed to be bounded. The stated approach is an extension of the method proposed in [Mitsos et al. (2008)](#_bookmark210) that dealt with purely continuous bilevel problems. In turn, the latter paper builds on theoretical developments in [Mitsos and Barton (2006)](#_bookmark208). The key idea is to exploit estimates on the optimal value function of the lower level, which requires the global so- lution of MINLPs as subproblems. The approach is shown to terminate in finite time and an implementation of the approach is evaluated on a

small test set.

In a series of papers, the so-called branch-and-sandwich approach for bilevel MINLPs is developed. The main idea is to subsequently compute tightened bounds on the optimal value function [(3)](#_bookmark9) and on the upper- level objective function value. Starting with continuous but nonconvex lower-level problems in [Kleniati and Adjiman (2014a)](#_bookmark202) and a numerical evaluation thereof in [Kleniati and Adjiman (2014b)](#_bookmark203), the approach is ex- tended to the mixed-integer case in [Kleniati and Adjiman (2015)](#_bookmark205). The approach stated in the latter paper is applicable to problems with twice

continuously differentiable functions *𝐹* , *𝑓* , *𝐺*, and *𝑔* and requires bounds

on all variables. In this setting, the branch-and-sandwich approach ter-

minates in finite time. Recently, novel bounding schemes for this ap- proach have been published in [Paulavičius and Adjiman (2020)](#_bookmark230) and fur- ther implementation details can be found in [Paulavičius et al. (2020)](#_bookmark233). Due to the general hardness of the problems under consideration, the computational study in [Kleniati and Adjiman (2015)](#_bookmark205) deals with rather small problems with up to 12 variables and 7 constraints.

functions *𝐹* , *𝑓* , *𝐺*, and *𝑔* are continuous but possibly nonconvex. In A different setting is considered in [Lozano and Smith (2017b)](#_bookmark193). All addition, the constraint functions *𝐺* and *𝑔* need to be separable in *𝑥* and *𝑦*, i.e., they have to be of the form *𝐺*(*𝑥, 𝑦*) = *𝐺*1(*𝑥*) + *𝐺*2(*𝑦*) and

*𝑔*(*𝑥, 𝑦*) = *𝑔*1(*𝑥*) + *𝑔*2(*𝑦*). Under the assumptions that (i) the upper- and lower-level feasible regions are compact, (ii) *𝑔*1(*𝑥*) is integer-valued for all *𝑥*, and (iii) all upper-level variables *𝑥* are integers, the authors derive

a finite solution approach based on the value-function reformulation [(4)](#_bookmark10). In particular, this approach is also capable of solving the pessimistic variant.

# Interdiction Problems

Interdiction games are a special class of bilevel problems that aim at monitoring or halting an adversarys activity in a given environment. They are used to model defender-attacker settings in which the attacker (the follower) optimizes some objective such as a shortest path or a maximum flow in a network (see, e.g., [Israeli and Wood (2002)](#_bookmark172)), or maximizes the profit of the items that can be packed in a knapsack ([Caprara et al., 2014; Fischetti et al., 2019](#_bookmark88)). The defender, who acts as the leader, has limited resources to protect the environment, e.g., by disabling the vertices/edges in a network or by changing their ca- pacity, or by removing the knapsack items, to achieve the worst possi- ble outcome for the attacker. Besides military applications, interdiction problems are extremely important in controlling the spread of infectious diseases ([Assimakopoulos, 1987; Furini et al., 2021; Shen et al., 2012](#_bookmark62)), spread of fake news in social networks ([Baggio et al., 2021](#_bookmark67)), in counter- [terrorism and in monitoring of communication networks (Wang et al., 2016).](#_bookmark249)

Interdiction problems follow the common structure of bilevel prob- lems without coupling constraints,

that the available budget of the leader is limited, so that the follower’s problem [(17)](#_bookmark42) stays feasible.

min

*𝑥*∈*𝑋,𝑦*∈*𝑆*(*𝑥*)

{*𝐹* (*𝑥, 𝑦*) ∶ *𝐺*(*𝑥*) ≥ 0}*,*

In some very special cases, when the interdiction problem can be modeled as a bilevel LP, the problem is polynomially solvable; see, e.g.,

where constraints *𝐺*(*𝑥*) describe some restrictions on the solution of

and *𝑆*(*𝑥*) represents the set of optimal solutions of the *𝑥*-parameterized the leader, typically including some budget or resource constraints,

lower-level problem. Interdiction problems model zero-sum Stackelberg games, i.e., they correspond to a competitive setting in which the leader and the follower have diametrically opposed objective functions:

*𝐹* (*𝑥, 𝑦*) = −*𝑓* (*𝑥, 𝑦*)*.*

This is why interdiction problems can be alternatively stated as

min {*𝜑*(*𝑥*) ∶ *𝐺*(*𝑥*) ≥ 0}*,* (16)

*𝑥*∈*𝑋*

where the follower’s problem is stated in its maximization form:

*𝜑*(*𝑥*) = max {*𝑓* (*𝑥, 𝑦*) ∶ *𝑔*(*𝑥, 𝑦*) ≥ 0}*.* (17)

*𝑦*∈*𝑌*

The leader prevents certain activities of the follower by reducing the availability of some objects or resources—for example, items or nodes/edges in a network. Based on the relationships between the func-

tions *𝑓* and *𝑔* and the nature of the leader’s variables *𝑥*, we make the

following distinction.

*Discrete Interdiction*

variables *𝑥𝑖* are binary, and they are set to one if and only if the respec- In the discrete interdiction setting given below, the linking tive object *𝑖* is unavailable for the follower. Thus, the objective function

*𝑓* (*𝑥, 𝑦*) = *𝑑⊤𝑦* is typically linear and the constraints *𝑔*(*𝑥, 𝑦*) ≥ 0 in [(17)](#_bookmark42) are

replaced by

*𝑦𝑖* ≤ *𝑈𝑖* (1 − *𝑥𝑖* )*, 𝑖* ∈ *𝑁𝑥,* (18a)

*𝑔̃*(*𝑦*) ≥ 0*,* (18b)

where *𝑈𝑖* represents the default upper bound for the follower variable *𝑦𝑖* (modeling the availability of object *𝑖* at the lower level), *𝑁𝑥 ⊆* {1*,* … *, 𝑛𝑥* } is the index set of the binary linking variables of the leader, and *𝑔̃* ∶ ℝ*𝑛𝑦* → ℝ𝓁 are constraints that impose further restrictions on the fol-

section we assume that *𝑛𝑦* = |*𝑁𝑥* |. lower’s solution. To simplify the exposition, in the remainder of this

*Continuous Interdiction*

In the continuous interdiction setting, the linking variables *𝑥𝑖* are continuous (i.e., 0 ≤ *𝑥𝑖* ≤ 1 for *𝑖* ∈ *𝑁𝑥* ) and they model a continuous in-

crease of costs (or reduction of available capacities) imposed on the interdicted objects. For example, in max-flow interdiction settings, the leader is given a limited budget to reduce the available capacity of arcs/vertices, while the follower tries to maximize the flow in the re- sulting network; see, e.g., [Lim and Smith (2007)](#_bookmark182); [Wood (1993)](#_bookmark258) and the further references therein. Alternatively, in the shortest-path interdic- tion (see, e.g., [Israeli and Wood (2002)](#_bookmark172)), the leader can increase the cost of arcs traversed by the follower and in this case, the constraints

*𝑔*(*𝑥, 𝑦*) ≥ 0 in [(17)](#_bookmark42) are replaced by *𝑔̃*(*𝑦*) ≥ 0 and the objective function of

the follower *𝑓* (*𝑥, 𝑦*) becomes bilinear, i.e.,

∑

*𝑓* (*𝑥, 𝑦*) = *𝑦𝑖* (*𝑑𝑖* + *𝛿𝑖 𝑥𝑖* )*,* (19)

*𝑖*∈*𝑁𝑥*

as it now encodes the increase of arc costs (modeled by *𝛿*) caused by

the leader. Finally, one can also consider discrete interdiction problems

with bilinear objective functions.

By the nature of the objective function, there is no distinction be- tween optimistic and pessimistic solutions. It is also commonly assumed

[Fulkerson and Harding (1977)](#_bookmark178), where it is shown that continuous in- terdiction of the shortest-path problem can be equivalently stated as a minimum cost flow problem. However, discrete interdiction of the

shortest-path problem in which the leader chooses *𝑘* arcs to interdict

[(i.e., *𝐺*(*𝑥*) ≥ 0 translates into 1*⊤𝑥* ≤ *𝑘*), already is NP-hard (Ball et al., 1989). The latter problem is frequently referred to as the *𝑘*-most-vital](#_bookmark70)

arcs problem. Just like for general bilevel MILPs, if the follower solves an NP-hard problem (e.g., the maximum knapsack problem or the maxi- mum clique problem), the corresponding interdiction problem turns out

to be Σ*𝑝*-hard ([Caprara et al., 2014; Rutenburg, 1994](#_bookmark88)).

2

* 1. *Commonly Studied Interdiction Problems*

We provide below a classification of interdiction problems based on the structures of the encompassed lower-level problems.

*Network Interdiction with Polynomial Lower-Level Problems*

These problems model some of the most traditional and oldest appli- cations arising in the areas of military or homeland security. Besides in- terdiction of shortest paths ([Israeli and Wood, 2002](#_bookmark172)) or maximum flows [(Akgün et al., 2011; Cormican et al., 1998; Janjarassuk and Linderoth, 2008; Wood, 1993), problems also have been studied in which the fol-](#_bookmark53) lower solves the spanning tree ([Bazgan et al., 2013; Lin and Chern, 1993](#_bookmark83)) or the maximum matching problem ([Zenklusen, 2010](#_bookmark275)).

*Mixed-Integer Linear System Interdiction Problems*

These are interdiction problems in which the lower level is an MILP. They were first studied in the PhD thesis of [Israeli (1999)](#_bookmark171) and later in the PhD thesis of [DeNegre (2011)](#_bookmark135). One of the most studied (and structurally easiest) variants is discrete interdiction of the maximum knapsack problem, in which the leader and the follower have a knap- sack of their own, and the follower can only choose from those items that are not taken by the leader. Complexity results for this prob- lem are given in [Caprara et al. (2014)](#_bookmark88), whereas tailored exact meth- [ods have been developed in](#_bookmark122) [Caprara et al. (2016)](#_bookmark91)[; Della Croce and Scatamacchia (2019);](#_bookmark122) [Fischetti](#_bookmark170) [et al. (2019); see also PhD thesis by](#_bookmark122) [Carvalho (2016)](#_bookmark97). The problem’s extension in which the leader and the follower solve a multidimensional knapsack problem has been addressed in [Fischetti et al. (2019)](#_bookmark170). For other variants of more general bilevel knap- sack problems (that do not belong to the interdiction setting) see the PhD thesis by [Carvalho (2016)](#_bookmark97) and the further references therein.

Facility location with interdiction has been studied as well. [Scaparra and Church (2008)](#_bookmark259) investigate the problem in which the leader is concerned with protecting a limited number of facilities, assuming the follower will attack a fixed number of them in order to maximize the transportation cost between the clients and the remaining operational facilities. In [Zhang et al. (2016)](#_bookmark270), the leader locates a fixed number of fa- cilities first, followed by the follower, who is prohibited to use the same location as the leader. Both players face disruption risks while trying to maximize the market share, assuming that each customer patronizes the nearest open facility.

Interdiction problems on networks in which the follower solves an NP-hard problem also fall into this category. These problems in- [clude interdiction of the clique number (Furini et al., 2019; 2020b; Rutenburg, 1994), or interdiction of independent sets and vertex cov-](#_bookmark134) ers ([Bazgan et al., 2011](#_bookmark81)).

*Blocking Problems*

Closely related to interdiction problems are the so-called blocking problems in which the leader wishes to minimize the cost of blocking the

will be bounded by a user-defined threshold *𝑟* ∈ ℝ: activities of the follower, while ensuring the optimal follower’s response

min {*𝑐⊤𝑥* ∶ *𝜑*(*𝑥*) ≤ *𝑟, 𝐺*(*𝑥*) ≥ 0}

*𝑥 .*

*𝑥*∈*𝑋*

When blocking the maximum number of cliques, e.g., the leader min- imizes the (un)weighted sum of vertices/edges to remove from the graph, so that the maximum (weighted) clique in the remaining graph [is bounded from above by a given integer (Pajouh, 2020; Pajouh et al., 2014). The blocking of vertices or edges has been studied with respect](#_bookmark221) to other graph optimization problems such as the maximum matchings ([Zenklusen et al., 2009](#_bookmark276)), shortest paths ([Golden, 1978](#_bookmark145)), spanning trees ([Bazgan et al., 2013](#_bookmark83)), or dominating sets ([Pajouh et al., 2015](#_bookmark224)).

Most exact methods for blocker problems share similarities with the methods derived for interdiction problems, which is why we focus on the latter ones in the remainder of this section.

* 1. *Methods*

When the lower-level problem is linear, this transformation allows one to use duality theory and reformulate the problem as a single-level problem, following the same approach as for the bilinear objective func- tion [(19)](#_bookmark45) described above. If the lower-level problem is discrete (and NP- hard), one can apply a Benders-like decomposition approach instead.

*Benders-like Decomposition*

For linear lower-level problems, the value of *𝑀𝑖* in the penalization approach is chosen as an upper bound of the dual variable associated to

constraint [(18a)](#_bookmark43); see, e.g., [Brown et al. (2006)](#_bookmark120); [Lim and Smith (2007)](#_bookmark182); [Wood (2011)](#_bookmark259).

