Electronic Notes in Theoretical Computer Science 210 (2008) 3–13 

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Axiomatic Description of Mixed States From Selinger’s CPM-construction

Bob Coecke[1](#_bookmark0)*,*[2](#_bookmark0)

*Oxford University Computing Laboratory, Wolfson Building, Parks Road,*

*OX1 3QD Oxford, UK.*

**Abstract**

We recast Selinger’s CPM-construction of mixed states completely positive maps [[11](#_bookmark23)] as an axiomatization of maximally mixed states. This axiomatization also guarantees categories of completely positive maps to satisfy the preparation-state agreement axiom of [[3](#_bookmark15)], and admits a physical interpretation in terms of purification of mixed states and CPMs. Internal traces, which are crucial in quantum information theory, are the adjoints to these maximally mixed states.

*Keywords: †*-compact category, completely positive maps, purification, internal trace.

# Introduction

In [[11](#_bookmark23)] Selinger proposed an intriguing construction of mixed states and completely positive maps given any *†*-compact category representing a semantics for pure state quantum informatics in the sense of Abramsky and the author [[1](#_bookmark13),[2](#_bookmark14)]. Conceptually speaking, in Selinger’s construction an ancillary system is introduced in such a way that the distinct possible interactions between pure quantum channels and this ancillary system exactly give rise to all CPMs, and hence also all mixed states, when considering their preparation procedures as a special case of quantum channels.

Since for each *†*-compact category Selinger’s construction provides another *†*- compact category, it doesn’t truly provide a profound structural grasp on quantum

1 B.C. is supported by EPSRC Advanced Research Fellowship EP/D072786/1 *The Structure of Quantum Information and its Ramifications for IT* and EPSRC Grant EP/C500032/1 *High-level methods in quantum computation and quantum information*. The motivation for this work arose partly from interaction with Yannick Delbecque, Dan Gottesman and Jon Yard at the *Quantum Information, Computation and Logic* workshop at the Perimeter Institute in July 2005, and benefited from my collaboration with Keye Martin on domain theory. Peter Selinger provided useful feedback on an earlier version of this paper.

2 Email: [coecke@comlab.ox.ac.uk](mailto:coecke@comlab.ox.ac.uk)

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doi:10.1016/j.entcs.2008.04.014

mixedness in the usual axiomatic sense. In this paper we observe that an (ad- mittedly quite minor) adjustment enables this construction to be recast as a true axiomatization. Moreover, this adjustment exactly imposes the preparation-state agreement axiom of [[3](#_bookmark15)] on the category of CPMs, that is, it explicitly requires that if two preparation procedures of pure states coincide then the resulting pure states should also coincide — note that while for **FdHilb** the category of finite-dimensional Hilbert spaces and linear maps **CPM**(**FdHilb**) does satisfy this requirement, for **C** an arbitrary *†*-compact category **CPM**(**C**) doesn’t (cf. [[3](#_bookmark15)]+[[11](#_bookmark23)]).

Let’s change perspective now. Given a *†*-symmetric monoidal category [[11](#_bookmark23)], passing to a *†*-compact category adjoins and hence axiomatizes *Bell-states* [[1](#_bookmark13),[2](#_bookmark14)], generating at its turn all entangled states and multi-partite operations. In the same vein, in this paper we adjoin and hence axiomatize *maximally mixed states*, generat- ing mixed states and CPMs. Moreover, the adjoints to the maximally mixed states provide and hence axiomatize the *internal traces*, which, rather than the Joyal- Street-Verity (JSV) partial traces [[7](#_bookmark16)] which in a *†*-compact category canonically arise as

*C A,B*

Tr

(*f* : *C ⊗ A → C ⊗ B*) :=

*λ† ◦* ( 1*C*’ *⊗* 1*B*)*†*

*B*

*◦* (1*C∗ ⊗ f* ) *◦* ( 1*C*’ *⊗* 1*A*) *◦ λA* : *A → B,*

play a crucial role in quantum information theory. To our knowledge, the need for an abstract notion of internal trace has so far only been indicated by Delbecque in [[5](#_bookmark17)], motivated by the fact that while in Selinger’s construction they arise from an underlying JSV-trace in some other categories they enjoy an autonomous existence. This same idea can also be implemented at the level of graphical calculus. While the passage from *†*-symmetric monoidal to *†*-compact introduces for each type a new primitive ingredient, e.g. ‘pink triangle’ in [[4](#_bookmark18)], which is subject to a yanking axiom, here we again introduce for each type a new primitive ingredient, which we will refer to as ‘black triangle’, which is again subject to some axiom. It remains to be seen how (dis)advantageous this graphical presentation is as compared to Selinger’s, but it does seem to have advantages when graphically trying to conceptualise the messy

zoo of all recently proposed quantum informatic quantities (e.g. [[9](#_bookmark20)]).

