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An Efficient Algorithm for Representing Piecewise Linear Functions into Logic

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**Abstract**

Rational McNaughton functions may be implicitly represented by logical formulas in L- ukasiewicz Infinitely- valued Logic by constraining the set of valuations to the ones that satisfy some specific formulas. This work investigates this implicit representation called representation modulo satisfiability and describes a polynomial algorithm that builds it — the representative formula and the constraining ones — for a given rational McNaughton function.

*Keywords:*

Functions.

L- ukasiewicz Infinitely-valued Logic, Rational McNaughton Functions, Piecewise Linear

# Introduction

The ability to represent any piecewise linear function with a logical formula allows us to apply automated reasoning techniques to the study of real systems, whose behavior is either modeled or approximated by such a function. However such an ability will only be effective if there are efficient ways to generate a formula in a target logic in which reasoning is not exceedingly complex. Classical logic with its binary semantics, despite of being a natural target for representing Boolean functions, may not be the natural way to represent continuous functions which inevitably would require some encoding of rational or real numbers; so we follow the path of electing some form of many-valued logic, whose semantics range over rational numbers, as a more adequate representation framework. That path has initially been explored by applications in fuzzy control [[5](#_bookmark72)].

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Neural network interpretability is a challenge to the development of artificial intelligence and is also another motivation for the representation of piecewise linear functions, as described by [[4](#_bookmark70)]. In fact, a neural network, depending on its class of activation functions, can be seen either as a piecewise linear function or as a continuous function that can be approximated by one [[12](#_bookmark79)].

The first candidate to consider as a target logic is L- ukasiewicz Infinitely-valued Logic (*L- ∞*), arguably one of the best studied many-valued logics [[7](#_bookmark74)]; it has sev- eral interesting properties such as continuous truth-functional semantics, classical logic as limit case, and well developed proof-theoretical and algebraic presentations. Its formulas are known to represent exactly the so-called McNaughton functions, consisting of [0*,* 1]-valued piecewise linear functions *with integer coefficients* over [0*,* 1]*n* [[13,](#_bookmark80)[15](#_bookmark82)]. This restriction to integer coefficients fails to fulfill, for instance, the hypotheses for the classic Stone-Weierstrass Approximation Theorem [[16](#_bookmark83)] and there is no known analogous to Proposition [1.1](#_bookmark1) below for McNaughton functions.

This issue is circumvented by slightly modifying McNaughton functions to al- low their linear pieces to have rational coefficients — the rational McNaughton functions; such generalization is enough to perform Weierstrass-like approxima- tions [[5,2](#_bookmark68)]. Figure [1](#_bookmark2) shows a continuous function *f* : [0*,* 1] *→* [0*,* 1] (a) with two possible approximations by rational McNaughton functions: with two (b) and five different linear pieces (c). Note that continuous functions with more general domain and range might be normalized in order to perform such approximations.

## Proposition 1.1 (Variation of Weierstrass Approximation Theorem [[5](#_bookmark72)])

*Let f* : [0*,* 1]*n →* [0*,* 1] *be a continuous function and ε >* 0*. Then there is a rational*

*McNaughton function*

**x** *∈* [0*,* 1]*n.*

*f*˜ : [0*,* 1]*n →* [0*,* 1] *such that |f* (**x**) *− f*˜(**x**)*| < ε, for all*

In this context, the target logic must be a system, preferably based on *L- ∞*, with semantics that comprehends all rational McNaughton functions and, given one such function, be endowed with an efficient algorithm that provides a formula that represents it. Furthermore, we highlight that it would be of little practical use if the reasoning complexity in such system is exceedingly high.

Esteva, Godo & Montagna propose logic *L-* Π 1 which extends *L- ∞* with a product operator, its residuum, and a constant expressing the truth value 1 , not directly expressible in *L- ∞* [[8](#_bookmark75)]. That logic not only allows for the expressivity of rational McNaughton functions but also expresses piecewise polynomials; as a consequence

2

2

satisfiability over *L-* Π 1

2

requires finding roots of polynomials of *n*-degree rendering

its complexity extremely high. Aguzzoli & Mundici propose logic *∃L-* which also

expresses rational McNaughton functions and has complexity Σ*p* for the satisfiability

2

problem [[2,](#_bookmark68)[3](#_bookmark71)]. Logic *∃L-* extends *L- ∞* and introduces rational numbers by providing

restricted form of propositional quantification whose semantic counterpart is the maximization of a set of *L- ∞*-valuations of a formula.

Gerla introduces Rational

L- ukasiewicz Logic by extending

*L- ∞* with division

operators *δn* that induces division by *n ∈* N*∗* in its semantics [[11](#_bookmark78)]; its associated tautology problem is coNP-complete, which is a reasonable complexity for this task.

*f* (*x*)

*f*˜1(*x*)

*x*

(a)

*x*

(b)

*f*˜2(*x*)

*x*

(c)

Fig. 1. Continuous one-variable function approximated by rational McNaughton functions.

This logic expresses all rational McNaughton functions however it was not provided an algorithm to build the representative formulas, and an attempt to derive one from the results in [[11](#_bookmark78)] would lead to the problem of representing McNaughton functions in *L- ∞*; it is known that this task may be done in polynomial time on the coefficients of some specific functions [[1](#_bookmark69)], however these methods lead to exponential time complexity if binary representation of the coefficients is used.

Finger & Preto provide a way to *implicitly* express rational McNaughton func- tions in *L- ∞* called representation modulo satisfiability (*L- ∞*-MODSAT) [[10](#_bookmark76)]. For that, in addition to a representative formula, it is introduced a set of formulas that constrains *L- ∞*-valuations to those that satisfy all formulas in the set; a ra- tional McNaughton function *f* is then represented by a pair *⟨ϕ,* Φ*⟩*, where *ϕ* is a formula that semantically acquires values *f* (**x**), for **x** *∈* [0*,* 1]*n*, from valuations in

*{v*(*ψ*) = 1 *| ψ ∈* Φ*}*, where Φ is a set of formulas. Instead of an extension of the logic, this proposal works in *L- ∞* itself, which has computational problems with rea- sonable complexity — e.g., satisfiability over *L- ∞* is NP-complete [[14](#_bookmark81)]. Also, there

already are available implementations of *L- ∞*-solvers which are discussed in the liter- ature and for which phase transition phenomenon is identified [[6,](#_bookmark73)[9](#_bookmark77)]. Unfortunately, an attempt to derive a representation builder algorithm from results in [[10](#_bookmark76)] would also lead to an exponential blow up, since the proposed pairs for representing only truncated linear functions are already exponential in the binary representation of their coefficients.

Our goal here is to provide an efficient algorithm that, given a rational Mc- Naughton function, outputs a pair *⟨ϕ,* Φ*⟩* that represents it in the target system *L- ∞*-MODSAT. We show that all rational McNaughton functions may be represented modulo satisfiability by a constructive proof from which we derive a polynomial al- gorithm that builds such representation.

This paper is organized as follows: Section [2](#_bookmark3) introduces all necessary concepts of L- ukasiewicz Infinitely-valued Logic and the definition of rational McNaughton functions; Section [3](#_bookmark6) has the formalization of the concept of representation modulo satisfiability; Section [4](#_bookmark8) provides a convention on rational McNaughton functions encoding for computation purposes as well as some results about these functions; Section [5](#_bookmark13) has a theoretical and algorithmic treatment of a particular case of represen- tation modulo satisfiability of rational McNaughton functions which are truncated linear functions; and Section [6](#_bookmark55) finally treats, also theoretically and algorithmically, the representation modulo satisfiability of general rational McNaughton functions.

# Preliminaries

The basic language *L* of L- ukasiewicz Infinitely-valued Logic (*L- ∞*) comprehends the formulas built from a countable set of propositional variables P, and disjunction (*⊕*) and negation (*¬*) operators. For the semantics, define a valuation as a function *v* : *L→* [0*,* 1], such that, for *ϕ, ψ ∈ L*:

*v*(*ϕ ⊕ ψ*)= min(1*, v*(*ϕ*)+ *v*(*ψ*)); (1)

*v*(*¬ϕ*)=1 *− v*(*ϕ*)*.* (2)

One may just give a function *v*P which maps propositional variables to a value in the interval [0*,* 1] and extend this function to a valuation by obeying ([1](#_bookmark4)) and ([2](#_bookmark5)). This extension is uniquely defined by such assignment to the variables in P given by *v*P.