[Israeli (1999)](#_bookmark171) proposes to use the penalty function to reformulate interdiction problems whose lower-level problem is an MILP. The lower-

does not depend on *𝑥* anymore and, hence, the value function can be level problem is then convexified using the fact that its feasible region

restated as

The specific structure of interdiction problems can be exploited in

{

*𝜑*(*𝑥*) = max

∑ *𝑦̄* (*𝑑* − *𝑀 𝑥* ) ∶ *𝑦̄* ∈ *𝑌̄*

}

*,* (21)

different ways to derive problem-tailored exact approaches. We summa-

*𝑖 𝑖*

*𝑖*∈*𝑁𝑥*

*𝑖 𝑖*

rize generic strategies used to solve interdiction problems to optimality.

*Dualization*

When the followers problem corresponds to a linear optimization problem, duality theory can be exploited to derive a single-level refor- mulation. If the leader influences the objective function of the follower,

where *𝑌̄* represents the set of extreme points of the polytope described

is well defined, so that the set *𝑌̄* is nonempty. Hence, the function *𝜑*(*𝑥*), by [(18b)](#_bookmark44) and [(20)](#_bookmark49). Recall that we assume that the lower-level problem

described as the maximum of a set of aﬃne functions given in [(21)](#_bookmark46), is convex and the starting problem, given in the Form [(16)](#_bookmark41), can be now

reformulated by projecting out the follower’s variables *𝑦* and by intro-

ducing an auxiliary variable *𝜃* as

level problem for a given value of *𝑥*. That way, we get rid of the bi- like in the bilinear objective function [(19)](#_bookmark45), we first dualize the lower-

min

{

*𝜃* ∶ *𝜃* ≥ ∑ *𝑦̄* (*𝑑* − *𝑀 𝑥* )*, 𝑦̄* ∈ *𝑌̄ , 𝐺*(*𝑥*) ≥ 0

}

*.* (22)

[Wood (2002) for the shortest path interdiction) involving variables *𝑥* of linear terms and obtain a single-level formulation (see, e.g., Israeli and](#_bookmark172)

*𝑥*∈*𝑋*

*𝑖 𝑖*

*𝑖*∈*𝑁𝑥*

*𝑖 𝑖*

the leader and dual variables associated to constraints of the follower’s problem [(17)](#_bookmark42).

If the feasible region of the lower level is influenced by the leader, such as in [(18)](#_bookmark43), after dualizing the lower-level problem, the resulting

single-level reformulation again optimizes over *𝑥* and dual variables as-

tive function involves bilinear terms in which *𝑥*-variables are multiplied sociated to each constraint of the follower’s problem. However, its objec-

with continuous variables of the follower’s dual problem. When dealing with discrete interdiction problems, these bilinear terms are typically linearized using McCormick’s inequalities ([McCormick, 1976](#_bookmark201)), result- ing in a single-level MILP problem reformulation; see, e.g., the seminal work by [Wood (1993)](#_bookmark258) where this technique is applied for the maximum-

flow interdiction problem. For continuous *𝑥*-variables, a specialized ex-

act method has been proposed by [Lim and Smith (2007)](#_bookmark182) assuming that

*𝐺*(*𝑥*) ≥ 0 models a budget constraint, exploiting the fact that at most one of the *𝑥*-components will be fractional in an optimal interdiction

strategy.

*Penalization*

An alternative way to deal with constraints of type [(18a)](#_bookmark43) in a discrete interdiction setting is to relax them into

*𝑦𝑖* ≤ *𝑈𝑖 , 𝑖* ∈ *𝑁𝑥,* (20)

and penalize the use of the object *𝑖*, whenever *𝑥𝑖* = 1 for an *𝑖* ∈ *𝑁𝑥* . This can be achieved by introducing coeﬃcients *𝑀𝑖* , *𝑖* ∈ *𝑁𝑥* , and by replacing the linear objective function *𝑑⊤𝑦* of the follower by

∑

*𝑦𝑖* (*𝑑𝑖* − *𝑀𝑖𝑥𝑖* )*.*

*𝑖*∈*𝑁𝑥*

Due to the fact that *𝑥𝑖* ∈ {0*,* 1}, the coeﬃcients *𝑀𝑖* have to be suﬃciently

ified follower’s problem in which *𝑥𝑖* = 1 implies *𝑦𝑖* = 0 for all *𝑖* ∈ *𝑁𝑥* ; large to ensure that there always exists an optimal solution of the mod-

see, e.g., [Smith and Song (2020)](#_bookmark235); [Wood (2011)](#_bookmark259) and further references therein.

ing interdiction problems are typically Σ*𝑝*-hard, which implies that there Recall that when the follower solves an NP-hard problem, the result-

is no way of formulating such problems as single-level integer programs of polynomial size unless the polynomial hierarchy collapses. This, in particular, means that separating Benders-like constraints

2

∑

*𝜃* ≥ *𝑦̄𝑖* (*𝑑𝑖* − *𝑀𝑖𝑥𝑖* ) (23)

*𝑖*∈*𝑁𝑥*

in [(22)](#_bookmark47) for any given solution (*𝜃̄, 𝑥̄*) of the leader requires solving the NP-hard follower’s problem defined by *𝜑*(*𝑥̄*) in [(21)](#_bookmark46). Nevertheless, when

effective algorithms are available for solving these lower-level prob- lems (rather than formulating them as MILPs and using general-purpose solvers), some recent results show that tight canonical single-level refor- mulations can be obtained. [Fischetti et al. (2019)](#_bookmark170) use dynamic program- [ming for the maximum knapsack interdiction and Furini et al. (2019, 2020b) use tailored branch-and-bound solvers for two variants of the](#_bookmark134) maximum clique interdiction problems. Moreover, any heuristic solu- tion of the lower-level problem also provides a valid Benders-like con- straint [(23)](#_bookmark48) and standard stabilization techniques for improving the con- vergence can be applied.

The choice of the coeﬃcients *𝑀𝑖* is crucial for the computational eﬃ-

ciency of the derived single-level reformulation. In some particular cases

example, [Cormican et al. (1998)](#_bookmark117) show that *𝑀𝑖* = 1 for the interdiction in which the follower solves an LP, such bounds can be very tight—for

that tight *𝑀𝑖* coeﬃcients (i.e., *𝑀𝑖* = *𝑑𝑖* for *𝑖* ∈ *𝑁𝑥* ) can also be derived of the maximum flow problem. Recently, [Fischetti et al. (2019)](#_bookmark170) show

for lower-level problems which are NP-hard, provided that lower-level

latter property assumes that if *𝑦̄* is a feasible lower-level solution for a constraints satisfy the so-called downward monotonicity property. The given *𝑥*, then any *𝑦̂* such that 0 ≤ *𝑦̂* ≤ *𝑦̄* is also feasible. This condition

is, for example, satisfied if the follower solves a variant of a set-packing problem ([Dinitz and Gupta, 2013](#_bookmark146)), including the maximum knapsack, multidimensional knapsack, or a graph optimization problem that satis- fies the hereditary property with respect to interdicted objects; e.g., the maximum matching problem if edges are interdicted or the maximum

clique problem if vertices are interdicted. [Furini et al. (2019)](#_bookmark134) show that for the maximum clique interdiction problem in which the leader re- moves the vertices of the graph, Constraints [(23)](#_bookmark48) are facet-defining with

*𝑀𝑖* = 1 under some mild conditions. Finally, [Fischetti et al. (2019)](#_bookmark170) pro-

vide further generalizations of their result that include settings in which

ables that are not influenced by the leader (i.e., *𝑛𝑦 >* |*𝑁𝑥* |) showing the follower’s MILP is an extended formulation, involving other vari-

that interdiction of some facility location problems and variants of the Steiner tree problem fall into this category.

*Other Approaches*

[Tang et al. (2016)](#_bookmark239) propose a generic exact method for solving dis- crete interdiction problems in which the feasible region of the lower- level MILP is influenced by the leader. The authors show that valid lower bounds are obtained by progressively building a convex inner approximation of the feasible solutions of the lower-level MILP. This inner approximation is modeled as an LP, and dualized to obtain a single-level MILP formulation. The solution of this formulation provides

a valid lower bound and a feasible solution *𝑥* for the leader, which can

be plugged in into *𝜑*(*𝑥*) to calculate a valid upper bound. The algorithm

ble solutions from {*𝑥* ∈ *𝑋* ∶ *𝐺*(*𝑥*) ≥ 0} have been exhaustively searched. terminates when the lower and upper bound coincide or when all feasi-

[Salmeron et al. (2009)](#_bookmark256) propose a global Benders decomposition method for discrete interdiction problems. The method alternates be- tween solving the master problem (containing discrete interdiction vari- ables) and LP subproblem(s), building a convex piecewise-linear approx-

imation of the function *𝜑*(*𝑥*). The method is more general in the sense

that it can be applied to interdiction problems for which the function

*𝜑*(*𝑥*) is not convex. A sequence of lower-bounding piecewise-linear ap- proximations of *𝜑*(*𝑥*) is built, which is tight for any given discrete choice of *𝑥*, which guarantees that in a finite number of iterations the opti-

mal solution can be found. [Salmeron and Wood (2015)](#_bookmark257) generalize this method for solving an interdiction problem of a power system whose lower-level problem is an MILP.

[Lozano and Smith (2017a)](#_bookmark189) introduce a backward-sampling approach for solving discrete interdiction problems. Sampling is used to create a subset of the followers solutions—optimizing over this subset gives the maximum perceived damage made by the leader and, hence, a valid lower bound. Similarly as in the method by [Tang et al. (2016)](#_bookmark239), the au- thors carefully extend the sampling set, while alternating between the calculations of the lower and upper bound, until the two values meet.

Finally, a combined column-and-row generation method, which [relies on Benders decomposition, has been proposed by Zhao and Zeng (2013). The MILP model is dynamically extended by new vari-](#_bookmark272)

decision *𝑥* and the newly added constraints ensure the optimality con- ables that correspond to the followers response given a fixed leaders

dition for the followers response.

* 1. *Critical Vertex/Edge Detection Problems in Graphs Seen Through the Lens of Bilevel Optimization*

Most of the existing literature dealing with detection of most vi- tal arcs/vertices, with respect to some given graph-functionality mea- sures, rely on extended MILP formulations; see, e.g., the recent survey by [Lalou et al. (2018)](#_bookmark222). Interesting applications include the maximiza- tion of the total number of pair-wise connected vertices while removing [a limited number of arcs/vertices from the network (Arulselvan et al., 2009; Di Summa et al., 2012), the maximization of the number of con-](#_bookmark63) nected components or the minimization of the size of the largest con- nected component ([Shen and Smith, 2012; Shen et al., 2012](#_bookmark264)). Only re- cently, a connection between critical vertex/edge problems in graphs and Stackelberg games has been exploited by [Furini et al. (2020a)](#_bookmark130) and [Furini et al. (2021)](#_bookmark132), where the authors derive canonical formulations in

the natural space of variables for the *𝑘*-vertex cut and the capacitated

vertex separator problem, respectively. The authors propose eﬃcient

B&C methods that beat the state-of-the-art thanks to the bilevel-like problem interpretation.

Finally, we point out that an up-to-date survey on network interdic- tion models and algorithms can be found in [Smith and Song (2020)](#_bookmark235). The survey also includes other aspects not covered in our survey such as interdiction under uncertainty, multilevel interdiction also known as defender-attacker-defender games ([Baggio et al., 2021](#_bookmark67)), as well as in- teresting problem extensions including situations in which both players act simultaneously, or problems with information asymmetry or infor- mation incompleteness (for either the leader or the follower).

# Possible Directions for Future Research

The variety of aspects discussed in this survey show the required broadness of techniques that have to be exploited to solve bilevel opti- mization problems effectively. Despite the large amount of research car- ried out in the recent years, there are still very many aspects that need further investigation. In this section, we sketch a few of the many pos- sible directions for future research in the field of computational bilevel optimization and begin with those aspects that are rather close to what is discussed in this paper.

1. The incorporation of integer variables in models is known to make the problem harder to solve. However, it is often easier to design provably correct algorithms for solving bilevel problems if all link- ing variables are integer. If this assumption does not hold, we have discussed in [Section 5.1](#_bookmark39) that optimal solutions may not be attain- able. This is the reason why more methods exist for ILP-ILP bilevel problems compared to what is published for the mixed-integer, i.e., the MILP-MILP, case. In this setting, solution methods need to deal

with *𝜀*-optimality of solutions. This might be one reason why the

performance of these methods is usually not comparable with the

performance of methods that rely on the assumption of integer link- ing variables.

1. In this survey, we mainly discussed branch-and-bound as well as branch-and-cut methods for solving bilevel problems. We also men- tioned a few primal heuristics. If one, however, compares the rich- ness of cutting planes used in the field of single-level mixed-integer optimization with the number of known valid inequalities for bilevel optimization, it is obvious that many branch-and-cut algorithms would benefit from a larger set of bilevel-specific cutting planes. Moreover, the entire field of presolve techniques is almost com- pletely unexplored in bilevel optimization, whereas the performance of state-of-the-art MILP solvers heavily relies on them.
2. If one compares the number of approaches for specific bilevel prob- lems such as pricing problems as well as Stackelberg or interdiction games with the number of general-purpose approaches for bilevel LPs or bilevel MI(N)LPs, it becomes obvious that there still is a lot to do with respect to developing general-purpose algorithms for larger classes of bilevel problems. Of course, both previously mentioned as- pects will also play a key role in developing further general-purpose methods.
3. In addition to the mathematical aspects mentioned so far, computa- tional bilevel optimization still suffers from the absence of a broad variety of well-curated instance libraries that can be used to test and tune specific implementations of newly developed algorithms. Al- [though some instance sets are already publicly available (Paulavičius and Adjiman, 2019; Ralphs, 2020; Sinnl, 2020; Zhou et al., 2020),](#_bookmark231) the community of computational bilevel optimization would greatly benefit from more, and in particular more diverse, instance sets.
4. In addition, the development of novel algorithmic techniques would also very much benefit from more mature open-source realizations of “classical” methods in the field. Today, new ideas, e.g., for a novel valid inequality or a new presolve technique can usually only be tested if a lot of other techniques have been implemented on top of a basic branch-and-bound scheme. Obviously, availability of such

open-source codes (of which MibS, see [Ralphs (2018)](#_bookmark240), is a notable example) would push the field significantly.