Finally, the notion of purification of mixed states and mixed channels, which plays an important role in the quantum information theory literature (e.g. [[8](#_bookmark21),[10](#_bookmark22)]), provides a simple physical interpretation for our adaptation of Selinger’s CPM- construction.

# Denoting types and variances

For the basic definitions of *†*-compact categories and their interpretation as se- mantics for quantum mechanics we refer to the existing literature [[1](#_bookmark13),[2](#_bookmark14),[3](#_bookmark15),[11](#_bookmark23)] and references therein. We will refer to *†*-symmetric monoidal categories as (*⊗, †*)- categories, to *†*-compact categories as (*⊗, †,* 1’)-categories, and to the categories which in addition to (*⊗, †,* 1’)-categories also contain maximally mixed states as (*⊗, †, ⊥*)-categories (see Definition [3.1](#_bookmark3) below).

When expressing naturality we will use indices on objects to refer to the involu- tions (*−*)*†*, (*−*)*∗* and (*−*)*∗* which alter the variance in that variable e.g. in the case of (*⊗, †,* 1’)-categories

**C**(I*, A∗ ⊗ B*)**C**(*A, B*)**C**(*A∗ ⊗ B†,* I)

stands for commutation of

**C**(I*, A∗ ⊗ B*) (**C**(*A, B*)) **C**(*A∗ ⊗ B,* I)

(*f∗ ⊗ g*) *◦−*

v

*g ◦−◦ f*

v

*−◦* (*f∗ ⊗ g†*)

v

**C**(I*, C∗ ⊗ D*) ( **C**(*C, D*) ) **C**(*C∗ ⊗ D,* I)

and hence in ordinary compact closed categories where we have

**C**(I*, A∗ ⊗ B*)**C**(*A, B*)**C**(*A ⊗ B∗,* I)

the *∗*-symbol now also specifies alteration of the variance (besides merely assigning the dual object). The same convention applies to typed expressions since *f* : *Aq → Bb* stands for *f ∈* **C**(*Aq, Bb*), and we can compress the size of the expression

*f* : *Aq → Bb* by setting *f*

*q→ b*

*A B*

. Dirac notations for *states |ψ⟩* and *co-states ⟨ψ|*

respectively arise as *ψ*I*→A* and *ψ†*

*A†→*I

so our notation is in fact a refinement of

Dirac’s by providing explicit *types* and additional data on *variances*.

When setting *C* := *A*, *D* := *C*, *f* := 1*A* and using *compositionality* [[1](#_bookmark13)]

*g* := *λC ◦* ( 1*B*’*∗ ⊗* 1*C*)

*†*

*†*

*◦* (1*B ⊗* *g*’) *◦ ρB* : *B → C*

in the left square of the above diagram we obtain a *natural propagation of compo- sition* diagram

**C**(*A, B*) *×* **C**(*B, C*) *−◦−* ) **C**(*A, C*)

(1)

v v

where

**C**(I*, A∗ ⊗ B*) *×* **C**(I*, B∗ ⊗ C*) ) **C**(I*, A∗ ⊗ C*)

*−Δ−*

*f* ’ *g*’ := (1*A∗ ⊗ λC*) *◦* (1*A∗ ⊗* 1*B*’*∗ ⊗* 1*C*) *◦* ( *f* ’ *⊗* *g*’) *◦ ρ*I

*Δ † †*

i.e. we obtain a CUT-like composition (cf. [[1](#_bookmark13)]).

# Maximally mixed states, internal trace, purification

The following definition introduces *maximally mixed states* (*⊥*-states) as the *gener- ator* of mixedness, in analogy to 1’-states constituting the *generator* of entangle- ment.