We denote by **Val** the set of all valuations; by Var(Φ) the set of all propositional

variables occurring in the formulas *ϕ ∈* Φ; and by **X***n* the set of propositional variables *{X*1*,..., Xn}⊂* P. A formula *ϕ* is *satisﬁable* if there exists a *v ∈* **Val** such that *v*(*ϕ*) = 1; otherwise it is *unsatisﬁable*. A set of formulas Φ is satisfiable if there exists a *v ∈* **Val** such that *v*(*ϕ*)= 1, for all *ϕ ∈* Φ. We denote by **Val**Φ the set of all valuations *v ∈* **Val** that satisfies a set of formulas Φ.

From disjunction and negation we derive the following operators:

Conjunction: *ϕ ⊙ ψ* =def *¬*(*¬ϕ ⊕ ¬ψ*) *v*(*ϕ ⊙ ψ*)= max(0*, v*(*ϕ*)+ *v*(*ψ*) *−* 1) Implication: *ϕ → ψ* =def *¬ϕ ⊕ ψ v*(*ϕ → ψ*)= min(1*,* 1 *− v*(*ϕ*)+ *v*(*ψ*)) Maximum: *ϕ ∨ ψ* =def *¬*(*¬ϕ ⊕ ψ*) *⊕ ψ v*(*ϕ ∨ ψ*)= max(*v*(*ϕ*)*, v*(*ψ*))

Minimum: *ϕ ∧ ψ* =def *¬*(*¬ϕ ∨ ¬ψ*) *v*(*ϕ ∧ ψ*)= min(*v*(*ϕ*)*, v*(*ψ*)) Bi-implication: *ϕ ↔ ψ* =def (*ϕ → ψ*) *∧* (*ψ → ϕ*) *v*(*ϕ ↔ ψ*)=1 *− |v*(*ϕ*) *− v*(*ψ*)*|*

Note that *v*(*ϕ → ψ*)=1 iff *v*(*ϕ*) *≤ v*(*ψ*); similarly, *v*(*ϕ ↔ ψ*)=1 iff *v*(*ϕ*)= *v*(*ψ*). Let *X* be a propositional variable, then, *v*(*X ⊙¬X*)= 0, for any *v ∈* **Val**; we define the constant **0** by *X ⊙ ¬X*. We also define 0*ϕ* =def **0** and *nϕ* =def *ϕ ⊕· · ·⊕ ϕ*, *n* times, for *n ∈* N*∗*; and *i∈*∅ *ϕi* =def **0**.

L

Adapting the definition in [[7](#_bookmark74)], a *rational McNaughton function f* : [0*,* 1]*n →* [0*,* 1]

is a function that satisfies the following conditions:

* *f* is continuous with respect to the usual topology of [0*,* 1] as an interval of the real number line;
* There are linear polynomials *p*1*,..., pm* over [0*,* 1]*n* with rational coefficients such that, for each point **x** *∈* [0*,* 1]*n*, there is an index *i ∈ {*1*,..., m}* with *f* (**x**)= *pi*(**x**). Polynomials *p*1*,..., pm* are the *linear pieces* of *f* .

# Representation Modulo Φ-Satisfiable

A *McNaughton function* is a rational McNaughton function whose linear pieces have integer coefficients. Let *ϕ* be a *L- ∞*-formula with Var(*ϕ*) *⊂* **X***n*, we inductively associate to *ϕ* a function *f**ϕ* : [0*,* 1]*n →* [0*,* 1] by:

* 1. *fXj* (*x*1*,..., xn*)= *xj*, for *j* = 1*,..., n*;
  2. *f¬ϕ*(*x*1*,..., xn*)=1 *− fϕ*(*x*1*,..., xn*);
  3. *fϕ*1*⊕ϕ*2 (*x*1*,..., xn*)= min(1*, fϕ*1 (*x*1*,..., xn*)+ *fϕ*2 (*x*1*,..., xn*)).

We have that *fϕ* is a McNaughton function such that

*fϕ*(*v*(*X*1)*,..., v*(*Xn*)) = *v*(*ϕ*)*,* for *v ∈* **Val**. (3) Reciprocally, McNaughton’s Theorem [[13](#_bookmark80)] states that, for any McNaughton function

*f* , there is a formula *ϕ* such that *f* = *fϕ*. We say that *ϕ represents f* .

Although formulas of *L- ∞* only represent (integer) McNaughton functions, we take the strategy of restricting the set **Val** of valuations in order to implicitly represent rational McNaughton functions. For that, we start by noting that value of a formula *ϕ* according to some valuation *v* is determined only by the values associated to a finite set of propositional variables **X** such that Var(*ϕ*) *⊂* **X**; this very property is the crux for the possibility that logical formulas represent functions. We next generalize this notion.

**Definition 3.1** Let *ϕ* be a formula and let Φ be a set of formulas. We say that a set of variables **X***n determines ϕ modulo* Φ*-satisﬁable* if:

* For any *⟨x*1*,..., xn⟩ ∈* [0*,* 1]*n*, there exists at least one valuation *v ∈* **Val**Φ, such that *v*(*Xj*)= *xj*, for *j* = 1*,..., n*;
* For any valuations *v, vj ∈* **Val**Φ, such that *v*(*Xj*) = *vj*(*Xj*), for *j* = 1*,..., n*, *v*(*ϕ*)= *vj*(*ϕ*).

For instance, for any formula *ϕ* such that Var(*ϕ*) *⊂* **X***n*, **X***n* determines *ϕ* modulo

∅-satisfiable, by truth functionality and the fact that **Val**∅ = **Val**.

It is important to note that any set **X***n* can now represent a rational fraction

1 by determining a propositional variable *Z* 1

*d d*

modulo *ϕ* 1

*d*

= *Z* 1

*d*

*↔ ¬*(*d −* 1)*Z* 1

*d*

satisfiable, with *d ∈* N*∗*. In fact, for any valuation *v ∈* **Val**, if *v*(*ϕ* 1 ) = 1, then

*d*

*v*(*Z* 1 ) = 1 . We define representation modulo satisfiability in a way that retrieves

*d d*

property ([3](#_bookmark7)).

**Definition 3.2** Let *f* : [0*,* 1]*n →* [0*,* 1] be a function, and *⟨ϕ,* Φ*⟩* be a pair where *ϕ* is a formula and Φ is a set of formulas. We say that *ϕ represents f modulo* Φ*-satisﬁable* or that *⟨ϕ,* Φ*⟩ represents f in the system L- ∞-MODSAT* if:

* **X***n* determines *ϕ* modulo Φ-satisfiable;
* *f* (*v*(*X*1)*,..., v*(*Xn*)) = *v*(*ϕ*), for *v ∈* **Val**Φ.

Representation modulo satisfiability presented in [[10](#_bookmark76)] has a different approach, which we call function-based and is more restrictive than the one presented here, which we call formula-based. However, the representation methods and algorithms we develop in this work apply to both approaches.

# Rational McNaughton Functions

Our algorithm uses a lattice representation of rational McNaughton functions; be- fore that we employ an encoding based in [[17,](#_bookmark84)[18](#_bookmark85)] as follows. Let Ω*○* be the interior of a set Ω *⊂* R*n*. A rational McNaughton function *f* : [0*,* 1]*n →* [0*,* 1] is given by *m* (not necessarily distinct) linear pieces

*pi*(**x**)= *γi*0 + *γi*1*x*1 + *···* + *γinxn,* (4) for **x** = *⟨x*1*,..., xn⟩ ∈* [0*,* 1]*n*, *γij ∈* Q and *i* = 1*,..., m*, with each linear piece *pi*

identical to *f* over a convex set Ω*i ⊂* [0*,* 1]*n* called *region* such that:

* *m*

*i*=1

Ω*i* = [0*,* 1]*n*;

* Ω*○ ∩* Ω*○* = ∅, for *ij /*= *ijj*;

*i′ i′′*

* Regions Ω*i* are given in such a way that there is a polynomial procedure to determine whether or not a linear piece *pk* is *above* other linear piece *pi* over region Ω*i*, that is whether or not *pk*(**x**) *≥ pi*(**x**), for all **x** *∈* Ω*i*.