There are also many sub-areas of bilevel optimization that need to be developed further—especially when it comes to algorithmic and compu- tational aspects. Let us exemplarily discuss two of them.

1. Most of the methods discussed in this survey tackle optimistic bilevel optimization problems. Although some important theoretical ad- vances have been made in the field of pessimistic problems, the al- gorithmic treatment of these models is still in its infancy.
2. Another field worth to be mentioned is bilevel optimization under uncertainty—let it be stochastic or robust optimization problems embedded in a bilevel context. This problem class obviously is of tremendous importance for practice but, on the other hand, is also very hard to solve. The main reason is that the incorporation of un- certainty usually introduces another level in the problem, which then directly leads to tri- or general multilevel models that we will also comment on below again.

To sum up, there are many important and insuﬃciently explored topics in the field of bilevel optimization that lead to open research questions and, thus, to possible topics of future work. The focus of this survey is on mixed-integer programming techniques for solving chal- lenging bilevel optimization problems. In this context, many interesting topics and questions arise that are at the interface of bilevel optimization and combinatorial optimization problems, problems from graph theory, algorithmic design, complexity theory, operations research, and appli- cations. We are convinced that these connections can help us to derive tighter models, faster exact or approximation algorithms, or new struc- tural properties. We sketch two exemplary problems at the interface of bilevel optimization as well as combinatorial optimization or graph the- ory:

1. The problem of generating a maximally violated valid inequality, i.e., the separation problem in a branch-and-cut context, can often be interpreted as a bilevel problem; see [Lodi et al. (2014)](#_bookmark190). In some cases, a compact single-level reformulation is not possible, and hence, any advancement in solving bilevel programs may have a significant im- pact on improving the performance of branch-and-cut based meth- ods for diﬃcult combinatorial optimization problems.

lems, minimum *𝑑*-blockers, or *𝑑*-transversals in graphs can be formu- 9. In graph theory, the families of critical vertex/edge detection prob-

lated as bilevel (interdiction-like) optimization problems; see, e.g., [Costa et al. (2011)](#_bookmark119). This allows to look at some of the classical prob- lems from graph theory from a different and fresh perspective and to possibly derive new mixed-integer formulations, which “live” in the canonical space of variables.

Moreover, many applications need to go beyond bilevel modeling and require tri- or even general multilevel models. This is, e.g., the case for interdiction problems with fortification ([Lozano and Smith, 2017a](#_bookmark189)), stochastic interdiction problems ([Cormican et al., 1998](#_bookmark117)), or for problems from energy market design ([Grimm et al., 2016](#_bookmark149)). In this context, rather small scale instances are usually solved by exploiting hand-crafted and highly problem-specific solution methods. Thus, applied bilevel opti- mization would very much benefit from algorithmic enhancements for general tri- or multilevel problems.

# Disclosure of conflicts of interest

The authors declare that there is no conflict of interest.

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# References

Achterberg, [T., 2019. What’s new in Gurobi 9.0. In: Webinar Talk. URL: https://www. gurobi.com/wp-content/uploads/2019/12/Gurobi-9.0-Overview-Webinar-Slides- 1.pdf.](https://www.gurobi.com/wp-content/uploads/2019/12/Gurobi-9.0-Overview-Webinar-Slides-1.pdf)

Akgün, I., Tansel, B.Ç., Wood, R.K., 2011. The multi-terminal maximum-flow network- interdiction problem. European Journal of Operational Research 211 (2), 241–251. doi:[10.1016/j.ejor.2010.12.011](https://doi.org/10.1016/j.ejor.2010.12.011).

Alguacil, N., Delgadillo, A., Arroyo, J.M., 2014. A trilevel programming approach for electric grid defense planning. Computers & Operations Research 41, 282–290. doi:[10.1016/j.cor.2013.06.009](https://doi.org/10.1016/j.cor.2013.06.009).

Ambrosius, M., Grimm, V., Kleinert, T., Liers, F., Schmidt, M., Zöttl, G., 2020. En- dogenous price zones and investment incentives in electricity markets: An appli- cation of multilevel optimization with graph partitioning. Energy Economics 92. doi:[10.1016/j.eneco.2020.104879](https://doi.org/10.1016/j.eneco.2020.104879).

Anandalingam, G., Friesz, T., 1992. Hierarchical optimization: An introduction. Annals of Operations Research 34 (1), 1–11. doi:[10.1007/BF02098169](https://doi.org/10.1007/BF02098169).

Anandalingam, G., White, D., 1990. A solution method for the linear static Stackelberg problem using penalty functions. IEEE Transactions on Automatic Control 35 (10), 1170–1173. doi:[10.1109/9.58565](https://doi.org/10.1109/9.58565).

Arroyo, J.M., 2010. Bilevel programming applied to power system vulnerability analysis under multiple contingencies. IET Generation, Transmission & Distribution 4 (2), 178–

190. doi:[10.1049/iet-gtd.2009.0098](https://doi.org/10.1049/iet-gtd.2009.0098).

Arulselvan, A., Commander, C.W., Elefteriadou, L., Pardalos, P.M., 2009. Detecting crit- ical nodes in sparse graphs. Computers & Operations Research 36 (7), 2193–2200. doi:[10.1016/j.cor.2008.08.016](https://doi.org/10.1016/j.cor.2008.08.016).

Assimakopoulos, N., 1987. A network interdiction model for hospital in- fection control. Computers in Biology and Medicine 17 (6), 413–422. doi:[10.1016/0010-4825(87)90060-6](https://doi.org/10.1016/0010-4825(87)90060-6).

Audet, C., Haddad, J., Savard, G., 2006. A note on the definition of a linear bilevel programming solution. Applied Mathematics and Computation 181 (1), 351–355. doi:[10.1016/j.amc.2006.01.043](https://doi.org/10.1016/j.amc.2006.01.043).

Audet, C., Haddad, J., Savard, G., 2007. Disjunctive cuts for continuous linear bilevel programming. Optimization Letters 1 (3), 259–267. doi:[10.1007/s11590-006-0024-3](https://doi.org/10.1007/s11590-006-0024-3). Audet, C., Hansen, P., Jaumard, B., Savard, G., 1997. Links between linear bilevel and mixed 0–1 programming problems. Journal of Optimization Theory and Applications

93 (2), 273–300. doi:10.1023/A:1022645805569.

Audet, C., Savard, G., Zghal, W., 2007. New Branch-and-Cut Algorithm for Bilevel Linear Programming. Journal of Optimization Theory and Applications 134 (2), 353–370. doi:[10.1007/s10957-007-9263-4](https://doi.org/10.1007/s10957-007-9263-4).

Aussel, D., Brotcorne, L., Lepaul, S., von Niederhäusern, L., 2020. A trilevel model for best response in energy demand-side management. European Journal of Operational Research 281 (2), 299–315. doi:[10.1016/j.ejor.2019.03.005](https://doi.org/10.1016/j.ejor.2019.03.005).

Avraamidou, S., Pistikopoulos, E.N., 2019. B-pop: Bi-level parametric op- timization toolbox. Computers & Chemical Engineering 122, 193–202. doi:[10.1016/j.compchemeng.2018.07.007](https://doi.org/10.1016/j.compchemeng.2018.07.007).

Avraamidou, S., Pistikopoulos, E.N., 2019. A multi-parametric optimization approach for bilevel mixed-integer linear and quadratic programming problems. Computers & Chemical Engineering 125, 98–113. doi:[10.1016/j.compchemeng.2019.01.021](https://doi.org/10.1016/j.compchemeng.2019.01.021).

[Baggio,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0017) [A., Carvalho, M., Lodi, A., Tramontani, A., 2021. Multilevel approaches for the critical node problem. Operations Research.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0017) [To appear](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0017)

Balas, E., 1971. Intersection cuts–a new type of cutting planes for integer programming.

Operations Research 19 (1), 19–39. doi:[10.1287/opre.19.1.19](https://doi.org/10.1287/opre.19.1.19).

Ball, M.O., Golden, B.L., Vohra, R.V., 1989. Finding the most vital arcs in a network.

Operations Research Letters 8 (2), 73–76. doi:[10.1016/0167-6377(89)90003-5](https://doi.org/10.1016/0167-6377(89)90003-5).

Bard, J.F., 1983. Coordination of a multidivisional organization through two levels of management. Omega 11 (5), 457–468. doi:[10.1016/0305-0483(83)90038-5](https://doi.org/10.1016/0305-0483(83)90038-5).

Bard, J.F., 1984. Optimality conditions for the bilevel programming problem. Naval Re- search Logistics Quarterly 31 (1), 13–26. doi:[10.1002/nav.3800310104](https://doi.org/10.1002/nav.3800310104).

Bard, J.F., 1988. Convex two-level optimization. Mathematical Programming 40 (1), 15–

27. doi:[10.1007/BF01580720](https://doi.org/10.1007/BF01580720).

Bard, J.F., 1991. Some properties of the bilevel programming problem. Journal of Opti- mization Theory and Applications 68 (2), 371–378. doi:[10.1007/BF00941574](https://doi.org/10.1007/BF00941574).

[Bard,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0024) [J.F., 1998. Practical bilevel optimization: algorithms and applications, 30. Springer Science & Business Media.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0024)

Bard, J.F., Moore, J.T., 1990. A branch and bound algorithm for the bilevel program- ming problem. SIAM Journal on Scientific and Statistical Computing 11 (2), 281–292. doi:[10.1137/0911017](https://doi.org/10.1137/0911017).

Bard, J.F., Moore, J.T., 1992. An algorithm for the discrete bilevel programming problem. Naval Research Logistics 39 (3), 419–435. doi:10.1002/1520-6750(199204)39:3<419::AID-NAV3220390310>3.0.CO;2-C.

Bard, J.F., Plummer, J., Sourie, J.C., 2000. A bilevel programming approach to determin- ing tax credits for biofuel production. European Journal of Operational Research 120 (1), 30–46. doi:[10.1016/S0377-2217(98)00373-7](https://doi.org/10.1016/S0377-2217(98)00373-7).

Baringo, L., Conejo, A.J., 2012. Transmission and wind power investment. IEEE Transac- tions on Power Systems 27 (2), 885–893. doi:[10.1109/TPWRS.2011.2170441](https://doi.org/10.1109/TPWRS.2011.2170441).

Basu, A., Ryan, C.T., Sankaranarayanan, S., 2021. Mixed-integer bilevel representability. Mathematical Programming 185 (1), 163–197. doi:[10.1007/s10107-019-01424-w](https://doi.org/10.1007/s10107-019-01424-w).

Bazgan, C., Toubaline, S., Tuza, Z., 2011. The most vital nodes with respect to inde- pendent set and vertex cover. Discrete Applied Mathematics 159 (17), 1933–1946. doi:[10.1016/j.dam.2011.06.023](https://doi.org/10.1016/j.dam.2011.06.023).

Bazgan, C., Toubaline, S., Vanderpooten, D., 2013. Critical edges/nodes for the minimum spanning tree problem: complexity and approximation. Journal of Combinatorial Op- timization 26 (1), 178–189. doi:[10.1007/s10878-011-9449-4](https://doi.org/10.1007/s10878-011-9449-4).

[Beale, E.M.L., Tomlin, J.A., 1970. Special facilities in a general mathematical program- ming system for non-convex problems using ordered sets of variables. In: Lawrence, J. (Ed.), Proceedings of the Fifth International Conference on Operational Research. Tavistock Publications, pp. 447–454.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0032)

Belotti, P., Kirches, C., Leyffer, S., Linderoth, J., Luedtke, J., Mahajan, A., 2013. Mixed-integer nonlinear optimization. Acta Numerica 22, 1–131. doi:[10.1017/S0962492913000032](https://doi.org/10.1017/S0962492913000032).

Ben-Ayed, O., 1993. Bilevel linear programming. Computers & Operations Research 20 (5), 485–501. doi:[10.1016/0305-0548(93)90013-9](https://doi.org/10.1016/0305-0548(93)90013-9).

Ben-Ayed, O., Blair, C.E., Boyce, D.E., LeBlanc, L.J., 1992. Construction of a real-world bilevel linear programming model of the highway network design problem. Annals of Operations Research 34 (1), 219–254. doi:[10.1007/BF02098181](https://doi.org/10.1007/BF02098181).

Ben-Ayed, O., Boyce, D.E., Blair, C.E., 1988. A general bilevel linear programming formu- lation of the network design problem. Transportation Research Part B: Methodological 22 (4), 311–318. doi:[10.1016/0191-2615(88)90006-9](https://doi.org/10.1016/0191-2615(88)90006-9).

Benders, J.F., 1962. Partitioning procedures for solving mixed-variables programming problems. Numerische Mathematik 4 (1), 238–252. doi:[10.1007/bf01386316](https://doi.org/10.1007/bf01386316).