**Definition 3.1** A *⊥-structure* on a (*⊗, †*)-category **C** comprises

* 1. a *maximally mixed state ⊥A* : I *→ A* for each object *A* which moreover satisfies

*⊥*I = 1I and *⊥A⊗B* = (*⊥A ⊗ ⊥B*) *◦ λ*I,

* 1. an all-objects-including sub-(*⊗, †*)-category **C**Σ of *pure states* which comes equipped with a 1’-structure,

which are such that for all *f, g ∈* **C**Σ we have

1. *f ◦ f†* = *g ◦ g† ⇐⇒ f ◦ ⊥*dom(*f* ) = *g ◦ ⊥*dom(*g*) *.*

In words, axiom ([2](#_bookmark4)) states when two mixed states *f ◦ ⊥*dom(*f* ) and *g ◦ ⊥*dom(*g*)

obtained by *acting with pure operations f and g on a maximally mixed state ⊥* coincide. There are two important special cases. **i.** Setting dom(*f* ) = dom(*g*) := I in axiom ([2](#_bookmark4)) and using *⊥*I = 1I we obtain

1. *ψ ◦ ψ†* = *φ ◦ φ†* =*⇒ ψ* = *φ*

i.e. the *preparation-state agreement axiom* [[3](#_bookmark15)]. **ii.** Setting *g* := 1codom(*f*) in axiom

([2](#_bookmark4)) we obtain

1. *f ◦ ⊥*dom(*f* ) = *⊥*codom(*f* ) *⇐⇒ f ◦ f†* = 1codom(*f* )

which expresses under which pure operations the maximally mixed state remains invariant, in particular including all unitary operations. Also, from naturality of *λA*, *ρA*, *σA,B*, *αA,B,C* and their coherence, together with *⊥A⊗B* = (*⊥A ⊗ ⊥B*) *◦ λ*I and *⊥*I = 1I we obtain

*⊥*I*⊗A* = *λA ◦ ⊥A ⊥B⊗A* = *σA,B ◦ ⊥A⊗B ⊥*(*A⊗B*)*⊗C* = *αA,B,C ◦ ⊥A⊗*(*B⊗C*) *.*

**Definition 3.2** In a (*⊗, †, ⊥*)-category the *partial internal trace* is the map

*C A,B*

tr

: **C**(*A, C ⊗ B*) *→* **C**(*A, B*) :: *f '→ λ†*

* (*⊥†*

*⊗* 1*B*) *◦ f*

for every three objects *A*, *B* and *C*, and the *full internal trace* is the map

*B*

*C*

tr*C* : **C**(I*, C*) *→* **C**(I*,* I) :: *ψ '→ ⊥† ◦ ψ*

*C*

for every two objects *A* and *B*.

Somewhere in the middle between the partial and the full trace we encounter the cases

t˜r*C* : **C**(*A, C*) *→* **C**(*A,* I) :: *f '→ ⊥† ◦ f*

and

*A C*

tr*C* : **C**(I*,C ⊗ A*) *→* **C**(I*, A*) :: Ψ *'→ λ† ◦* (*⊥† ⊗* 1*A*) *◦* Ψ *.*

*A A C*

**Definition 3.3** In a (*⊗, †, ⊥*)-category define a *puriﬁcation* of an operation *f* : *A →*

*B* to be a pure operation *g* : *A → C ⊗B* (i.e. in **C**Σ) which is such that *f* = tr*C* (*g*).

*A,B*

An operation is *puriﬁable* if it admits a purification. A purifiable operation can (and usually does) admit many different purifications, even many different purifications of the same type. A special case of purifications are purifications Ψ*ρ* : I *→ C ⊗ A* of mixed states *ρ* : I *→ A*, which play an important role in the standard quantum information theory literature.

Next we generalize the canonical JSV-traces which exist in (*⊗, †,* 1’)-categories by relaxing the unit of compactness 1*A*’ : I *→ A∗ ⊗A* to the name *f* ’ : I *→ C ⊗A* of arbitrary morphisms *f* : *C∗ → A*, or equivalently, by compactness, to arbitrary bipartite states Ψ : I *→ C ⊗ A*.