A rational McNaughton function as above is said to be in *regional format*. Note that

0*.*6

0

0*.*5

0

0*.*2

0*.*4

0*.*6

0*.*8

1 1

*x*1

0*.*4

*x*2

Fig. 2. Graph of rational McNaughton function with three linear pieces over [0*,* 1]2.

*x*2

Ω1

Ω2

Ω3

*x*1

*x*2

*P*321

*P*312

*P*231

*P*132

*P*213

*x*1

*P*123

(a) (b)

Fig. 3. Some possible region configurations for function *f* in Example [4.1](#_bookmark11).

in this format the number of linear pieces is equal to the number of regions; in this case, the size of a function is the sum of the number of bits necessary to represent its linear pieces coefficients as fractions *a* plus the space necessary for representing

*b*

its regions in some assumed encoding.

**Example 4.1** Rational McNaughton function *f* with graph in Figure [2](#_bookmark9) may be given by the linear pieces:

* *p*1(*x*1*, x*2)= 4 + 2 *x*2;

9 3

* *p*2(*x*1*, x*2)= 5 *−* 1 *x*2;

6 2

* *p*3(*x*1*, x*2)= 4 *− x*1.

3

Regions associated to each linear piece are depicted in Figure [3](#_bookmark10)(a). Determining if some linear piece *pk* is above other linear piece *pi* over Ω*i* is equivalent to determining if the system of corresponding inequalities in Table [1](#_bookmark12) with added equation *pk−pi* =0 has no solution and *pk*(**x**0) *> pi*(**x**0), for some point **x**0 *∈* Ω*○*, a tractable process.

*i*

Let *f* : [0*,* 1]*n →* [0*,* 1] be a rational McNaughton function as defined in Section [2,](#_bookmark3)

*○*

Ω

1

8 *−* 9*x*1 *−* 6*x*2 *>* 0

1 *− x*2 *>* 0

3

*x*1 *>* 0

*x*2 *>* 0

*○*

2

Ω

1 *−* 2*x*1 + *x*2 *>* 0

*−* 1 + *x*2 *>* 0

3

*x*1 *>* 0

1 *− x*2 *>* 0

*○*

3

Ω

*−*8+ 9*x*1 + 6*x*2 *>* 0

*−*1+ 2*x*1 *− x*2 *>* 0

1 *− x*1 *>* 0

*x*2 *>* 0

Table 1

Sets Ω*◦* for function *f* in Example [4.1](#_bookmark11).

*i*

with distinct linear pieces *p*1*,..., pm*. For each permutation *ρ* of the set *{*1*,..., m}*, we define the polyhedron

*Pρ* = *{***x** *∈* [0*,* 1]*n | pρ*(1)(**x**) *≥ · · ·≥ pρ*(*m*)(**x**)*}.* (5) Let *C* be the set of *n*-dimensional polyhedra *Pρ*, for some permutation *ρ*.

**Proposition 4.2** *The set C has the following properties:*

1. *C* = [0*,* 1]*n;*
2. *For polyhedron P ∈ C and indexes ij, ijj ∈ {*1*,..., m} with ij /*= *ijj, pi′* (**x**) */*=

*pi′′* (**x**)*, for any* **x** *∈ P○;*

1. *Pj○ ∩ P jj○* = ∅*, for Pj,Pjj ∈C such that Pj /*= *P jj;*
2. *For each polyhedron P ∈ C, there is an index iP ∈ {*1*,..., m} such that f* (**x**)=

*pi* (**x**)*, for* **x** *∈* [0*,* 1]*n.*

*P*

## Proof.

1. For any **x** *∈ P ∈ C*, **x** *∈* [0*,* 1]*n*. On the other hand, for any **x** *∈* [0*,* 1]*n*, there is a permutation *ρ* for which *Pρ* is *n*-dimensional and **x** *∈ Pρ*.
2. Let **x** *∈ P○* and let *ij, ijj ∈ {*1*,..., m}* be indexes such that *ij /*= *ijj*. Since *pi′* and *pi′′* are linear pieces, if *pi′* (**x**) = *pi′′* (**x**), for some **x** *∈ P○*, there would be points **x**1*,* **x**2 *∈ P○* in a neighborhood of **x** such that *pi′* (**x**1) *< pi′′* (**x**1) and *pi′′* (**x**2) *< pi′* (**x**2), contrary to the definition of *P* .
3. Let **x** *∈ Pj○∩Pjj○*. Then, by definitions of *Pj* and *P jj*, there are *ij, ijj ∈ {*1*,..., m}*

such that *pi′* (**x**)= *pi′′* (**x**), contrary to item (b).

1. Let *{i*1*,..., ik} ⊂ {*1*,..., m}* be a non-singleton set of indexes such that, for

any **x** *∈ P○*, there is *l ∈ {*1*,..., k}*, such that *f* (**x**) = *pi* (**x**) and *Ui*

*l*

*l*

= *{***x** *∈*

*P○ | f* (**x**)= *pi* (**x**)*} /*= ∅, for *l* = 1*,..., k*. We have that *∪k Ui* = *P○* and, by

*l l*=1 *l*

item (b), *Ui ′ ∩Ui ′′* = ∅, for *lj /*= *ljj*. As *P○* is a connected set, there are distinct

*l l*

*ij, ijj ∈ {i*1*,..., ik}* and **b** *∈ P○* such that **b** *∈ ∂Ui′* and **b** *∈ Ui′′* . As *pi′* restricted

to *Ui′ ∪ {***b***}* is continuous, for any sequence *{***b***n} ⊂ Ui′* such that lim **b***n* = **b** (which exists since **b** *∈ ∂Ui′* ), we have that lim *f* (**b***n*) = lim *pi′* (**b***n*) = *pi′* (**b**). However, *f* (**b**)= *pi′′* (**b**) */*= *pi′* (**b**), by item (b), contrary to the continuity of *f* .

*2*

Polyhedra in *C* may play the role of regions since they are convex sets with the properties above; determining whether a linear piece *pk* is above other linear piece *pi* over *P ∈ C* comes down to comparing their values for some point **x** *∈ P○*. Note that the same linear piece *pi* may be associated to many distinct polyhedra. Thus, any rational McNaughton function may be encoded in regional format. Figure [3](#_bookmark10)(b) shows the polyhedra-based configuration *C* for the function in Example [4.1](#_bookmark11).

The setback with describing a rational McNaughton function using the set *C* of polyhedra is that in the worst case *|C|* = *m*!. However, in general there are smaller sets of regions that comply with representation restrictions above [[18](#_bookmark85)].

# A Particular Case: Truncated Linear Functions

Let us show the possibility of representing a rational McNaughton function modulo satisfiability and develop a polynomial algorithm for computing such representation in the particular case that function is a truncated linear polynomial with rational coefficients.

Let *p* : [0*,* 1]*n →* R be a nonzero linear polynomial given by

*p*(**x**)= *a*0 + *a*1 *x*

*b*0 *b*1 1

+ *···* + *an x*

*bn n*

*,* (6)

for **x** = *⟨x*1*,..., xn⟩ ∈* [0*,* 1]*n*, *aj ∈* Z, and *bj ∈* Z*∗* . We want to build a representa- tion for the function *p*# : [0*,* 1]*n →* [0*,* 1] given by

+

*p*#(**x**)= min 1*,* max 0*, p*(**x**) *.* (7)

We have that *p*#(**x**) = 0, if *p*(**x**) *<* 0; *p*#(**x**) = 1, if *p*(**x**) *>* 1; and *p*#(**x**) = *p*(**x**), otherwise.