Bennett, K.P., Jing Hu, Xiaoyun Ji, Kunapuli, G., Jong-Shi Pang, 2006. Model selection via bilevel optimization. In: The 2006 IEEE International Joint Conference on Neural Network Proceedings, pp. 1922–1929. doi:[10.1109/IJCNN.2006.246935](https://doi.org/10.1109/IJCNN.2006.246935).

Bennett, K.P., Kunapuli, G., Hu, J., Pang, J.-S., 2008. Bilevel optimization and machine learning. In: IEEE World Congress on Computational Intelligence. Springer, pp. 25–47. doi:[10.1007/978-3-540-68860-0\_2](https://doi.org/10.1007/978-3-540-68860-0_2).

Besançon, M., Anjos, M. F., Brotcorne, L., 2019. Near-optimal robust bilevel optimization.

URL: <https://hal.archives-ouvertes.fr/hal-02414848>.

Besançon, M., Anjos, M.F., Brotcorne, L.. Complexity of near-optimal robust versions of multilevel optimization problems. [arXiv:2011.00824](http://arxiv.org/abs/2011.00824).

Bialas, W.F., Karwan, M.H., 1984. Two-level linear programming. Management Science 30 (8), 1004–1020. doi:[10.1287/mnsc.30.8.1004](https://doi.org/10.1287/mnsc.30.8.1004).

Bolusani, S., Coniglio, S., Ralphs, T.K., Tahernejad, S., 2020. A Unified Framework for Mul- tistage Mixed Integer Linear Optimization. Springer International Publishing, pp. 513–

560. doi:[10.1007/978-3-030-52119-6\_18](https://doi.org/10.1007/978-3-030-52119-6_18).

and its application to multilevel optimization. Tech. rep., COR@L Technical Report Bolusani, S., Ralphs, T.K., 2020. A framework for generalized Benders’ decomposition

20T-004. URL: <http://www.optimization-online.org/DB_HTML/2020/04/7755.html>. Bonami, P., Biegler, L.T., Conn, A.R., Cornuéjols, G., Grossmann, I.E., Laird, C.D.,

Lee, J., Lodi, A., Margot, F., Sawaya, N., et al., 2008. An algorithmic framework for convex mixed integer nonlinear programs. Discrete Optimization 5 (2), 186–204. doi:[10.1016/j.disopt.2006.10.011](https://doi.org/10.1016/j.disopt.2006.10.011).

Borrero, J.S., Prokopyev, O.A., Sauré, D., 2019. Sequential interdiction with incomplete information and learning. Operations Research 67 (1), 72–89. doi:[10.1287/opre.2018.1773](https://doi.org/10.1287/opre.2018.1773).

Böttger, T., Grimm, V., Kleinert, T., Schmidt, M., 2021. The cost of decoupling trade and transport in the European entry-exit gas market. European Journal of Operational Research doi:[10.1016/j.ejor.2021.06.034](https://doi.org/10.1016/j.ejor.2021.06.034).

Bouhtou, M., van Hoesel, S., van der Kraaij, A.F., Lutton, J.-L., 2007. Tariff op- timization in networks. INFORMS Journal on Computing 19 (3), 458–469. doi:[10.1287/ijoc.1060.0177](https://doi.org/10.1287/ijoc.1060.0177).

Bracken, J., McGill, J.T., 1973. Mathematical programs with optimization problems in the constraints. Operations Research 21 (1), 37–44. doi:[10.1287/opre.21.1.37](https://doi.org/10.1287/opre.21.1.37).

Brotcorne, L., Cirinei, F., Marcotte, P., Savard, G., 2011. An exact algorithm for the network pricing problem. Discrete Optimization 8 (2), 246–258. doi:[10.1016/j.disopt.2010.09.003](https://doi.org/10.1016/j.disopt.2010.09.003).

Brotcorne, L., Labbé, M., Marcotte, P., Savard, G., 2000. A bilevel model and solution algorithm for a freight tariff-setting problem. Transportation Science 34 (3), 289–302. doi:[10.1287/trsc.34.3.289.12299](https://doi.org/10.1287/trsc.34.3.289.12299).

Brotcorne, L., Labbé, M., Marcotte, P., Savard, G., 2001. A bilevel model for toll opti- mization on a multicommodity transportation network. Transportation Science 35 (4), 345–358. doi:[10.1287/trsc.35.4.345.10433](https://doi.org/10.1287/trsc.35.4.345.10433).

Brotcorne, L., Labbé, M., Marcotte, P., Savard, G., 2008. Joint design and pricing on a network. Operations Research 56 (5), 1104–1115. doi:[10.1287/opre.1080.0617](https://doi.org/10.1287/opre.1080.0617).

Brown, G., Carlyle, M., Salmerón, J., Wood, R.K., 2006. Defending critical in- frastructure. INFORMS Journal on Applied Analytics 36 (6), 530–544. doi:[10.1287/inte.1060.0252](https://doi.org/10.1287/inte.1060.0252).

Bucarey, V., Casorrán, C., Labbé, M., Ordoñez, F., Figueroa, O., 2019. Coordinating re- sources in Stackelberg security games. European Journal of Operational Research doi:[10.1016/j.ejor.2019.11.002](https://doi.org/10.1016/j.ejor.2019.11.002).

Burtscheidt, J., Claus, M., 2020. Bilevel Linear Optimization Under Uncertainty. Springer International Publishing, pp. 485–511. doi:[10.1007/978-3-030-52119-6\_17](https://doi.org/10.1007/978-3-030-52119-6_17).

Burtscheidt, J., Claus, M., Dempe, S., 2020. Risk-averse models in bilevel stochas- tic linear programming. SIAM Journal on Optimization 30 (1), 377–406. doi:[10.1137/19M1242240](https://doi.org/10.1137/19M1242240).

Bylling, H.C., Boomsma, T.K., Gabriel, S.A., 2020. A Parametric Programming Ap- proach to Bilevel Merchant Electricity Transmission Investment Problems. In: Hesamzadeh, M.R., Rosellón, J., Vogelsang, I. (Eds.), Transmission Network Invest- ment in Liberalized Power Markets. Lecture Notes in Energy, vol 79. Springer, Cham,

pp. 237–254. doi:[10.1007/978-3-030-47929-9\_8](https://doi.org/10.1007/978-3-030-47929-9_8).

Calvete, H.I., Galé, C., 1999. The bilevel linear/linear fractional programming problem. European Journal of Operational Research 114 (1), 188–197. doi:[10.1016/S0377-2217(98)00078-2](https://doi.org/10.1016/S0377-2217(98)00078-2).

Calvete, H.I., Galé, C., 2004. Solving linear fractional bilevel programs. Operations Re- search Letters 32 (2), 143–151. doi:[10.1016/j.orl.2003.07.003](https://doi.org/10.1016/j.orl.2003.07.003).

Calvete, H.I., Galé, C., 2020. Algorithms for Linear Bilevel Optimization. Springer Inter- national Publishing, pp. 293–312. doi:[10.1007/978-3-030-52119-6\_10](https://doi.org/10.1007/978-3-030-52119-6_10).

Campelo, M., Dantas, S., Scheimberg, S., 2000. A note on a penalty function approach for solving bilevel linear programs. Journal of Global Optimization 16 (3), 245–255. doi:10.1023/A:1008308218364.

Candler, W., Fortuny-Amat, J., McCarl, B., 1981. The potential role of multilevel program- ming in agricultural economics. American Journal of Agricultural Economics 63 (3), 521–531. doi:[10.2307/1240543](https://doi.org/10.2307/1240543).

[Candler,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0064) [W., Norton, R., 1977. Multi-level Programming. Discussion Papers, Development Research Center, International Bank for Reconstruction and Development. World Bank.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0064)

Caprara, A., Carvalho, M., Lodi, A., Woeginger, G.J., 2014. A study on the computational complexity of the bilevel knapsack problem. SIAM Journal on Optimization 24 (2), 823–838. doi:[10.1137/130906593](https://doi.org/10.1137/130906593).

Caprara, A., Carvalho, M., Lodi, A., Woeginger, G.J., 2016. Bilevel knapsack with interdiction constraints. INFORMS Journal on Computing 28 (2), 319–333. doi:[10.1287/ijoc.2015.0676](https://doi.org/10.1287/ijoc.2015.0676).

Caramia, M., Mari, R., 2015. Enhanced exact algorithms for discrete bilevel linear prob- lems. Optimization Letters 9 (7), 1447–1468. doi:[10.1007/s11590-015-0872-9](https://doi.org/10.1007/s11590-015-0872-9).

Cardinal, J., Demaine, E.D., Fiorini, S., Joret, G., Langerman, S., Newman, I., Weimann, O., 2011. The Stackelberg minimum spanning tree game. Algorithmica 59 (2), 129–144. doi:[10.1007/s00453-009-9299-y](https://doi.org/10.1007/s00453-009-9299-y).

Carvalho, M., 2016. Computation of equilibria on integer programming games. Faculdade de Ciências da Universidade do Porto. PhD thesis. URL: <https://repositorio-aberto.up.pt/bitstream/10216/83362/2/126961.pdf>

Casorrán, C., Fortz, B., Labbé, M., Ordóñez, F., 2019. A study of general and security Stackelberg game formulations. European Journal of Operational Research 278 (3), 855–868. doi:[10.1016/j.ejor.2019.05.012](https://doi.org/10.1016/j.ejor.2019.05.012).

Castelli, L., Labbé, M., Violin, A., 2013. A network pricing formulation for the revenue maximization of european air navigation service providers. Transportation Research Part C: Emerging Technologies 33, 214–226. doi:[10.1016/j.trc.2012.04.013](https://doi.org/10.1016/j.trc.2012.04.013).

Castelli, L., Labbé, M., Violin, A., 2017. Network pricing problem with unit toll. Networks 69 (1), 83–93. doi:[10.1002/net.21701](https://doi.org/10.1002/net.21701).

Cerulli, M., D’Ambrosio, C., Liberti, L., 2019. Flying safely by bilevel programming. In: Advances in Optimization and Decision Science for Society, Services and Enterprises. Springer, pp. 197–206. doi:[10.1007/978-3-030-34960-8\_18](https://doi.org/10.1007/978-3-030-34960-8_18).

Cerulli, M., d’Ambrosio, C., Liberti, L., Pelegrín, M., 2020. Detecting and solv- ing aircraft conflicts using bilevel programming. Technical report. URL: <https://hal.archives-ouvertes.fr/hal-02869699>

Clark, F.E., 1961. Remark on the constraint sets in linear programming. The American Mathematical Monthly 68 (4), 351–352. doi:[10.2307/2311583](https://doi.org/10.2307/2311583).

Clark, P.A., 1990. Bilevel programming for steady-state chemical process design-ii. Per- formance study for nondegenerate problems. Computers & Chemical Engineering 14 (1), 99–109. doi:[10.1016/0098-1354(90)87008-D](https://doi.org/10.1016/0098-1354(90)87008-D).

Clark, P.A., Westerberg, A.W., 1983. Optimization for design problems having more than one objective. Computers & Chemical Engineering 7 (4), 259–278. doi:[10.1016/0098-1354(83)80015-5](https://doi.org/10.1016/0098-1354(83)80015-5).

Clark, P.A., Westerberg, A.W., 1990. Bilevel programming for steady-state chemical pro- cess design-i. Fundamentals and algorithms. Computers & Chemical Engineering 14 (1), 87–97. doi:[10.1016/0098-1354(90)87007-C](https://doi.org/10.1016/0098-1354(90)87007-C).

Colson, B., Marcotte, P., Savard, G., 2005. Bilevel programming: A survey. 4OR 3 (2), 87–107. doi:[10.1007/s10288-005-0071-0](https://doi.org/10.1007/s10288-005-0071-0).

[Colson,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0080) [B., Marcotte, P., Savard, G., 2007. An overview of bilevel optimization. Annals of Operations Research 153 (1), 235–256.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0080)

Conforti, M., Cornuejols, G., Zambelli, G., 2014. Integer Programming. Graduate Texts in Mathematics, 271. Springer doi:[10.1007/978-3-319-11008-0](https://doi.org/10.1007/978-3-319-11008-0).

Conitzer, V., Sandholm, T., 2006. Computing the optimal strategy to commit to. In: Proceedings of the 7th ACM conference on Electronic commerce, pp. 82–90. doi:[10.1145/1134707.1134717](https://doi.org/10.1145/1134707.1134717).

Cormican, K.J., Morton, D.P., Wood, R.K., 1998. Stochastic network interdiction. Opera- tions Research 46 (2), 184–197. doi:[10.1287/opre.46.2.184](https://doi.org/10.1287/opre.46.2.184).

Costa, M.-C., de Werra, D., Picouleau, C., 2011. Minimum *𝑑*-blockers and *𝑑*-

transversals in graphs. Journal of Combinatorial Optimization 22 (4), 857–872. doi:[10.1007/s10878-010-9334-6](https://doi.org/10.1007/s10878-010-9334-6).

Dan, T., Lodi, A., Marcotte, P., 2020. Joint location and pricing within a user- optimized environment. EURO Journal on Computational Optimization 8 (1), 61–84. doi:[10.1007/s13675-019-00120-w](https://doi.org/10.1007/s13675-019-00120-w).

Dan, T., Marcotte, P., 2019. Competitive facility location with selfish users and queues.

Operations Research 67 (2), 479–497. doi:[10.1287/opre.2018.1781](https://doi.org/10.1287/opre.2018.1781).

Della Croce, F., Scatamacchia, R., 2019. Lower bounds and a new exact approach for the bilevel knapsack with interdiction constraints. In: Lodi, A., Nagarajan, V. (Eds.), Inte- ger Programming and Combinatorial Optimization. Springer International Publishing,

pp. 155–167. doi:[10.1007/978-3-030-17953-3\_12](https://doi.org/10.1007/978-3-030-17953-3_12).