**Definition 3.4** Given Ψ : I *→ C ⊗ A* in a (*⊗, †, ⊥*)-category the Ψ*-trace* is

Tr(Ψ) : **C**(*A ⊗ E, A ⊗ E'*) *→* **C**(*E, E'*) ::

*f '→ λE' ◦* (Ψ *⊗* 1*E'* )

*†*

*†*

* (1*C ⊗ f* ) *◦* (Ψ *⊗* 1*E*) *◦ λE .*

Denote by *ϕρ* : *C∗ → A* the pure operation which is such that *ϕρ*’ = Ψ*ρ*, where Ψ*ρ* is a purification of a mixed state *ρ*. Below read “ Ψ*ρ*” as “some purification of *ρ* ”, with obvious analogue for “ *ϕρ*”, to which we, in the vein of *†*-compactness, will also refer to as a purification of *ρ*.

The following result provides a physical interpretation for axiom [2](#_bookmark4).

**Proposition 3.5** *With the assumptions of Deﬁnition* [*3.1*](#_bookmark3) *the following are equiva- lent* :

* 1. *axiom* ([2](#_bookmark4))*,*
  2. *for all puriﬁable ρ, ρ'* : I *→ A we have*

*† † '*

*ϕρ ◦ ϕρ* = *ϕρ' ◦ ϕρ'* =*⇒ ρ* = *ρ ,*

* 1. *for all puriﬁable ρ, ρ'* : I *→ A we have*

Tr(Ψ*ρ*) = Tr(Ψ*ρ'* ) =*⇒ ρ* = *ρ .*

*'*

**Proof:** We have *i ⇔ ii* by the definition of *ϕρ* and *ii ⇔ iii* since by

Tr(Ψ*ρ*) = *λE' ◦* ( 1*A*’ *⊗* 1*E'* )

*†*

*†*

* ((*ϕρ ◦ ϕ†* )*∗*

*⊗ −*) *◦* ( 1*A*’ *⊗* 1*E*) *◦ λE*

and *ϕ*

*ρ*

* + *ϕ†* = Tr(Ψ )(*σ*

) it follows that Tr(Ψ ) and *ϕ ◦ ϕ†* are in bijective cor-

*ρ ρ ρ*

*A,A*

*ρ ρ ρ*

respondence.

The last implication expresses that Tr(Ψ*ρ*) does not depend on the particular choice of purification. This for example implies that Schumacher’s [[10](#_bookmark22)] *entanglement ﬁdelity* of a state *ρ* with respect to channel/operation *f* : *A → A*, in our language defined as Tr(Ψ*ρ*)(*f* ), does not depend on the “particular details of the purification process”.

# Properties of purifiable operations

Denote by **C**purif the ‘(*⊗, †,* 1’*, ⊥*)-category’ of all purifiable operations (see Propo- sition [4.1](#_bookmark6) below).

**Proposition 4.1** *In a* (*⊗, †, ⊥*)*-category* **C** *the* 1’*-structure of* **C**Σ *and the fact that* **C**Σ *satisﬁes the preparation-state agreement axiom lift to* **C**purif*, which also inherits the ⊥-structure from* **C***.*

**Proof:** One easily verifies that ‘purifiability’ is closed under *◦*, *⊗* and *†*, that operations in **C**Σ are trivially purifiable, and in particular that 1*A*’ is a purification of *⊥A*. Hence the only non-trivial part of the proof constitutes satisfaction of preparation-state agreement. It suffices to show that for all *ρ, ρ'* : I *→ A* we have *ρ ⊗ ρ∗* = *ρ' ⊗ ρ'∗ ⇒ ρ* = *ρ'* (see [[3](#_bookmark15)]). For *ϕρ* (resp. *ϕρ'* ) a purification for *ρ* (resp. *ρ'*) we have that *ϕρ ⊗* (*ϕρ*)*∗* (resp. *ϕρ' ⊗* (*ϕρ'* )*∗*) is a purification of *ρ ⊗ ρ∗* (resp. *ρ' ⊗ ρ'* )

*∗*

‘up to *⊗*-natural isomorphisms’. We have

*ρ ⊗ ρ∗* = *ρ' ⊗ ρ'∗*

*⇔* (*ϕρ ⊗* (*ϕρ*)*∗*) *◦* (*ϕρ ⊗* (*ϕρ*)*∗*) = (*ϕρ' ⊗* (*ϕρ'* )*∗*) *◦* (*ϕρ' ⊗* (*ϕρ'* )*∗*)