In order to rewrite expression ([6](#_bookmark14)), we define:

*αj* = *aj,* for *j ∈ P* ;

*αj* = *−aj,* for *j ∈ N* ;

*βj* = *β · bj,* for *j* = 0*,..., n*;

where *j ∈ P* , if *aj >* 0, and *j ∈ N* , if *aj <* 0, with *P ∪ N ⊂ {*0*,..., n}*, and *β* is the least integer greater than or equal to

max ,⎨Σ *aj , −* Σ *aj* ,⎬ *.*

,*j∈P bj j∈N bj* ,

We have that *αj ∈* Z+ and *βj ∈* Z*∗* , for *j* = 0*,..., n*. Let *x*0 = 1 and define functions *pP* : [0*,* 1]*n →* R and *pN* : [0*,* 1]*n →* R, for **x** = *⟨x*1*,..., xn⟩∈* [0*,* 1]*n*, by:

+

*p* (**x**)= Σ *αj x* ; *p* (**x**)= Σ *αj x .* (8)

*P*

*βj*

*j*

*N*

*βj*

*j*

*j∈P*

*j∈N*

**Lemma 5.1** *Functions p, pP , and pN in* ([6](#_bookmark14)) *and* ([8](#_bookmark16)) *have the following properties, for* **x** *∈* [0*,* 1]*n:*

1. *p*(**x**)= *β · pP* (**x**) *− pN* (**x**) *;*
2. 0 *≤ pP* (**x**)*, pN* (**x**) *≤* 1*.*

**Proof.** By elementary algebraic manipulation. *2*

Lemma above decomposes function *p* in terms of *pP* and *pN* , let us represent

the latter ones. Let *Zp,Z*  1 *∈* P. For a set of indexes *J ∈ {P, N}*, define:

*j β**j*

*ϕ*˜*J* =  *αjZp*; Φ˜*J* = *ϕ* 1 *, βjZp ↔ Xj, Zp → Z* 1 *.*

*j*

*j∈J\{*0*}*

*βj j*

*j∈J\{*0*}*

*j βj*

And then, define:

*ϕ*¯*J* = *ϕ*˜*J* ;

*ϕ*¯*J* = *α*0*Z*  1

*β*0

*⊕ ϕ*˜*J* ;

Φ¯ *J* = Φ˜*J ,* if 0 *∈/ J*; Φ¯ *J* = Φ˜*J ∪ {ϕ*  1 *},* otherwise.

*β*0

(9)

**Lemma 5.2** *Functions pP and pN in* ([8](#_bookmark16)) *may respectively be represented by*

*⟨ϕ*¯*P ,* Φ¯ *P ⟩ and ⟨ϕ*¯*N ,* Φ¯ *N ⟩ in* ([9](#_bookmark18))*.*

**Proof.** Let *J ∈ {P, N}*. If *J* = ∅, then *⟨ϕ*¯*J ,* Φ¯ *J⟩* = *⟨***0***,* ∅*⟩* represents *pJ* . For

*⟨x*1*,..., xn⟩∈* [0*,* 1]*n*, define a valuation *v ∈* **Val** such that *v*(*Xj*)= *xj* and *v*(*Zp*)=

*j*

*xj* , for *j ∈ J \ {*0*}*, and *v*(*Z*  1 ) =  1 , for *j ∈ J*. We have that *v ∈* **Val** ¯ .

*βj βj βj* Φ*J*

Now, let *v, vj ∈* **Val**Φ¯ such that *v*(*Xj*) = *vj*(*Xj*), for *j* = 1*,..., n*. By rational

*J j*  1

*p*  1

constant representation, *v*(*Z*  1 ) = *v* (*Z*  1 ) =

*βj* , for *j ∈ J*. Thus *v*(*Zj* ) *≤ βj*

*βj βj*

and *vj*(*Zp*) *≤*  1 , which implies that *βj · v*(*Zp*) = *v*(*βjZp*) = *v*(*Xj*) = *vj*(*Xj*) =

*j βj j j*

*vj*(*βjZp*) = *βj · vj*(*Zp*) and, then, *v*(*Zp*) = *vj*(*Zp*), for *j ∈ J \ {*0*}*. Therefore,

*j j j j*

*v*(*ϕ*¯*J* ) = *vj*(*ϕ*¯*J* ) and **X***n* determines *ϕ*¯*J* modulo Φ¯ *J* -satisfiable. Finally, suppose

*v ∈* **Val**Φ¯ *J* . In the case where 0 *∈ J*,

*j*

*pJ* (*v*(*X*1)*,..., v*(*Xn*)) = *α*0 *· v*(*Z*  1 )+

Σ

*β*0

*j∈J\{*0*}*

*αj · v*(*Zp*)= *v*(*ϕ*¯*J* )*,*

by Lemma [5.1](#_bookmark17) and aforementioned equations *v*(*Z*  1 )=  1 and *βj · v*(*Zp*)= *v*(*Xj*).

*β*0 *β*0 *j*

The case where 0 *∈/ J* is similar. *2*

For the final step towards a representation for *p*#, we define:

*ϕ*¯*p* = *β*[*¬*(*ϕ*¯*P → ϕ*¯*N* )];

Φ¯ *p* = Φ¯ *P ∪* Φ¯ *N .* (10)

**Theorem 5.3** *Function p*# *in* ([7](#_bookmark15)) *may be represented by ⟨ϕ*¯*p,* Φ¯ *p⟩ in* ([10](#_bookmark20))*.*

**Proof.** For *⟨x*1*,..., xn⟩ ∈* [0*,* 1]*n*, there exists *v ∈* **Val**Φ¯ such that *v*(*Xj*) = *xj* as

*p*

in the proof of Lemma [5.2](#_bookmark19) with *J* = *P ∪ N* . Now, let *v, vj ∈* **Val**Φ¯

*p*

such that

*v*(*Xj*)= *vj*(*Xj*), for *j* = 1*,..., n*. In particular, *v, vj ∈* **Val**Φ¯ and, by Lemma [5.2](#_bookmark19),

*J*

*v*(*ϕ*¯*J* )= *vj*(*ϕ*¯*J* ), for *J ∈ {P, N}*. Therefore, *v*(*ϕ*¯*p*)= *vj*(*ϕ*¯*p*) and **X***n* determines *ϕ*¯*p* modulo Φ¯ *p*-satisfiable. Finally, suppose *v ∈* **Val**Φ¯ *p* . In particular, *v ∈* **Val**Φ¯ *P* and *v ∈* **Val**Φ¯ *N* . If *p*(*v*(*X*1)*,..., v*(*Xn*)) *≤* 0, by Lemma [5.1](#_bookmark17), *pP* (*v*(*X*1)*,..., v*(*Xn*)) *≤*

*pN* (*v*(*X*1)*,..., v*(*Xn*)). Therefore, by Lemma [5.2](#_bookmark19), *v*(*ϕ*¯*P* ) *≤ v*(*ϕ*¯*N* ) and, then,

*v*(*ϕ*¯*p*) = 0. On the other hand, if *p*(*v*(*X*1)*,..., v*(*Xn*)) *≥* 0, by Lemma [5.1](#_bookmark17), *pP* (*v*(*X*1)*,..., v*(*Xn*)) *≥ pN* (*v*(*X*1)*,..., v*(*Xn*)). Therefore, by Lemma [5.2](#_bookmark19), *v*(*ϕ*¯*P* ) *≥ v*(*ϕ*¯*N* ) and, then, *v*(*¬*(*ϕ*¯*P → ϕ*¯*N* )) = 1*−*min(1*,* 1*−v*(*ϕ*¯*P* )+*v*(*ϕ*¯*N* )) = *v*(*ϕ*¯*P* )*−v*(*ϕ*¯*N* ). Finally, by Lemmas [5.1](#_bookmark17) and [5.2](#_bookmark19), *p*(*v*(*X*1)*,..., v*(*Xn*)) = *β ·* (*v*(*ϕ*¯*P* ) *− v*(*ϕ*¯*N* )), hence *p*#(*v*(*X*1)*,..., v*(*Xn*)) = *v*(*ϕ*¯*p*) in any case. *2*

Table [2](#_bookmark25) shows how functions in Example [4.1](#_bookmark11) can be represented as in Theorem [5.3](#_bookmark21).