Dempe, S., 1987. A simple algorithm for the linear bilevel programming problem. Opti- mization 18 (3), 373–385. doi:[10.1080/02331938708843247](https://doi.org/10.1080/02331938708843247).

Dempe, S., 2002. Foundations of Bilevel Programming. Springer. doi:[10.1007/b101970](https://doi.org/10.1007/b101970).

Dempe, S., 2019. Computing locally optimal solutions of the bilevel optimization problem using the KKT approach. In: Khachay, M., Kochetov, Y., Pardalos, P. (Eds.), Mathemat- ical Optimization Theory and Operations Research. Springer International Publishing,

pp. 147–157. doi:[10.1007/978-3-030-22629-9\_11](https://doi.org/10.1007/978-3-030-22629-9_11).

Dempe, S., 2020. Bilevel Optimization: Theory, Algorithms, Applications and a Bibliography. Springer International Publishing, pp. 581–672. doi:[10.1007/978-3-030-52119-6\_20](https://doi.org/10.1007/978-3-030-52119-6_20).

Dempe, S., Dutta, J., 2012. Is bilevel programming a special case of a mathematical pro- gram with complementarity constraints? Mathematical Programming 131 (1-2), 37–

48. doi:[10.1007/s10107-010-0342-1](https://doi.org/10.1007/s10107-010-0342-1).

Dempe, S., Franke, S., 2019. Solution of bilevel optimization problems using the KKT approach. Optimization 68 (8), 1471–1489. doi:[10.1080/02331934.2019.1581192](https://doi.org/10.1080/02331934.2019.1581192).

Dempe, S., Ivanov, S., Naumov, A., 2017. Reduction of the bilevel stochastic optimiza- tion problem with quantile objective function to a mixed-integer problem. Applied Stochastic Models in Business and Industry 33 (5), 544–554. doi:[10.1002/asmb.2254](https://doi.org/10.1002/asmb.2254). Dempe, S., Kalashnikov, V., Pérez-Valdés, G.A., Kalashnykova, N., 2015. Bilevel Program-

ming Problems. Springer. doi:[10.1007/978-3-662-45827-3](https://doi.org/10.1007/978-3-662-45827-3).

Dempe, S., Kue, F.M., 2017. Solving discrete linear bilevel optimization problems using the optimal value reformulation. Journal of Global Optimization 68 (2), 255–277. doi:[10.1007/s10898-016-0478-5](https://doi.org/10.1007/s10898-016-0478-5).

Dempe, S., Mordukhovich, B.S., Zemkoho, A.B., 2019. Two-level value function approach to non-smooth optimistic and pessimistic bilevel programs. Optimization 68 (2-3), 433–455. doi:[10.1080/02331934.2018.1543294](https://doi.org/10.1080/02331934.2018.1543294).

DeNegre, S.T., 2011. Interdiction and discrete bilevel linear programming. [Lehigh University. PhD thesis. URL: http://coral.ie.lehigh.edu/~ted/files/ papers/ScottDeNegreDissertation11.pdf](http://coral.ie.lehigh.edu/~ted/files/papers/ScottDeNegreDissertation11.pdf)

DeNegre, S.T., Ralphs, T.K., 2009. A branch-and-cut algorithm for integer bilevel lin- ear programs. In: Operations research and cyber-infrastructure. Springer, pp. 65–78. doi:[10.1007/978-0-387-88843-9\_4](https://doi.org/10.1007/978-0-387-88843-9_4).

Deng, X., 1998. Complexity Issues in Bilevel Linear Programming. Springer US, pp. 149–

164. doi:[10.1007/978-1-4613-0307-7\_6](https://doi.org/10.1007/978-1-4613-0307-7_6).

Dewez, S., Labbé, M., Marcotte, P., Savard, G., 2008. New formulations and valid in- equalities for a bilevel pricing problem. Operations Research Letters 36 (2), 141–149. doi:[10.1016/j.orl.2007.03.005](https://doi.org/10.1016/j.orl.2007.03.005).

Di Summa, M., Grosso, A., Locatelli, M., 2012. Branch and cut algorithms for detecting critical nodes in undirected graphs. Computational Optimization and Applications 53 (3), 649–680. doi:[10.1007/s10589-012-9458-y](https://doi.org/10.1007/s10589-012-9458-y).

Didi-Biha, M., Marcotte, P., Savard, G., 2006. Path-based formulations of a bilevel toll setting problem. In: Optimization with Multivalued Mappings. Springer, pp. 29–50. doi:[10.1007/0-387-34221-4\_2](https://doi.org/10.1007/0-387-34221-4_2).

Dinitz, M., Gupta, A., 2013. Packing interdiction and partial covering prob- lems. In: Proceedings of the 16th International Conference on Integer Pro- gramming and Combinatorial Optimization. Springer-Verlag, pp. 157–168. doi:[10.1007/978-3-642-36694-9\_14](https://doi.org/10.1007/978-3-642-36694-9_14).

Dobson, G., Kalish, S., 1988. Positioning and pricing a product line. Marketing Science 7 (2), 107–125. doi:[10.1287/mksc.7.2.107](https://doi.org/10.1287/mksc.7.2.107).

Dobson, G., Kalish, S., 1993. Heuristics for pricing and positioning a product- line using conjoint and cost data. Management Science 39 (2), 160–175. doi:[10.1287/mnsc.39.2.160](https://doi.org/10.1287/mnsc.39.2.160).

Duran, M.A., Grossmann, I.E., 1986. An outer-approximation algorithm for a class of mixed-integer nonlinear programs. Mathematical Programming 36 (3), 307–339. doi:[10.1007/BF02592064](https://doi.org/10.1007/BF02592064).

Edmunds, T.A., Bard, J.F., 1991. Algorithms for nonlinear bilevel mathematical pro- grams. IEEE Transactions on Systems, Man, and Cybernetics 21 (1), 83–89. doi:[10.1109/21.101139](https://doi.org/10.1109/21.101139).

Faísca, N.P., Dua, V., Rustem, B., Saraiva, P.M., Pistikopoulos, E.N., 2007. Parametric global optimisation for bilevel programming. Journal of Global Optimization 38 (4), 609–623. doi:[10.1007/s10898-006-9100-6](https://doi.org/10.1007/s10898-006-9100-6).

Fampa, M., Barroso, L., Candal, D., Simonetti, L., 2008. Bilevel optimization applied to strategic pricing in competitive electricity markets. Computational Optimization and Applications 39 (2), 121–142. doi:[10.1007/s10589-007-9066-4](https://doi.org/10.1007/s10589-007-9066-4).

Fanghänel, D., Dempe, S., 2009. Bilevel programming with discrete lower level problems.

Optimization 58 (8), 1029–1047. doi:[10.1080/02331930701763389](https://doi.org/10.1080/02331930701763389).

Fernandes, C.G., Ferreira, C.E., Franco, A.J., Schouery, R.C., 2016. The envy-free pricing problem, unit-demand markets and connections with the network pricing problem. Discrete Optimization 22, 141–161. doi:[10.1016/j.disopt.2015.09.003](https://doi.org/10.1016/j.disopt.2015.09.003).

Fioretto, F., Mak, T.W., Van Hentenryck, P., 2019-07. Privacy-preserving obfuscation of critical infrastructure networks. In: Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI-19. International Joint Conferences on Artificial Intelligence Organization, pp. 1086–1092. doi:[10.24963/ijcai.2019/152](https://doi.org/10.24963/ijcai.2019/152).

Fischetti, M., Ljubić, I., Monaci, M., Sinnl, M., 2017a. Instances and solver soft- ware for mixed-integer bilevel linear problems. Last accessed 2020/12/21, URL: <https://msinnl.github.io/pages/bilevel.html>.

Fischetti, M., Ljubić, I., Monaci, M., Sinnl, M., 2017. A new general-purpose algorithm for mixed-integer bilevel linear programs. Operations Research 65 (6), 1615–1637. doi:[10.1287/opre.2017.1650](https://doi.org/10.1287/opre.2017.1650).

Fischetti, M., Ljubić, I., Monaci, M., Sinnl, M., 2018. On the use of intersection cuts for bilevel optimization. Mathematical Programming 172 (1-2), 77–103. doi:[10.1007/s10107-017-1189-5](https://doi.org/10.1007/s10107-017-1189-5).

Fischetti, M., Ljubić, I., Monaci, M., Sinnl, M., 2019. Interdiction games and monotonicity, with application to knapsack problems. INFORMS Journal on Computing 31 (2), 390–

410. doi:[10.1287/ijoc.2018.0831](https://doi.org/10.1287/ijoc.2018.0831).

Fletcher, R., Leyffer, S., 1994. Solving mixed integer nonlinear programs by outer approx- imation. Mathematical Programming 66 (1), 327–349. doi:[10.1007/BF01581153](https://doi.org/10.1007/BF01581153).

Fliege, J., Tin, A., Zemkoho, A., 2020. Gauss-newton-type methods for bilevel optimiza- tion. Technical report. [arXiv:2003.03128](http://arxiv.org/abs/2003.03128).

Fontaine, P., Minner, S., 2014. Benders decomposition for discrete-continuous linear bilevel problems with application to traﬃc network design. Transportation Research Part B: Methodological 70, 163–172. doi:[10.1016/j.trb.2014.09.007](https://doi.org/10.1016/j.trb.2014.09.007).

Fortuny-Amat, J., McCarl, B., 1981. A representation and economic interpretation of a two-level programming problem. The Journal of the Operational Research Society 32 (9), 783–792. doi:[10.1057/jors.1981.156](https://doi.org/10.1057/jors.1981.156).

[Franceschi, L., Frasconi, P., Salzo, S., Grazzi, R., Pontil, M., 2018. Bilevel program- ming for hyperparameter optimization and meta-learning. In: Dy, J., Krause, A. (Eds.), Proceedings of the 35th International Conference on Machine Learning. PMLR,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0122)

[pp. 1568–1577.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0122)

Fulkerson, D.R., Harding, G.C., 1977. Maximizing the minimum source-sink path subject to a budget constraint. Mathematical Programming 13 (1), 116–118. doi:[10.1007/BF01584329](https://doi.org/10.1007/BF01584329).

for the *𝑘*-vertex cut problem. Mathematical Programming Computation 12, 133–164. Furini, F., Ljubić, I., Malaguti, E., Paronuzzi, P., 2020. On integer and bilevel formulations

doi:[10.1007/s12532-019-00167-1](https://doi.org/10.1007/s12532-019-00167-1).

Furini, F., Ljubić, I., Malaguti, E., Paronuzzi, P., 2021. Casting light on the hidden bilevel combinatorial structure of the capacitated vertex separator problem. Operations Re- search doi:[10.1287/opre.2021.2110](https://doi.org/10.1287/opre.2021.2110).

Furini, F., Ljubić, I., San Segundo, P., Martin, S., 2019. The maximum clique in- terdiction problem. European Journal of Operational Research 277 (1), 112–127. doi:[10.1016/j.ejor.2019.02.028](https://doi.org/10.1016/j.ejor.2019.02.028).

Furini, F., Ljubić, I., San Segundo, P., Zhao, Y., 2020. A branch-and-cut algorithm for the edge interdiction clique problem. European Journal of Operational Research 294 (1), 54–69. doi:[10.1016/j.ejor.2021.01.030](https://doi.org/10.1016/j.ejor.2021.01.030).

[Gabriel,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0128) [S.A., Conejo, A.J., Fuller, J.D., Hobbs, B.F., Ruiz, C., 2012. Complementarity mod- eling in energy markets, 180. Springer Science & Business Media.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0128)

Gairing, M., Harks, T., Klimm, M., 2017. Complexity and approximation of the contin- uous network design problem. SIAM Journal on Optimization 27 (3), 1554–1582. doi:[10.1137/15M1016461](https://doi.org/10.1137/15M1016461).

Garcés, L.P., Conejo, A.J., García-Bertrand, R., Romero, R., 2009. A bilevel approach to transmission expansion planning within a market environment. IEEE Transactions on Power Systems 24 (3), 1513–1522. doi:[10.1109/TPWRS.2009.2021230](https://doi.org/10.1109/TPWRS.2009.2021230).

Garcia-Herreros, P., Zhang, L., Misra, P., Arslan, E., Mehta, S., Grossmann, I.E., 2016. Mixed-integer bilevel optimization for capacity planning with rational markets. Com- puters & Chemical Engineering 86, 33–47. doi:[10.1016/j.compchemeng.2015.12.007](https://doi.org/10.1016/j.compchemeng.2015.12.007). Geoffrion, A.M., 1972. Generalized Benders decomposition. Journal of Optimization The-

ory and Applications 10 (4), 237–260. doi:[10.1007/BF00934810](https://doi.org/10.1007/BF00934810).

Golden, B., 1978. A problem in network interdiction. Naval Research Logistics Quarterly 25 (4), 711–713. doi:[10.1002/nav.3800250412](https://doi.org/10.1002/nav.3800250412).

González-Díaz, J., González-Rodríguez, B., Leal, M., Puerto, J., 2020. Global optimization for bilevel portfolio design: Economic insights from the Dow Jones index. Omega 102353. doi:[10.1016/j.omega.2020.102353](https://doi.org/10.1016/j.omega.2020.102353).

Grimm, V., Kleinert, T., Liers, F., Schmidt, M., Zöttl, G., 2019. Optimal price zones of electricity markets: a mixed-integer multilevel model and global solution approaches. Optimization Methods and Software 34 (2), 406–436. doi:[10.1080/10556788.2017.1401069](https://doi.org/10.1080/10556788.2017.1401069).

Grimm, V., Martin, A., Schmidt, M., Weibelzahl, M., Zöttl, G., 2016. Transmission and generation investment in electricity markets: The effects of market splitting and network fee regimes. European Journal of Operational Research 254 (2), 493–509. doi:[10.1016/j.ejor.2016.03.044](https://doi.org/10.1016/j.ejor.2016.03.044).

Grimm, V., Orlinskaya, G., Schewe, L., Schmidt, M., Zöttl, G., 2020. Optimal de- sign of retailer-prosumer electricity tariffs using bilevel optimization. Omega doi:[10.1016/j.omega.2020.102327](https://doi.org/10.1016/j.omega.2020.102327).

Grimm, V., Schewe, L., Schmidt, M., Zöttl, G., 2019. A multilevel model of the European entry-exit gas market. Mathematical Methods of Operations Research 89 (2), 223–255. doi:[10.1007/s00186-018-0647-z](https://doi.org/10.1007/s00186-018-0647-z).

Gümüş, Z.H., Ciric, A.R., 1997. Reactive distillation column design with va- por/liquid/liquid equilibria. Computers & Chemical Engineering 21, S983–S988. doi:[10.1016/S0098-1354(97)87630-2](https://doi.org/10.1016/S0098-1354(97)87630-2).

[Guruswami,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0140) [V., Hartline, J.D., Karlin, A.R., Kempe, D., Kenyon, C., McSherry, F., 2005.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0140)

[On profit-maximizing envy-free pricing. In: SODA, 5, pp. 1164–1173.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0140)

Hansen, P., Jaumard, B., Savard, G., 1992. New branch-and-bound rules for linear bilevel programming. SIAM Journal on scientific and Statistical Computing 13 (5), 1194– 1217. doi:[10.1137/0913069](https://doi.org/10.1137/0913069).

Harsanyi, J.C., Selten, R., 1972. A generalized Nash solution for two-person bargain- ing games with incomplete information. Management Science 18 (5-part-2), 80–106. doi:[10.1287/mnsc.18.5.80](https://doi.org/10.1287/mnsc.18.5.80).

Heilporn, G., Labbé, M., Marcotte, P., Savard, G., 2010. A parallel between two classes of pricing problems in transportation and marketing. Journal of Revenue and Pricing Management 9 (1-2), 110–125. doi:[10.1057/rpm.2009.39](https://doi.org/10.1057/rpm.2009.39).

Heilporn, G., Labbé, M., Marcotte, P., Savard, G., 2010. A polyhedral study of the network pricing problem with connected toll arcs. Networks 55 (3), 234–246. doi:[10.1002/net.20368](https://doi.org/10.1002/net.20368).

Heilporn, G., Labbé, M., Marcotte, P., Savard, G., 2011. Valid inequalities and branch- and-cut for the clique pricing problem. Discrete Optimization 8 (3), 393–410. doi:[10.1016/j.disopt.2011.01.001](https://doi.org/10.1016/j.disopt.2011.01.001).

Hoheisel, T., Kanzow, C., Schwartz, A., 2013. Theoretical and numerical comparison of relaxation methods for mathematical programs with complementarity constraints. Mathematical Programming 137 (1), 257–288. doi:[10.1007/s10107-011-0488-5](https://doi.org/10.1007/s10107-011-0488-5).

Horst, R., Tuy, H., 2013. Global optimization: Deterministic approaches. Springer Science & Business Media doi:[10.1007/978-3-662-03199-5](https://doi.org/10.1007/978-3-662-03199-5).

[Israeli, E., 1999. System interdiction and defense. Naval Postgraduate School. PhD thesis](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0148). Israeli, E., Wood, R.K., 2002. Shortest-path network interdiction. Networks 40 (2), 97–

111. doi:[10.1002/net.10039](https://doi.org/10.1002/net.10039).

Ivanov, S.V., 2018. A bilevel stochastic programming problem with random parameters in the follower’s objective function. Journal of Applied and Industrial Mathematics 12 (4), 658–667. doi:[10.1134/S1990478918040063](https://doi.org/10.1134/S1990478918040063).

Jain, M., An, B., Tambe, M., 2013. Security games applied to real-world: Research contributions and challenges. In: Moving Target Defense II. Springer, pp. 15–39. doi:[10.1007/978-1-4614-5416-8\_2](https://doi.org/10.1007/978-1-4614-5416-8_2).

[Jain,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0152) [M., Kardes, E., Kiekintveld, C., Ordónez, F., Tambe, M., 2010. Security games with arbitrary schedules: A branch and price approach. In: AAAI.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0152)

[Jain, M., Ordonez, F., Pita, J., Portway, C., Tambe, M., Western, C., Paruchuri, P., Kraus, S., 2008. Robust solutions in Stackelberg games: Addressing boundedly rational human](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0153)

[preference models. In: Association for the Advancement of Artificial Intelligence 4th Multidiciplinary Workshop on Advances in Preference Handling.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0153)

Janjarassuk, U., Linderoth, J.T., 2008. Reformulation and sampling to solve a stochastic network interdiction problem. Networks 52, 120–132. doi:[10.1002/net.20237](https://doi.org/10.1002/net.20237).

Jenabi, M., Fatemi Ghomi, S.M.T., Smeers, Y., 2013. Bi-level game approaches for coor- dination of generation and transmission expansion planning within a market envi- [ronment. IEEE Transactions on Power Systems 28 (3), 2639–2650. doi:10.1109/TP- WRS.2012.2236110.](https://doi.org/10.1109/TPWRS.2012.2236110)

Jeroslow, R.G., 1985. The polynomial hierarchy and a simple model for competitive anal- ysis. Mathematical Programming 32 (2), 146–164. doi:[10.1007/BF01586088](https://doi.org/10.1007/BF01586088).

Jin, S., Ryan, S.M., 2011. Capacity expansion in the integrated supply network for an electricity market. IEEE Transactions on Power Systems 26 (4), 2275–2284. doi:[10.1109/TPWRS.2011.2107531](https://doi.org/10.1109/TPWRS.2011.2107531).

Joret, G., 2011. Stackelberg network pricing is hard to approximate. Networks 57 (2), 117–120. doi:[10.1002/net.20391](https://doi.org/10.1002/net.20391).

Jünger, M., Liebling, T.M., Naddef, D., Nemhauser, G.L., Pulleyblank, W.R., Reinelt, G., Rinaldi, G., Wolsey, L.A. (Eds.), 2010. 50 Years of integer programming 1958-2008: From the early years to the state-of-the-art. Springer doi:[10.1007/978-3-540-68279-0](https://doi.org/10.1007/978-3-540-68279-0). Kelley Jr, J.E., 1960. The cutting-plane method for solving convex programs. Jour- nal of the Society for Industrial and Applied Mathematics 8 (4), 703–712.

doi:[10.1137/0108053](https://doi.org/10.1137/0108053).

[Kiekintveld, C., Jain, M., Tsai, J., Pita, J., Ordóñez, F., Tambe, M., 2009. Computing op- timal randomized resource allocations for massive security games. In: Proceedings of The 8th International Conference on Autonomous Agents and Multiagent System- s-Volume 1, pp. 689–696.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0161)

[Kleinert, T., Grimm, V., Schmidt, M., 2021. Outer approximation for global optimization of mixed-integer quadratic bilevel problems. Mathematical Programming 188, 461–521.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0162) Kleinert, T., Labbé, M., Plein, F., Schmidt, M., 2020b. Closing the gap in linear bilevel op- timization: A new valid primal-dual inequality. Optimization Letters 15, 1027–1040.

doi:[10.1007/s11590-020-01660-6](https://doi.org/10.1007/s11590-020-01660-6).

Kleinert, T., Labbé, M., Plein, F., Schmidt, M., 2020. There’s no free lunch: On the hardness of choosing a correct big-*M* in bilevel optimization. Operations Research 68 (6), 1716– 1721. doi:[10.1287/opre.2019.1944](https://doi.org/10.1287/opre.2019.1944).

Kleinert, T., Schmidt, M., 2019. Computing stationary points of bilevel problems with a penalty alternating direction method. INFORMS Journal on Computing 33 (1), 198–

215. doi:[10.1287/ijoc.2019.0945](https://doi.org/10.1287/ijoc.2019.0945).

Kleinert, T., Schmidt, M., 2019. Global optimization of multilevel electricity market mod- els including network design and graph partitioning. Discrete Optimization 33, 43–69. doi:[10.1016/j.disopt.2019.02.002](https://doi.org/10.1016/j.disopt.2019.02.002).

Kleinert, T., Schmidt, M., 2020. Why there is no need to use a big-*𝑀* in linear bilevel op-

timization: A computational study of two ready-to-use approaches. Technical report. URL: <http://www.optimization-online.org/DB_HTML/2020/10/8065.html>

Kleniati, P.-M., Adjiman, C.S., 2014. Branch-and-sandwich: a deterministic global optimization algorithm for optimistic bilevel programming problems. Part I: Theoretical development. Journal of Global Optimization 60 (3), 425–458. doi:[10.1007/s10898-013-0121-7](https://doi.org/10.1007/s10898-013-0121-7).

Kleniati, P.-M., Adjiman, C.S., 2014. Branch-and-sandwich: a deterministic global opti- mization algorithm for optimistic bilevel programming problems. Part II: Conver- gence analysis and numerical results. Journal of Global Optimization 60 (3), 459–481. doi:[10.1007/s10898-013-0120-8](https://doi.org/10.1007/s10898-013-0120-8).

Kleniati, P.-M., Adjiman, C.S., 2015. A generalization of the branch-and-sandwich algo- rithm: From continuous to mixed-integer nonlinear bilevel problems. Computers & Chemical Engineering 72, 373–386. doi:[10.1016/j.compchemeng.2014.06.004](https://doi.org/10.1016/j.compchemeng.2014.06.004).

Klotz, E., 2017. Performance tuning for CPLEX’s spatial branch-and-bound solver for global nonconvex (mixed integer) quadratic programs. URL: <http://orwe-conference.mines.edu/files/IOS2018SpatialPerfTuning.pdf>

[Kolstad, C.D., 1985. A review of the literature on bi-level mathematical programming.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0172)

[Technical report. Los Alamos National Laboratory Los Alamos, NM.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0172)

Köppe, M., Queyranne, M., Ryan, C.T., 2010. Parametric integer programming algorithm for bilevel mixed integer programs. Journal of Optimization Theory and Applications 146 (1), 137–150. doi:[10.1007/s10957-010-9668-3](https://doi.org/10.1007/s10957-010-9668-3).

[Korzhyk, D., Conitzer, V., Parr, R., 2010. Complexity of computing optimal Stackelberg strategies in security resource allocation games. In: Twenty-Fourth AAAI Conference on Artificial Intelligence.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0174)

Korzhyk, D., Yin, Z., Kiekintveld, C., Conitzer, V., Tambe, M., 2011. Stackelberg vs. Nash in security games: An extended investigation of interchangeability, equiv- alence, and uniqueness. Journal of Artificial Intelligence Research 41, 297–327. doi:[10.1613/jair.3269](https://doi.org/10.1613/jair.3269).

Labbé, M., Marcotte, P., Savard, G., 1998. A bilevel model of taxation and its applica- tion to optimal highway pricing. Management Science 44 (12-part-1), 1608–1622. doi:[10.1287/mnsc.44.12.1608](https://doi.org/10.1287/mnsc.44.12.1608).

Labbé, M., Pozo, M., Puerto, J., 2021. Computational comparisons of different formula- tions for the Stackelberg minimum spanning tree game. International Transactions in Operational Research 28 (1), 48–69. doi:[10.1111/itor.12680](https://doi.org/10.1111/itor.12680).

Labbé, M., Violin, A., 2013. Bilevel programming and price setting problems. 4OR 11 (1), 1–30. doi:[10.1007/s10288-012-0213-0](https://doi.org/10.1007/s10288-012-0213-0).

Lagos, F., Ordóñez, F., Labbé, M., 2017. A branch and price algorithm for a Stackelberg security game. Computers & Industrial Engineering 111, 216–227. doi:[10.1016/j.cie.2017.06.034](https://doi.org/10.1016/j.cie.2017.06.034).

Lalou, M., Tahraoui, M.A., Kheddouci, H., 2018. The critical node detection problem in networks: A survey. Computer Science Review 28, 92–117. doi:[10.1016/j.cosrev.2018.02.002](https://doi.org/10.1016/j.cosrev.2018.02.002).

[Land, A.H., Doig, A.G., 1960. An automatic method of solving discrete programming prob- lems. Econometrica 28 (3), 497–520.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0181)

Leal, M., Ponce, D., Puerto, J., 2020. Portfolio problems with two levels decision-makers:

Optimal portfolio selection with pricing decisions on transaction costs. European Jour- nal of Operational Research 284 (2), 712–727. doi:[10.1016/j.ejor.2019.12.039](https://doi.org/10.1016/j.ejor.2019.12.039).

LeBlanc, L.J., Boyce, D.E., 1986. A bilevel programming algorithm for exact solution of the network design problem with user-optimal flows. Transportation Research Part B: Methodological 20 (3), 259–265. doi:[10.1016/0191-2615(86)90021-4](https://doi.org/10.1016/0191-2615(86)90021-4).

Lee, J., Leyffer, S. (Eds.). 2012. Mixed integer nonlinear programming. The IMA Volumes in Mathematics and its Applications, 154. Springer New York. doi:[10.1007/978-1-4614-1927-3](https://doi.org/10.1007/978-1-4614-1927-3).