In order to set up a polynomial algorithm for computing a representation *⟨ϕp,* Φ*p⟩* for *p*#, we analyze more closely expressions *nψ*, which show up in *ϕ*¯*p* and in formulas in Φ¯ *p*. These expressions are exponential in the binary representation of *n* since it denotes an *n*-fold repetition of formula *ψ*. We deviate from this situation by

using *[*log *n♩* + 1 new propositional variables *ξ*0 *, ξ*1 *,.* *, ξ[*log *n♩* and replacing every

*ψ ψ ψ*

occurrence of *nψ*, where *n ∈* N *\ {*0*,* 1*}*, with the formula

*ξnψ* =def *ψ*

|  |  |  |
| --- | --- | --- |
| *[*log *n♩* |  | |
|  | *ξk ,* | (11) |
| *k*=0 *nk*=1 |  |  |

where *nk ∈ {*0*,* 1*}* comes from the binary representation Σ*[*log *n♩* 2*knk* of *n*, and by

*k*=0

adding the following formulas to Φ¯ *p*:

0 *↔ ψ*;

*ξ*

*ψ*

*ξk ↔ ξk—*1 *⊕ ξk—*1*,* for *k* = 1*,..., [*log *n♩.*

(12)

*ψ ψ ψ*

These formulas define the propositional variables *ξk* and we call Ξ*nψ* the set that

*ψ*

comprehends them. In this way we avoid exponential blow up as shown in Theorem [5.5](#_bookmark26).

**Lemma 5.4** *Let n ∈* N *\ {*0*,* 1*}, ψ be a formula, and ξnψ and* Ξ*nψ be respectively a formula as in* ([11](#_bookmark22)) *and a set as in* ([12](#_bookmark23)) *built from n and ψ. For any valuation v ∈* **Val**Ξ*nψ , v*(*nψ*)= *v*(*ξnψ*)*.*

*¬* *Z*  1

*⊕ Z*  1 *⊕ Z*  1

*⊕ Z*  1 *⊕ Zp*1 *⊕ Zp*1 *→* **0**

*ϕ*¯*p*1 :

*⊕¬* *Z*

18

18

*⊕ Z ⊕ Z*

1 18

1 18

18

18

*⊕ Z ⊕ Zp*1 *⊕ Zp*1 *→* **0**

2

2

1 18

1 18

2

2

Φ¯ *p*1 :

*Z*  1

18

*↔ ¬ Z*  1 *⊕ Z*  1

18 18

*⊕ Z*  1 *⊕ Z*  1

18 18

*⊕ Z*  1 *⊕ Z*  1

18 18

*⊕ Z*  1 *⊕ Z*  1

18 18

*⊕ Z*  1

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*⊕ Z* 1

18

*⊕ Z* 1 *⊕ Z* 1

18 18

*⊕ Z* 1 *⊕ Z* 1

18 18

*⊕ Z* 1 *⊕ Z* 1

18 18

*⊕ Z* 1

18

*Z* 1 *↔ ¬* *Z* 1 *⊕ Z* 1 *⊕ Z* 1 *⊕ Z* 1 *⊕ Z* 1

6

6

6

6

6

6

*Zp*1 *⊕ Zp*1 *⊕ Zp*1 *⊕ Zp*1 *⊕ Zp*1 *⊕ Zp*1 *↔ X*2

2 2 2 2 2 2

*Zp*1 *→ Z* 1

2 6

2

6

6

6

6

6

2

*ϕ*¯*p* : *¬* *Z* 1 *⊕ Z* 1 *⊕ Z* 1 *⊕ Z* 1 *⊕ Z* 1 *→ Zp*2

Φ¯ *p*2 : *Z* 1 *↔ ¬* *Z* 1 *⊕ Z* 1 *⊕ Z* 1 *⊕ Z* 1 *⊕ Z* 1

*Z* 1 *↔ ¬Z* 1

6

6

6

6

6

6

2 2

*Zp*2 *⊕ Zp*2 *↔ X*2

2 2

*Zp*2 *→ Z* 1

2 2

3

6

6

6

6

1

6

6

6

6

1

*ϕ*¯*p* : *¬* *Z* 1 *⊕ Z* 1 *⊕ Z* 1 *⊕ Z* 1 *→ Zp*3 *⊕¬* *Z* 1 *⊕ Z* 1 *⊕ Z* 1 *⊕ Z* 1 *→ Zp*3

Φ¯ *p*3 : *Z* 1 *↔ ¬*  *Z* 1 *⊕ Z* 1 *⊕ Z* 1 *⊕ Z* 1 *⊕ Z* 1

*Z* 1 *↔ ¬Z* 1

6

6

6

6

6

6

2 2

*Zp*3 *⊕ Zp*3 *↔ X*1

1 1

*Zp*3 *→ Z* 1

1 2

Table 2

Representations as in ([10](#_bookmark20)) for functions *p*#, *p*# and *p*#, where functions *p*1, *p*2 and *p*3 are from Example

1 2 3

[4.1](#_bookmark11).

*k k*

**Proof.** For *v ∈* **Val**Ξ*nψ* and *k* = 0*,..., [*log *n♩*, *v*(*ξ* )= min(1*,* 2 *v*(*ψ*)). Then,

*ψ*

*v*(*nψ*)= min ,1*,*

⎝

,

*[*log *n♩ k*=0

*[*log *n♩*

Σ

Σ

2*knkv*(*ψ*)⎞

⎠

⎞

*v*(*ξ* )*nk*⎠ = *v* ⎝

, ⎞

= min ⎝1*,*

*k*=0

*k ψ*

*nk*=1

*ψ*

*ξk* ⎠ = *v*(*ξnψ*)*,*

where *nk ∈ {*0*,* 1*}* in the binary representation *n* = Σ*[*log *n♩* 2*knk*. *2*

*k*=0

**Theorem 5.5** *Let n ∈* N *\ {*0*,* 1*}, ψ be a formula, and ⟨ϕp,* Φ*p⟩ be a pair deﬁned from representation ⟨ϕ*¯*p,* Φ¯ *p⟩ in* ([10](#_bookmark20)) *by replacing any occurrence of nψ in ϕ*¯*p and* Φ¯ *p with ξnψ in* ([11](#_bookmark22)) *and by adding formulas in set* Ξ*nψ in* ([12](#_bookmark23)) *to* Φ¯ *p. Then, ⟨ϕp,* Φ*p⟩*

*is also a representation for p*# *in* ([7](#_bookmark15))*. Furthermore, ⟨ϕp,* Φ*p⟩ is a representation*

*for p*# *even if it is deﬁned by multiple suitable replacements of expressions nlψl, for*

*l* = 1*,.* *, L.*

**Proof.** For *⟨x*1*,..., xn⟩ ∈* [0*,* 1]*n*, define a valuation *v* such that *v*(*Xj*) = *xj* and

*v*(*Zp*) = *xj* , for *j* = 1*,..., n*, *v*(*Z*  1 ) =  1 , for *j* = 0*,..., n*, *v*(*ξ*0 ) = *v*(*ψ*), and

*j βj*

*βj βj ψ*

*v*(*ξk* ) = min(1*, v*(*ξk—*1)+ *v*(*ξk—*1)), for *k* = 1*,..., [*log *n♩*. Note that *v ∈* **Val**Φ¯

*ψ ψ ψ p*

and *v ∈* **Val**Ξ*nψ* , then, by Lemma [5.4](#_bookmark24), as Ξ*nψ ⊂* Φ*p*, we have that *v ∈* **Val**Φ*p* .

Still, for any *v ∈* Φ*p*, we have that *v ∈* **Val**Ξ*nψ* and, by Lemma [5.4](#_bookmark24), *v ∈* Φ¯ *p*.

Therefore, again by Lemma [5.4](#_bookmark24), for *v, vj ∈* Φ*p* such that *v*(*Xj*) = *vj*(*Xj*), for *j* =

1*,..., n*, it follows that *v*(*ϕp*) = *vj*(*ϕp*), **X***n* determines *ϕp* modulo Φ*p*-satisfiable, and *p*#(*v*(*X*1)*,..., v*(*Xn*)) = *v*(*ϕp*). This argument still holds when considering multiple replacements. *2*

We set *⟨ϕp,* Φ*p⟩* from *⟨ϕ*¯*p,* Φ¯ *p⟩* in ([10](#_bookmark20)) by properly replacing all occurrences of

*nlψl* as stated in the above theorem. By construction, *⟨ϕp,* Φ*p⟩* is given by

*ϕp* = *β*[*¬*(*ϕP → ϕN* )]; Φ*p* = Φ*P ∪* Φ*N* ; (13)

where *ϕP* , *ϕN* , Φ*P* , and Φ*N* are properly defined from their barred correspondents in

([9](#_bookmark18)). Table [3](#_bookmark28) shows how functions in Example [4.1](#_bookmark11) can be represented as in Theorem [5.5](#_bookmark26).