[Letchford,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0185) [J., Conitzer, V., 2013. Solving security games on graphs via marginal probabil- ities. In: AAAI.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0185)

Lim, C., Smith, J.C., 2007. Algorithms for discrete and continuous multicom- modity flow network interdiction problems. IIE Transactions 39 (1), 15–26. doi:[10.1080/07408170600729192](https://doi.org/10.1080/07408170600729192).

Lin, K.-C., Chern, M.-S., 1993. The most vital edges in the minimum spanning tree problem. Information Processing Letters 45 (1), 25–31. doi:[10.1016/0020-0190(93)90247-7](https://doi.org/10.1016/0020-0190(93)90247-7).

Liu, J., Fan, Y., Chen, Z., Zheng, Y., 2018. Pessimistic bilevel optimization: A survey. Inter- [national Journal of Computational Intelligence Systems 11, 725–736. doi:10.2991/ij- cis.11.1.56.](https://doi.org/10.2991/ijcis.11.1.56)

Liu, J., Fan, Y., Chen, Z., Zheng, Y., 2020. Methods for Pessimistic Bilevel Optimization.

Springer, Cham, pp. 403–420. doi:[10.1007/978-3-030-52119-6\_14](https://doi.org/10.1007/978-3-030-52119-6_14).

Liu, S., Wang, M., Kong, N., Hu, X., 2020. An enhanced branch-and-bound algorithm for bilevel integer linear programming. European Journal of Operational Research doi:[10.1016/j.ejor.2020.10.002](https://doi.org/10.1016/j.ejor.2020.10.002). Online first

Lodi, A., Ralphs, T.K., Woeginger, G.J., 2014. Bilevel programming and the separation problem. Mathematical Programming 146 (1), 437–458. doi:[10.1007/s10107-013-0700-x](https://doi.org/10.1007/s10107-013-0700-x).

Lozano, L., Smith, J.C., 2017. A backward sampling framework for interdiction problems with fortification. INFORMS Journal on Computing 29 (1), 123–139. doi:[10.1287/ijoc.2016.0721](https://doi.org/10.1287/ijoc.2016.0721).

Lozano, L., Smith, J.C., 2017. A value-function-based exact approach for the bilevel mixed-integer programming problem. Operations Research 65 (3), 768–786. doi:[10.1287/opre.2017.1589](https://doi.org/10.1287/opre.2017.1589).

[Luo,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0194) [Z.-Q., Pang, J.-S., Ralph, D., 1996. Mathematical programs with equilibrium con- straints. Cambridge University Press.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0194)

Lv, Y., Hu, T., Wang, G., Wan, Z., 2007. A penalty function method based on Kuhn-Tucker condition for solving linear bilevel programming. Applied Mathematics and Compu- tation 188 (1), 808–813. doi:[10.1016/j.amc.2006.10.045](https://doi.org/10.1016/j.amc.2006.10.045).

Marcotte, P., 1986. Network design problem with congestion effects: A case of bilevel pro- gramming. Mathematical Programming 34 (2), 142–162. doi:[10.1007/BF01580580](https://doi.org/10.1007/BF01580580). Marcotte, P., Mercier, A., Savard, G., Verter, V., 2009. Toll policies for mitigat- ing hazardous materials transport risk. Transportation Science 43 (2), 228–243.

doi:[10.1287/trsc.1080.0236](https://doi.org/10.1287/trsc.1080.0236).

Marcotte, P., Savard, G., 1991. A note on the pareto optimality of solutions to the linear bilevel programming problem. Computers and Operations Research 18 (4), 355–359. doi:[10.1016/0305-0548(91)90096-A](https://doi.org/10.1016/0305-0548(91)90096-A).

McCormick, G.P., 1976. Computability of global solutions to factorable nonconvex pro- grams: Part i-convex underestimating problems. Mathematical Programming 10 (1), 147–175. doi:[10.1007/BF01580665](https://doi.org/10.1007/BF01580665).

McNaughton, R., 1959. Scheduling with deadlines and loss functions. Management Sci- ence 6 (1), 1–12. doi:[10.1287/mnsc.6.1.1](https://doi.org/10.1287/mnsc.6.1.1).

Mersha, A.G., Dempe, S., 2006. Linear bilevel programming with upper level constraints depending on the lower level solution. Applied Mathematics and Computation 180 (1), 247–254. doi:[10.1016/j.amc.2005.11.134](https://doi.org/10.1016/j.amc.2005.11.134).

Migdalas, A., 1995. Bilevel programming in traﬃc planning: Models, methods and chal- lenge. Journal of Global Optimization 7 (4), 381–405. doi:[10.1007/BF01099649](https://doi.org/10.1007/BF01099649).

Mitsos, A., 2010. Global solution of nonlinear mixed-integer bilevel programs. Journal of Global Optimization 47 (4), 557–582. doi:[10.1007/s10898-009-9479-y](https://doi.org/10.1007/s10898-009-9479-y).

Mitsos, A., Barton, P.I., 2006. Issues in the development of global optimization algo- rithms for bilevel programs with a nonconvex inner program. Technical report.

[URL: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.703.4195&rep=](http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.703.4195\04526rep%3Drep1\04526type%3Dpdf)

[rep1&type=pdf](http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.703.4195\04526rep%3Drep1\04526type%3Dpdf)

Mitsos, A., Lemonidis, P., Barton, P.I., 2008. Global solution of bilevel programs with a nonconvex inner program. Journal of Global Optimization 42 (4), 475–513. doi:[10.1007/s10898-007-9260-z](https://doi.org/10.1007/s10898-007-9260-z).

Moore, J.T., Bard, J.F., 1990. The mixed integer linear bilevel programming problem.

Operations Research 38 (5), 911–921. doi:[10.1287/opre.38.5.911](https://doi.org/10.1287/opre.38.5.911).

Morais, V., da Cunha, A.S., Mahey, P., 2016. A branch-and-cut-and-price algorithm for the Stackelberg minimum spanning tree game. Electronic Notes in Discrete Mathematics 52, 309–316. doi:[10.1016/j.endm.2016.03.041](https://doi.org/10.1016/j.endm.2016.03.041).

Morales, J.M., Pinson, P., Madsen, H., 2012. A transmission-cost-based model to estimate the amount of market-integrable wind resources. IEEE Transactions on Power Systems 27 (2), 1060–1069. doi:[10.1109/TPWRS.2011.2177281](https://doi.org/10.1109/TPWRS.2011.2177281).

Motto, A.L., Arroyo, J.M., Galiana, F.D., 2005. A mixed-integer LP procedure for the anal- ysis of electric grid security under disruptive threat. IEEE Transactions on Power Sys- tems 20 (3), 1357–1365. doi:[10.1109/TPWRS.2005.851942](https://doi.org/10.1109/TPWRS.2005.851942).

Myklebust, T.G., Sharpe, M., Tunçel, L., 2016. Eﬃcient heuristic algorithms for maxi- mum utility product pricing problems. Computers & Operations Research 69, 25–39. doi:[10.1016/j.cor.2015.11.013](https://doi.org/10.1016/j.cor.2015.11.013).

Nocedal, J., Wright, S.J., 2006. Numerical Optimization, 2nd Springer doi:[10.1007/978-0-387-40065-5](https://doi.org/10.1007/978-0-387-40065-5).

Pajouh, F.M., 2020. Minimum cost edge blocker clique problem. Annals of Operations Research 294 (1), 345–376. doi:[10.1007/s10479-019-03315-x](https://doi.org/10.1007/s10479-019-03315-x).

Pajouh, F.M., Boginski, V., Pasiliao, E.L., 2014. Minimum vertex blocker clique problem.

Networks 64 (1), 48–64. doi:[10.1002/net.21556](https://doi.org/10.1002/net.21556).

Pajouh, F.M., Walteros, J.L., Boginski, V., Pasiliao, E.L., 2015. Minimum edge blocker

dominating set problem. European Journal of Operational Research 247 (1), 16–26. doi:[10.1016/j.ejor.2015.05.037](https://doi.org/10.1016/j.ejor.2015.05.037).

Pandzic, H., Conejo, A.J., Kuzle, I., Caro, E., 2012. Yearly maintenance scheduling of transmission lines within a market environment. IEEE Transactions on Power Systems 27 (1), 407–415. doi:[10.1109/TPWRS.2011.2159743](https://doi.org/10.1109/TPWRS.2011.2159743).

[Paruchuri, P., Kraus, S., Pearce, J.P., Marecki, J., Tambe, M., Ordonez, F., 2008. Play- ing games for security: An eﬃcient exact algorithm for solving bayesian stackelberg games. In: Padgham, Parkes, Mueller, Parsons (Eds.), Proc. of AAMAS 2008. Citeseer.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0216) Paulavičius, R., Adjiman, C.S., 2020. New bounding schemes and algorithmic op- tions for the branch-and-sandwich algorithm. Journal of Global Optimization 1–29.

doi:[10.1007/s10898-020-00874-3](https://doi.org/10.1007/s10898-020-00874-3).

Paulavičius, R., Gao, J., Kleniati, P.-M., Adjiman, C., 2020. Basbl: Branch-and-sandwich bilevel solver. implementation and computational study with the basblib test set. Com- puters & Chemical Engineering 106609. doi:[10.1016/j.compchemeng.2019.106609](https://doi.org/10.1016/j.compchemeng.2019.106609).

Paulavičius, R., Adjiman, C.S., 2019. Basblib - a library of bilevel test problems (version v2.3). Zenodo. doi:[10.5281/zenodo.3266835](https://doi.org/10.5281/zenodo.3266835).

Pineda, S., Bylling, H., Morales, J., 2018. Eﬃciently solving linear bilevel programming problems using off-the-shelf optimization software. Optimization and Engineering 19 (1), 187–211. doi:[10.1007/s11081-017-9369-y](https://doi.org/10.1007/s11081-017-9369-y).

Pineda, S., Morales, J.M., 2019. Solving linear bilevel problems using big-*M*s: Not [all that glitters is gold. IEEE Transactions on Power Systems doi:10.1109/TP- WRS.2019.2892607.](https://doi.org/10.1109/TPWRS.2019.2892607)

[Pita, J.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0222), [Jain, M.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0222), [Marecki, J.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0222), [Ordóñez, F.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0222), [Portway, C.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0222), [Tambe, M.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0222), [Western, C.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0222), [Paruchuri, P., Kraus, S., 2008. Deployed armor protection: the application of a game theoretic model for security at the los angeles international airport. In: Proceedings of the 7th international joint conference on Autonomous agents and multiagent systems: industrial track, pp. 125–132.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0222)

Pita, J., Jain, M., Tambe, M., Ordóñez, F., Kraus, S., 2010. Robust solutions to Stackelberg games: Addressing bounded rationality and limited observations in human cognition. Artificial Intelligence 174 (15), 1142–1171. doi:[10.1016/j.artint.2010.07.002](https://doi.org/10.1016/j.artint.2010.07.002).

Poirion, P., Toubaline, S., D’Ambrosio, C., Liberti, L., 2020. Algorithms and appli- cations for a class of bilevel milps. Discrete Applied Mathematics 272, 75–89. doi:[10.1016/j.dam.2018.02.015](https://doi.org/10.1016/j.dam.2018.02.015).

Ralphs, T. K., 2018. Mibs. Last accessed 2020/12/21, URL: <https://msinnl.github.io/pages/bilevel.html>.

Ralphs, T. K., 2020. Cor@l: Bilevel optimization problem library. URL: <http://coral.ise.lehigh.edu/data-sets/bilevel-instances>

Reisi, M., Gabriel, S.A., Fahimnia, B., 2019. Supply chain competition on shelf space and pricing for soft drinks: A bilevel optimization approach. International Journal of Pro- duction Economics 211, 237–250. doi:[10.1016/j.ijpe.2018.12.018](https://doi.org/10.1016/j.ijpe.2018.12.018).

Roch, S., Savard, G., Marcotte, P., 2005. An approximation algorithm for Stackelberg net- work pricing. Networks 46 (1), 57–67. doi:[10.1002/net.20074](https://doi.org/10.1002/net.20074).

Ruiz, C., Conejo, A.J., 2009. Pool strategy of a producer with endogenous formation of locational marginal prices. IEEE Transactions on Power Systems 24 (4), 1855–1866. doi:[10.1109/TPWRS.2009.2030378](https://doi.org/10.1109/TPWRS.2009.2030378).

Ruiz, C., Conejo, A.J., Smeers, Y., 2012. Equilibria in an oligopolistic electricity pool with stepwise offer curves. IEEE Transactions on Power Systems 27 (2), 752–761. doi:[10.1109/TPWRS.2011.2170439](https://doi.org/10.1109/TPWRS.2011.2170439).

Rutenburg, V., 1994. Propositional truth maintenance systems: Classification and com- plexity analysis. Annals of Mathematics and Artificial Intelligence 10 (3), 207–231. doi:[10.1007/BF01530952](https://doi.org/10.1007/BF01530952).

Ryu, J.-H., Dua, V., Pistikopoulos, E.N., 2004. A bilevel programming framework for enterprise-wide process networks under uncertainty. Computers & Chemical Engineer- ing 28 (6), 1121–1129. doi:[10.1016/j.compchemeng.2003.09.021](https://doi.org/10.1016/j.compchemeng.2003.09.021). FOCAPO 2003

Special issue

Saharidis, G.K., Ierapetritou, M.G., 2009. Resolution method for mixed integer bi-level linear problems based on decomposition technique. Journal of Global Optimization 44 (1), 29–51. doi:[10.1007/s10898-008-9291-0](https://doi.org/10.1007/s10898-008-9291-0).

Salmeron, J., Wood, K., Baldick, R., 2009. Worst-case interdiction analysis of large- scale electric power grids. IEEE Transactions on Power Systems 24 (1), 96–104. doi:[10.1109/TPWRS.2008.2004825](https://doi.org/10.1109/TPWRS.2008.2004825).