Algorithms [1](#_bookmark29) and [2](#_bookmark37) compute the representation modulo satisfiability of *nψ*. Algorithm [1](#_bookmark29) returns **0** and *ψ* in the limit cases *n* = 0 and *n* = 1 (lines [1](#_bookmark30) to [5](#_bookmark31)); when *n ∈* N *\ {*0*,* 1*}*, it returns formula *ξnψ* in ([11](#_bookmark22)) by building it in line [6](#_bookmark32) plus a *[*log *n♩* + 1 iteration loop (lines [7](#_bookmark33) to [13](#_bookmark36)) where the *nk*’s in the binary representation of *n* are calculated by the routine in lines [8](#_bookmark34) and [9](#_bookmark35). Algorithm [2](#_bookmark37) returns ∅ in the limit cases *n* = 0 and *n* = 1 (lines [1](#_bookmark38) to [3](#_bookmark40)); when *n ∈* N *\ {*0*,* 1*}*, it returns set Ξ*nψ* that comprehends formulas ([12](#_bookmark23)) by building it in line [4](#_bookmark39) plus a *[*log *n♩* iteration loop (lines [5](#_bookmark41) to [7](#_bookmark42)). Both algorithms terminate in time *O*(log *n*) assuming propositional variables are all represented with a constant size.

*ϕp* :*ξ*1 2 1

1 *¬*(*ξZ*

1

18

*⊕ξ p*1 *→***0**)

2

*Z*

Φ*p* :*Z*  1

*↔¬* *ξ*4 *⊕ ξ*0 *ξ*0 *p*

*↔ Zp*1

1 18

*Z* 1 *Z* 1

18 18

*Z*2 1 2

*ξ*0 *↔ Z*  1 *ξ*1 *p*1 *↔ ξ*0 *p*1 *⊕ ξ*0 *p*1

*Z*  1 18

18

*Z*2 *Z*2 *Z*2

*ξ*1 *↔ ξ*0

*⊕ ξ*0

*ξ*2 *p*1 *↔ ξ*1 *p*1 *⊕ ξ*1 *p*1

*Z*  1 18

2

*ξ*

*Z*  1 18

*ξ*3

*Z*  1 18

1

*↔ ξ*

*Z*  1 18

*↔ ξ*2

*Z*  1 18

1

*⊕ ξ*

*Z*  1 18

*⊕ ξ*2

*Z*2 *Z*2

0 *↔ Z* 1

*ξ*

*Z*

1 6

6

*ξ*1 *↔ ξ*0

*Z*2

*⊕ ξ*0

*Z*  1 18

4

*ξ*

*Z*  1

18

*Z*  1 18

3

*↔ ξ*

*Z*  1

18

*Z*  1 18

3

*⊕ ξ*

*Z*  1

18

*Z* 1 *Z* 1

6 6

2 *↔ ξ*1

*ξ*

*Z*

*Z*

1 1

6 6

*Z* 1

6

*⊕ ξ*1

*Z*

1

6

*Z* 1 *↔¬* *ξ*2 *⊕ ξ*0

*ξ*0 2 1

*↔¬* *ξ*2 *⊕ ξ*1 *p*1 *→* **0**

6 *Z* 1 *Z* 1

*¬*(*ξZ*  1 *⊕ξ p*1 *→***0**)

*Z*  1 *Z*2

6 6 18 *Z*2 18

*ξ*2 *p*1 *⊕ ξ*1 *p*1 *↔ X*2 *ξ*1 2 1

*↔ ξ*0 2 1

*⊕ ξ*0 2 1

*Z*2 *Z*2

*¬*(*ξZ*  1 *⊕ξZp*1 *→***0**)

*¬*(*ξZ*  1 *⊕ξZp*1 *→***0**)

*¬*(*ξZ*  1 *⊕ξZp*1 *→***0**)

*Zp*1 *→ Z* 1

18 2

18 2

18 2

2

*ϕp* :*¬* *ξ*2

6

*⊕ ξ*0

*→ Zp*2

2 *Z* 1 *Z* 1 2

6 6

Φ*p* :*Z* 1 *↔¬* *ξ*2

*⊕ ξ*0

*ξ*1

*↔ ξ*0

*⊕ ξ*0

2 6 *Z* 1 *Z* 1

6 6

*Z* 1 *↔ ¬Z* 1

*Z* 1 *Z* 1

6 6

*ξ*2 *↔ ξ*1

*Z* 1

6

*⊕ ξ*1

2 2 *Z* 1

6

*Z* 1 *Z* 1

6 6

*ξ*1 *p*2

*Z*2

*↔ X*2 *ξ*0 *p*2

*Z*2

*↔ Zp*2

*Zp*2 *→ Z* 1

2

*ξ*1 *p*

*↔ ξ*0 *p*

*⊕ ξ*0 *p*

2 2

0 *↔ Z* 1

*ξ*

*Z*

1 6

6

*Z*2 2

*Z*2 2

*Z*2 2

*ϕp* :*ξ*1 2 *p*3

3 *¬*(*ξ*

*Z*

1

6

*→Z*1 )

Φ*p* :*Z* 1 *↔¬* *ξ*2

*⊕ ξ*0

*Zp*3 *→ Z* 1

3 6 *Z* 1 *Z* 1

6 6

0 *↔ Z* 1

*ξ*

*Z*

1 6

6

1

*ξ*0 *p*3

*Z*1

2

*↔ Zp*3

1

*ξ*1 *↔ ξ*0

*⊕ ξ*0

*ξ*1 *p*3 *↔ ξ*0 *p*3 *⊕ ξ*0 *p*3

*Z* 1 *Z* 1

6 6

2 *↔ ξ*1

*ξ*

*Z*

*Z*

1 1

6 6

*Z* 1

6

*⊕ ξ*1

*Z*

1

6

*Z*1

*ξ*0 2

*¬*(*ξZ* 1

6

*Z*1

*→Zp*3 )

1

*Z*1

2 *→ Zp*3

*↔¬* *ξ*

1

*Z* 1

6

*Z* 1 *↔ ¬Z* 1

*ξ*1 2

*p*3 *↔ ξ*0 2

*p*3 *⊕ ξ*0 2 *p*3

2 2 *¬*(*ξZ* 1 *→Z*1 )

6

*¬*(*ξZ* 1 *→Z*1 )

6

*¬*(*ξZ* 1 *→Z*1 )

6

*ξ*1 *p*3 *↔ X*1

*Z*1

Table 3

Representations as in (13) for functions *p*#, *p*# and *p*#, where functions *p*1, *p*2 and *p*3 are from Example

1 2 3

4.1.

**Algorithm 1** BINARY-F: computes formula *ξnψ* in ([11](#_bookmark22)) or **0** or *ψ*

**Input:** A natural number *n* and a formula *ψ*.

**Output:** Formula *ξnψ*.

1: **if** *n* =0 **then**

2: **return 0**;

3: **else if** *n* =1 **then**

4: **return** *ψ*;

5: **end if**

6: *q* := *n*, *nk* := 0, *ξnψ* := **0**;

7: **for** *k* = 0*,..., [*log *n♩* **do**

8: *nk* := remainder from division of *q* by 2;

9: *q* := quotient from division of *q* by 2;

10: **if** *nk* =1 **then**

11: *ξnψ* := *ξk ⊕ ξnψ*;

*ψ*

12: **end if**

13: **end for**

14: **return** *ξnψ*;

**Algorithm 2** BINARY-S: computes set Ξ*nψ* in ([12](#_bookmark23)) or ∅

**Input:** A natural number *n* and a formula *ψ*.

**Output:** Set Ξ*nψ*.

1: **if** *n* =0 or *n* =1 **then**

2: **return** ∅;

3: **end if**

4: Ξ*nψ* := *{ξ*0 *↔ ψ}*;

*ψ*

5: **for** *k* = 1*,..., [*log *n♩* **do**

6: Ξ*nψ* := Ξ*nψ ∪ {ξk ↔ ξk—*1 *⊕ ξk—*1*}*;

*ψ ψ ψ*

7: **end for**

8: **return** Ξ*nψ*;

Algorithm [3](#_bookmark44) computes a representation modulo satisfiability for *p*#. It returns

*⟨***0***,* ∅*⟩* in the limit case *a*0 = *···* = *an* = 0 (lines [1](#_bookmark45) to [3](#_bookmark40)); otherwise it returns representation *⟨ϕp,* Φ*p⟩* given in ([13](#_bookmark27)). From line [4](#_bookmark46) to line [15](#_bookmark47), the algorithm sets all *P* , *N* , *α**j*, *βj*, and *β*, for *j* = 0*,..., n*, which are used to rewrite function *p* in terms of *pP* and *pN* as in Lemma [5.1](#_bookmark17). From line [16](#_bookmark48) to line [26](#_bookmark52), it writes formulas *ϕP* and *ϕN* and adds formulas in Φ*P* and Φ*N* to Φ*p*. For *J ∈ {P, N}*, it works throughout

a *|J|* iteration loop where each iteration takes a coefficient *aj*

*b*

*j*

into account, where

it treats *a*0

*b*

0

(lines [18](#_bookmark49) to [21](#_bookmark50)) separately from the others (lines [22](#_bookmark51) to [25](#_bookmark53)). In lines [27](#_bookmark52)

and [28](#_bookmark54) it finally writes formula *ϕp* and completes set Φ*p*.

**Theorem 5.6** *Given a rational linear function p by its coefficients, a representation*

*⟨ϕp,* Φ*p⟩ for p*# *may be computed in polynomial time by Algorithm* [*3*](#_bookmark44)*.*

**Proof.** Algorithm [3](#_bookmark44) builds representation *⟨ϕp,* Φ*p⟩* in ([13](#_bookmark27)). So, its correctness fol- lows from Theorem [5.5](#_bookmark26). Let [0*,* 1]*n* be the domain of *p* and *M* the maximum size

**Algorithm 3** REPRESENT-TL: computing representations for truncated linear functions

**Input:** A linear function *p* given by its rational coefficients *a*0 *, a*1 *,..., an* .

*b*0 *b*1 *bn*

**Output:** A representation *⟨ϕp,* Φ*p⟩* for the truncated function *p*#.

1: **if** *a*1 = *···* = *an* =0 **then**

2: **return** *⟨***0***,* ∅*⟩*;

3: **end if**

4: *P* := ∅, *N* := ∅;

5: **for** *j* := 0*,...,n* **do**

6: **if** *aj >* 0 **then**

7: *P* := *P ∪ {j}*, *αj* := *aj*;

8: **else if** *aj <* 0 **then**

9: *N* := *N ∪ {j}*, *αj* := *−aj*;

10: **end if**

11: **end for**

Σ

*, −* Σ

12: *β* := least integer greater than or equal to max*{*

13: **for** *j ∈ P ∪ N* **do**

14: *βj* := *β · bj*;

15: **end for**

16: *ϕP* := **0**, *ϕN* := **0**, Φ*p* := ∅;

17: **for** *J* = *P, N* **do**

18: **if** 0 *∈ J* **then**

19: *ϕJ* := *ϕJ ⊕* BINARY-F(*α*0*,Z*  1 );

*β*0

*aj*

*j∈P bj*

*aj*

*j∈N bj*

*}*;

20: Φ*p* := Φ*p ∪ {Z*  1 *↔ ¬*BINARY-F(*β*0 *−* 1*,Z*  1 )*}∪* BINARY-S(*α*0*,Z*  1 ) *∪*

*β*0 *β*0 *β*0

BINARY-S(*β*0 *−* 1*,Z*  1 );

*β*0

21: **end if**

22: **for** *j ∈ J \ {*0*}* **do**

23: *ϕJ* := *ϕJ ⊕* BINARY-F(*αj, Zp*);

*j*

*p*

24: Φ*p* := Φ*p ∪ {Z*  1

*βj*

*↔ ¬*BINARY-F(*βj −* 1*,Z*  1 )*,* BINARY-F(*βj, Zj* ) *↔*

*βj*

*Xj, Zp → Z*  1 *} ∪* BINARY-S(*αj, Zp*) *∪* BINARY-S(*βj −* 1*,Z*  1 ) *∪*

*j βj j βj*

BINARY-S(*βj, Zp*);

*j*

25: **end for**

26: **end for**

27: *ϕp* := BINARY-F(*β, ¬*(*ϕP → ϕN* ));

28: Φ*p* := Φ*p ∪* BINARY-S(*β, ¬*(*ϕP → ϕN* ));

29: **return** *⟨ϕp,* Φ*p⟩*;

of a binary representation for numbers among *aj* and *bj*; then the input size of *p* is at most 2(*n* + 1)*M* . The algorithm first calculates in polynomial time all *β*, *αj* and *βj*; let *μ* be the maximum size of a binary representation for numbers among *β*, *αj* and *βj*. Then, it proceeds to writing the representation which is made up of at

most 3(*n* + 1) propositional variables of the type *Xj*, *Zp* and *Z*  1 , and 2(*n* + 1)*μ* + *μ*

*j βj*

propositional variables of the type *ξk* , a quantity polynomially proportional to the size of the input. Thus, the size of the representation for each propositional variable may be assumed to be a constant *π* also polynomially proportional to the size of the input. Next, the algorithm calculates formulas *ϕP* and *ϕN* and sets Φ*P* and Φ*N* in *n* + 1 steps; in each one it calculates the part associated to a coefficient *αi* . For

*ψ*

*β*

*i*

each part, computation takes polynomial time on *π* and at most three executions

of routines BINARY-F (Algorithm [2](#_bookmark37)) and BINARY-S (Algorithm [1](#_bookmark29)) with argument

*⟨ν, P⟩*, where *ν* is *αi*, *βi* or *βi−*1, which are already or may be quickly computed, and *P* is a propositional variable. In these cases BINARY-F and BINARY-S run in poly- nomial time on *μ* and *π*. The algorithm finishes calculating *ϕp* and Φ*p* by running BINARY-F and BINARY-S with argument *⟨β, ¬*(*ϕP → ϕN* )*⟩*. Now, BINARY-F runs in polynomial time on *μ* and *π* and BINARY-S runs in polynomial time on *μ*, *π* and the size of *¬*(*ϕP → ϕN* ). After all, Algorithm [4](#_bookmark61) terminates in polynomial time. *2*

We call REPRESENT-TL-F and REPRESENT-TL-S the routines that sepa- rately compute *ϕp* and Φ*p*, respectively. Both may be easily derived from routine REPRESENT-TL in Algorithm [3.](#_bookmark44)

# The General Case

Given a rational McNaughton function formatted as in Section [4](#_bookmark8), we now compute a logical representation for it. Let *f* : [0*,* 1]*n →* [0*,* 1] be a rational McNaughton function in regional format with linear pieces:

*p* (**x**)= *ai*0 + *ai*1 *x*

+ *···* + *ain x*

*,* (14)

*i bi*0 *bi*1 1

*bin n*

for **x** = *⟨x*1*,..., xn⟩ ∈* [0*,* 1]*n*, *aij ∈* Z, *bij ∈* Z*∗*

+

and *i* = 1*,..., m*, with each piece

identical to *f* in region Ω*i*, for *i* = 1*,..., m*. We call ABOVE(*pk*,*pi*) the polynomial

routine that decides if linear piece *pk* is above a different linear piece *pi* over Ω*i*.

Let *⟨ϕp ,* Φ*p ⟩* be the representation for *p*#

given by Theorem [5.5](#_bookmark26), for *i* =

*i i* *i*

1*,..., m*. We define:

*m m*

*ϕ* = *ϕ*Ω*i ,* with *ϕ*Ω*i* = *ϕpk* ; Φ = Φ*pi* ; (15)

*i*=1

*k∈K*

*i*=1

where *k ∈ K* if *pk*(**x**) *≥ pi*(**x**), for all **x** *∈* Ω*i*. We are able to state the following representation result which is adapted from [[17,](#_bookmark84)[18](#_bookmark85)].