Salmeron, J., Wood, R.K., 2015. The value of recovery transformers in protecting an elec- tric transmission grid against attack. IEEE Transactions on Power Systems 30 (5), 2396–2403. doi:[10.1109/TPWRS.2014.2360401](https://doi.org/10.1109/TPWRS.2014.2360401).

Scaparra, M.P., Church, R.L., 2008. A bilevel mixed-integer program for critical infras- tructure protection planning. Computers & Operations Research 35 (6), 1905–1923. doi:[10.1016/j.cor.2006.09.019](https://doi.org/10.1016/j.cor.2006.09.019).

[Schewe, L., Schmidt, M., Thürauf, J., 2021. Global optimization for the multilevel Eu- ropean gas market system with nonlinear flow models on trees. Journal of Global Optimization forthcoming.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0237)

Scholtes, S., 2001. Convergence properties of a regularization scheme for mathematical programs with complementarity constraints. SIAM Journal on Optimization 11 (4), 918–936. doi:[10.1137/S1052623499361233](https://doi.org/10.1137/S1052623499361233).

Shen, S., Smith, J.C., 2012. Polynomial-time algorithms for solving a class of criti- cal node problems on trees and series-parallel graphs. Networks 60 (2), 103–119. doi:[10.1002/net.20464](https://doi.org/10.1002/net.20464).

Shen, S., Smith, J.C., Goli, R., 2012. Exact interdiction models and algorithms for disconnecting networks via node deletions. Discrete Optimization 9 (3), 172–188. doi:[10.1016/j.disopt.2012.07.001](https://doi.org/10.1016/j.disopt.2012.07.001).

tion with *𝑘*-optimal follower: A hierarchy of bounds. Technical report. URL: Shi, X., Prokopyev, O., Ralphs, T.K., 2020. Mixed integer bilevel optimiza-

<http://www.optimization-online.org/DB_HTML/2020/06/7874.html>

[Shieh, E., An, B., Yang, R., Tambe, M., Baldwin, C., DiRenzo, J., Maule, B., Meyer, G., 2012. Protect: A deployed game theoretic system to protect the ports of the united states. In: Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems-Volume 1, pp. 13–20.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0242)

Shioda, R., Tunçel, L., Myklebust, T.G., 2011. Maximum utility product pricing models and algorithms based on reservation price. Computational Optimization and Applications 48 (2), 157–198. doi:[10.1007/s10589-009-9254-5](https://doi.org/10.1007/s10589-009-9254-5).

Siddiqui, S., Gabriel, S.A., 2013. An SOS1-based approach for solving MPECs with a natural gas market application. Networks and Spatial Economics 13 (2), 205–227. doi:[10.1007/s11067-012-9178-y](https://doi.org/10.1007/s11067-012-9178-y).

Sinha, A., Fang, F., An, B., Kiekintveld, C., Tambe, M., 2018. Stackelberg security games: Looking beyond a decade of success. In: Proceedings of the Twenty-Seventh Interna- tional Joint Conference on Artificial Intelligence (IJCAI-18),Stockholm, Sweden, July 13-19. IJCAI, pp. 5494–5501. doi:[10.24963/ijcai.2018/775](https://doi.org/10.24963/ijcai.2018/775).

Sinha, A., Malo, P., Deb, K., 2018. A review on bilevel optimization: From classical to evolutionary approaches and applications. IEEE Transactions on Evolutionary Com- putation 22 (2), 276–295. doi:[10.1109/TEVC.2017.2712906](https://doi.org/10.1109/TEVC.2017.2712906).

Sinha, A., Malo, P., Xu, P., Deb, K., 2014. A bilevel optimization approach to auto- mated parameter tuning. In: Proceedings of the 2014 Annual Conference on Genetic and Evolutionary Computation. Association for Computing Machinery, pp. 847–854. doi:[10.1145/2576768.2598221](https://doi.org/10.1145/2576768.2598221).

Sinnl, M., 2020. Bilevel integer programming and interdiction problems. Accessed: 2020- 12-21, URL: <https://msinnl.github.io/pages/bilevel.html>.

Smith, J.C., Song, Y., 2020. A survey of network interdiction models and al- gorithms. European Journal of Operational Research 283 (3), 797–811. doi:[10.1016/j.ejor.2019.06.024](https://doi.org/10.1016/j.ejor.2019.06.024).

Still, G., 2002. Linear bilevel problems: Genericity results and an eﬃcient method for computing local minima. Mathematical Methods of Operations Research 55 (3), 383–

400. doi:[10.1007/s001860200](https://doi.org/10.1007/s001860200).

[Tahernejad,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0251) [S., Ralphs, T.K., DeNegre, S.T., 2020. A Branch-and-Cut Algorithm for Mixed Integer Bilevel Linear Optimization Problems and Its Implementation. Mathematical Programming Computation (12) 529–568.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0251)

[Tambe,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0252) [M., 2011. Security and game theory: algorithms, deployed systems, lessons learned. Cambridge university press.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0252)

Tang, Y., Richard, J.P., Smith, J.C., 2016. A class of algorithms for mixed-integer bilevel min-max optimization. Journal of Global Optimization 66 (2), 225–262. doi:[10.1007/s10898-015-0274-7](https://doi.org/10.1007/s10898-015-0274-7).

Van Hoesel, S., 2008. An overview of Stackelberg pricing in networks. European Journal of Operational Research 189 (3), 1393–1402. doi:[10.1016/j.ejor.2006.08.064](https://doi.org/10.1016/j.ejor.2006.08.064).

[Vicente,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0255) [L., Calamai, P.H., 1994. Bilevel and multilevel programming: A bibliography review. Journal of Global optimization 5 (3), 291–306.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0255)

[Vicente,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0256) [L., Savard, G., Júdice, J., 1994. Descent approaches for quadratic bilevel pro- gramming. Journal of Optimization Theory and Applications 81 (2), 379–399.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0256)

[Vicente,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0257) [L., Savard, G., Júdice, J., 1996. Discrete linear bilevel programming problem.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0257)

[Journal of Optimization Theory and Applications 89 (3), 597–614.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0257) [von Stackelberg, H., 1934. Marktform und Gleichgewicht. Springer](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0258).

[von Stackelberg, H., 1952. Theory of the market economy. Oxford University Press](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0259). Wang, L., Xu, P., 2017. The watermelon algorithm for the bilevel integer lin-

ear programming problem. SIAM Journal on Optimization 27 (3), 1403–1430. doi:[10.1137/15M1051592](https://doi.org/10.1137/15M1051592).

[Wang,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0261) [Z., Yin, Y., An, B., 2016. Computing optimal monitoring strategy for detecting terrorist plots. In: Schuurmans, D., Wellman, M.P. (Eds.), Proceedings of the Thirtieth AAAI Conference on Artificial Intelligence, February 12-17, 2016, Phoenix, Arizona, USA. AAAI Press, pp. 637–643.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0261)

Washburn, A., Wood, R.K., 1995. Two-person zero-sum games for network interdiction.

Operations Research 43 (2), 243–251. doi:[10.1287/opre.43.2.243](https://doi.org/10.1287/opre.43.2.243).

[Wen,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0263) [U.-P., Hsu, S.-T., 1991. Linear bi-level programming problems – a review. The Jour- nal of the Operational Research Society 42 (2), 125–133.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0263)

Wiesemann, W., Tsoukalas, A., Kleniati, P., Rustem, B., 2013. Pessimistic bilevel optimiza- tion. SIAM Journal on Optimization 23 (1), 353–380. doi:[10.1137/120864015](https://doi.org/10.1137/120864015).

Williams, A., 1970. Boundedness relations for linear constraint sets. Linear Algebra and its Applications 3 (2), 129–141. doi:[10.1016/0024-3795(70)90009-1](https://doi.org/10.1016/0024-3795(70)90009-1).

Wogrin, S., Pineda, S., Tejada-Arango, D.A., 2020. Applications of Bilevel Optimization in Energy and Electricity Markets. Springer International Publishing, pp. 139–168. doi:[10.1007/978-3-030-52119-6\_5](https://doi.org/10.1007/978-3-030-52119-6_5).

[Wolsey, L.A., 1998. Integer programming. John Wiley & Sons](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0267).

Wood, R.K., 1993. Deterministic network interdiction. Mathematical and Computer Mod- elling 17 (2), 1–18. doi:[10.1016/0895-7177(93)90236-R](https://doi.org/10.1016/0895-7177(93)90236-R).

Wood, R.K., 2011. Bilevel Network Interdiction Models: Formulations and Solutions.

American Cancer Society doi:[10.1002/9780470400531.eorms0932](https://doi.org/10.1002/9780470400531.eorms0932).

[Wu,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0270) [S., Marcotte, P., Chen, Y., 1998. A cutting plane method for linear bilevel programs.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0270)

[Systems Science and Mathematical Sciences 11, 125–133.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0270)

[Xu,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0271) [P., 2012. Three essays on bilevel optimization algorithms and applications. Iowa State University. PhD thesis.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0271)

Xu, P., Wang, L., 2014. An exact algorithm for the bilevel mixed integer linear program- ming problem under three simplifying assumptions. Computers & Operations Research 41, 309–318. doi:[10.1016/j.cor.2013.07.016](https://doi.org/10.1016/j.cor.2013.07.016).

[Yang,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0273) [R., Ford, B.J., Tambe, M., Lemieux, A., 2014. Adaptive resource allocation for wildlife protection against illegal poachers. In: AAMAS, pp. 453–460.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0273)

Yanikoglu, I., Kuhn, D., 2018. Decision rule bounds for two-stage stochastic bilevel pro- grams. SIAM Journal on Optimization 28 (1), 198–222. doi:[10.1137/16M1098486](https://doi.org/10.1137/16M1098486).

Ye, J.J., Zhu, D.L., 1995. Optimality conditions for bilevel programming problems. Opti- mization 33 (1), 9–27. doi:[10.1080/02331939508844060](https://doi.org/10.1080/02331939508844060).

[Yin, Z.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0276), [Jiang, A.X.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0276), [Johnson, M.P.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0276), [Kiekintveld, C.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0276), [Leyton-Brown, K.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0276), [Sandholm, T.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0276), [Tambe, M., Sullivan, J.P., 2012. Trusts: Scheduling randomized patrols for fare in- spection in transit systems. In: IAAI.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0276)

[Yin,](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0277) [Z., Tambe, M., 2012. A unified method for handling discrete and continuous un- certainty in bayesian Stackelberg games. In: Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems-Volume 2, pp. 855–862.](http://refhub.elsevier.com/S2192-4406(21)00134-9/sbref0277) Yue, D., Gao, J., Zeng, B., You, F., 2019. A projection-based reformulation and

decomposition algorithm for global optimization of a class of mixed inte- ger bilevel linear programs. Journal of Global Optimization 73 (1), 27–57. doi:[10.1007/s10898-018-0679-1](https://doi.org/10.1007/s10898-018-0679-1).

Yue, D., You, F., 2017. Stackelberg-game-based modeling and optimization for supply chain design and operations: A mixed integer bilevel pro- gramming framework. Computers & Chemical Engineering 102, 81–95. doi:[10.1016/j.compchemeng.2016.07.026](https://doi.org/10.1016/j.compchemeng.2016.07.026). Sustainability & Energy Systems

Zare, M.H., Borrero, J.S., Zeng, B., Prokopyev, O.A., 2019. A note on linearized reformu- lations for a class of bilevel linear integer problems. Annals of Operations Research 272 (1-2), 99–117. doi:[10.1007/s10479-017-2694-x](https://doi.org/10.1007/s10479-017-2694-x).

Zeng, B., An, Y., 2014. Solving bilevel mixed integer program by reformulations and decomposition. Technical report. URL: <http://www.optimization-online.org/DB_FILE/2014/07/4455.pdf>

Zenklusen, R., 2010. Matching interdiction. Discrete Applied Mathematics 158 (15), 1676–

1690. doi:[10.1016/j.dam.2010.06.006](https://doi.org/10.1016/j.dam.2010.06.006).

Zenklusen, R., Ries, B., Picouleau, C., de Werra, D., Costa, M.-C., Bentz, C., 2009. Blockers and transversals. Discrete Mathematics 309 (13), 4306–4314. doi:[10.1016/j.disc.2009.01.006](https://doi.org/10.1016/j.disc.2009.01.006).

Zhang, Y., Snyder, L.V., Ralphs, T.K., Xue, Z., 2016. The competitive facility location prob- lem under disruption risks. Transportation Research Part E: Logistics and Transporta- tion Review 93, 453–473. doi:[10.1016/j.tre.2016.07.002](https://doi.org/10.1016/j.tre.2016.07.002).

Zhao, L., Zeng, B., 2013. Vulnerability analysis of power grids with line switch- [ing. IEEE Transactions on Power Systems 28 (3), 2727–2736. doi:10.1109/TP- WRS.2013.2256374.](https://doi.org/10.1109/TPWRS.2013.2256374)

Zhou, S., Zemkoho, A.B., Tin, A., 2020. BOLIB: Bilevel Optimization LI- Brary of test problems. Springer International Publishing, pp. 513–560. doi:[10.1007/978-3-030-52119-6\_19](https://doi.org/10.1007/978-3-030-52119-6_19).

Zugno, M., Morales, J.M., Pinson, P., Madsen, H., 2013. A bilevel model for electricity retailers’ participation in a demand response market environment. Energy Economics 36, 182–197. doi:[10.1016/j.eneco.2012.12.010](https://doi.org/10.1016/j.eneco.2012.12.010).