**Lemma 6.1** *Let f be a rational McNaughton function in regional format with linear pieces given by* ([14](#_bookmark56))*, and let ϕ*Ω*i be a formula and* Φ *a set as in* ([15](#_bookmark57))*. Then, v*(*ϕ*Ω*i* ) *≤ f* (*v*(*X*1)*,..., v*(*Xn*))*, for v ∈* **Val**Φ *and i* = 1*,..., m.*

**Proof.** Let *v ∈* **Val**Φ and **x**0 = *⟨v*(*X*1)*,..., v*(*Xn*)*⟩*. In particular, *v ∈* **Val**Φ*pi* , for

*ϕ*: (*ϕp*1 *∧ ϕp*2 *∧ ϕp*3 ) *∨* (*ϕp*1 *∧ ϕp*2 *∧ ϕp*3 ) *∨* (*ϕp*1 *∧ ϕp*2 *∧ ϕp*3 )

Φ: Φ*p*1 *∪* Φ*p*2 *∪* Φ*p*3

Table 4

Representation as in ([15](#_bookmark57)) for function *f* from Example [4.1](#_bookmark11).

*i ∈ K* and, by Theorem [5.5](#_bookmark26),

*v*(*ϕ*Ω )= min *p*#(**x**0)*.*

*i k∈K k*

If **x**0 *∈* Ω*i*, then *v*(*ϕ*Ω ) *≤ p*#(**x**0)= *pi*(**x**0)= *f* (**x**0). On the other hand, if **x**0 *∈/* Ω*i*,

*i* *i*

there is some *k*0 such that **x**0 *∈* Ω*k*0 . In the case *pk*0 (**x**) *≥ pi*(**x**), for all **x** *∈* Ω*i*,

then *k*0 *∈ K* and *v*(*ϕ*Ω ) *≤ p*# (**x**0)= *pk* (**x**0)= *f* (**x**0). In the case there is **x***j ∈* Ω*i*

*i k*0 0

such that *pk* (**x***j*) *< pi*(**x***j*), continuity of *f* yields that there is *t ∈ K* such that

0

*pt*(**x**) *≥ pi*(**x**), for all **x** *∈* Ω*i* and *pt*(**x**) *≤ pk*0 (**x**), for all **x** *∈* Ω*k*0 . Therefore,

*v*(*ϕ*Ω ) *≤ p*#(**x**0) *≤ pt*(**x**0) *≤ pk* (**x**0)= *p*# (**x**0)= *f* (**x**0). *2*

*i t* 0 *k*0

**Theorem 6.2** *Any rational McNaughton function may be represented by ⟨ϕ,* Φ*⟩*

*in* ([15](#_bookmark57))*.*

**Proof.** First note that any rational McNaughton function may be put in regional format as showed in Section [4](#_bookmark8). For *⟨x*1*,..., xn⟩∈* [0*,* 1]*n*, define a valuation *v ∈* **Val**Φ

such that *v*(*Xj*)= *xj* and *v*(*Zpi* )= *xj* , for *i* = 1*,..., m,j* = 1*,..., n*, *v*(*Z*  1 )= 1 ,

*j βij*

*βij*

*βij*

for *i* = 1*,..., m,j* = 0*,..., n*, *v*(*ξ*0 )= *v*(*ψ*), and *v*(*ξk* )= min(1*, v*(*ξk—*1)+ *v*(*ξk—*1)),

*ψ ψ ψ ψ*

for *k* = 1*,..., [*log *n♩*, for any *nψ* that occurs in *ϕ* and Φ. Now, let *v, vj ∈* **Val**Φ

such that *v*(*Xj*) = *vj*(*Xj*), for *j* = 1*,..., n*. In particular, *v, vj ∈* **Val**Φ , for

*pi*

*i* = 1*,..., m*, and, by Theorem [5.5](#_bookmark26), *v*(*ϕp* ) = *vj*(*ϕp* ), for *i* = 1*,..., m*. Therefore,

*i* *i*

*v*(*ϕ*)= *vj*(*ϕ*) and **X***n* determines *ϕ* modulo Φ-satisfiable. Finally, suppose *v ∈* **Val**Φ.

There is some *k*0 *∈ K* such that *⟨v*(*X*1)*,..., v*(*Xn*)*⟩ ∈* Ω*k*0 . Note that *v*(*ϕ*Ω*k* ) =

0

*f* (*v*(*X*1)*,..., v*(*Xn*)). Therefore,

*f* (*v*(*X*1)*,..., v*(*Xn*)) = max

*i*=1*,...,m*

*v*(*ϕ*Ω*i* )= *v*(*ϕ*Ω*k* )*,*

by Lemma [6.1](#_bookmark58). *2*

0

Table [4](#_bookmark59) shows how function *f* in Example [4.1](#_bookmark11) can be represented as in Theo- rem [6.2](#_bookmark60).

Algorithm [4](#_bookmark61) returns representation *⟨ϕ,* Φ*⟩* for function *f* with linear pieces given in ([14](#_bookmark56)). From line [1](#_bookmark62) to line [13](#_bookmark66), the algorithm writes formulas *ϕ*Ω*i* and the set Φ: it first computes formulas *ϕpi* (lines [2](#_bookmark62) to [5](#_bookmark63)) by means of routine REPRESENT-TL-F and then it writes *ϕ*Ω*i* (lines [7](#_bookmark64) to [11](#_bookmark65)) by means of routine ABOVE. It writes set Φ computing each Φ*pi* by means of routine REPRESENT-TL-S (line [12](#_bookmark65)). In line [14](#_bookmark67) it writes formula *ϕ*.

**Algorithm 4** REPRESENT: computing representations for rational McNaughton functions **Input:** A rational McNaughton function *f* in regional format given by its linear

pieces coefficients *a*10 *,..., a*1*n* ,. . . , *am*0 *,..., amn*

and regions Ω1*,...,* Ω*m*.

*b*10

*b*1*n*

*bm*0

*bmn*

**Output:** A representation *⟨ϕ,* Φ*⟩* for the rational McNaughton function *f* .

1: Φ := ∅;

2: **for** *i* = 1*,...,m* **do**

3: *ϕp* := REPRESENT-TL-F( *ai*0 *,..., ain* );

*i*

4: *ϕ*Ω*i* := *ϕpi* ;

5: **end for**

6: **for** *i* = 1*,...,m* **do**

*bi*0

*bin*

7: **for** *k* = 1*,...,i −* 1*,i* + 1*,...,m* **do** 8: **if** ABOVE(*pk, pi*)= **true then** 9: *ϕ*Ω*i* = *ϕ*Ω*i ∧ ϕpk* ;

10: **end if**

11: **end for**

12: Φ :=Φ *∪* REPRESENT-TL-S( *ai*0 *,..., ain* );

13: **end for**

14: *ϕ* := *ϕ*Ω1 *∨· · · ∨ ϕ*Ω*m* ;

15: **return** *⟨ϕ,* Φ*⟩*;

*bi*0

*bin*

**Theorem 6.3** *Given a rational McNaughton function f in regional format, a log- ical representation for it may be computed in polynomial time on the size of f by Algorithm* [*4*](#_bookmark61)*.*

**Proof.** Algorithm [4](#_bookmark61) builds representation *⟨ϕ,* Φ*⟩* in ([15](#_bookmark57)). So, the algorithm correct- ness follows from Theorem [6.2.](#_bookmark60) The size of *f* is the space necessary to storage the coefficients of its *m* linear pieces *p*1*,..., pm* and the regions Ω1*,...,* Ω*m*. The algo- rithm first calculates *m* representative formulas *ϕpi* by REPRESENT-TL-F, which takes polynomial time on the size of *f* by Theorem [5.6](#_bookmark43). Then, it builds formulas *ϕ*Ω*i* from the already built representative formulas in *m*2 steps; in each of these steps it runs routine ABOVE in assumed polynomial time. Along with the above computa- tion, the algorithm also builds set Φ in *m* steps; in each one it calculates set Φ*pi* by REPRESENT-TL-S, which takes polynomial time on the size of *f* by Theorem [5.6](#_bookmark43). Finally, the algorithm calculates *ϕ* from formulas *ϕ*Ω*i* already computed. After all, Algorithm [4](#_bookmark61) terminates in polynomial time. *2*

# Conclusions

We introduced a way to represent functions by logical formulas in L- ukasiewicz

Infinitely-valued Logic — the representation modulo satisfiability —, and we showed by a constructive proof that all rational McNaughton functions can be represented this way. Moreover, we derive an algorithm that builds such a representation in polynomial time on the size of the function. For the future, we hope to couple this algorithm with algorithms that approximate (normalized) continuous functions by

rational McNaughton functions; also, apply these approximations to the study of real systems such as neural networks through automated reasoning techniques.

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