*† †*

*† † † †*

*⇔* (*ϕρ ◦ ϕρ*) *⊗* (*ϕρ ◦ ϕρ'* )*∗* = (*ϕρ ◦ ϕρ'* ) *⊗* (*ϕρ' ◦ ϕρ'* )*∗*

*⇔ ϕ ◦ ϕ†* = *ϕ ◦ ϕ†*

*⇔ ρ* = *ρ'*

*ρ ρ ρ ρ'*

by Proposition [3.5](#_bookmark5), bifunctoriality, preparation-state agreement for **C**Σ and again Proposition [3.5](#_bookmark5) respectively, what completes this proof.

Since Tr(Ψ*ρ*) does not depend on the choice of purification we can denote it by Tr*ρ*. More generally, due to the 1’-structure, also for any purifiable operation *g* : *B → A* the mapping

Tr*g* : **C**(*A ⊗ E, A ⊗ E'*) *→* **C**(*B ⊗ E, B ⊗ E'*) ::

*f '→* (*h ⊗* 1*E'* ) *◦* (1*C ⊗ f* ) *◦* (*h ⊗* 1*E*)

*†*

where *h* : *B → C ⊗ A* is any purification of *g* : *B → A* is well-defined. Recall from [[3](#_bookmark15),[11](#_bookmark23)] that a morphism *f* : *A → A* in a (*⊗, †*)-category is *positive* if it decomposes as *f* = *g† ◦ g* for some morphism *g* : *A → B*. Denote all purifiable states of type I *→ A* by **C**purif(I*, A*) and all positive morphisms in **C**Σ of type *A → A* by **C**pos(*A†, A*). We will use the notationΣ to denote naturality with respect to composition with pure operations.

Σ

**Proposition 4.2** *Axiom* ([2](#_bookmark4)) *is equivalent to the existence of a monoidal natural bijection*

mix : **C**pos(*A†, A*)Σ **C**purif(I*, A*) *.*

Σ

*This monoidal natural bijection moreover induces commutation of*

**C**purif (I

*, A∗*

*⊗ B*) *×* **C**purif

(I*, B∗*

*⊗ C*)

*−Δ−* )

**C**purif

(I*, A∗*

*⊗ C*)

(5)

mix*−*1

v

mix*−*1

v

**C**pos(*A∗ ⊗ B, A∗ ⊗ B*) *×* **C**pos(*B∗ ⊗ C, B∗ ⊗ C*) ) **C**pos(*A∗ ⊗ C, A∗ ⊗ C*)

Σ Σ

*where f♦g is deﬁned to be*

*−♦−* Σ

(1*A∗ ⊗ λC*)*† ◦* (1*A∗ ⊗* 1*B*’*∗ ⊗* 1*C*)*† ◦* (*f ⊗ g*) *◦* (1*A∗ ⊗* 1*B*’*∗ ⊗* 1*C*) *◦* (1*A∗ ⊗ λC*) *.*

**Proof:** Setting

mix : *f ◦ f† '→ f ◦ ⊥*dom(*f* ) *,*

the restriction to **C**pos assures totality, the forward implication of axiom ([2](#_bookmark4)) assures well-definedness, the backward direction assures injectivity, restriction to **C**purif as- sures surjectivity, and monoidal naturality, i.e. commutation of

Σ

**C**pos(*A, A*) m)ix

Σ

**C**purif(I*, A*)

*g ◦−◦ g†*

v

*g ◦−*

v

**C**pos(*B, B*) ) **C**purif(I*, B*)

Σ mix

where *g* is pure together with ‘good’ behavior of mix w.r.t. *⊗*, follow straightfor- wardly — note in particular that the action *g ◦−◦ g†* : **C**pos(*A†, A*) *→* **C**pos(*B†, B*) indeed preserves positivity of morphisms. When setting *⊥A* := mix(1*A*) the converse is also straightforward. For *f* : *D → A⊗C* pure and *h* : *B → C⊗A* a co-purification of *g* : *B → A* we have

Σ

mix(*h ◦* (1*C ⊗ f ◦ f†*) *◦ h†*) = *h ◦* (1*C ⊗ f* ) *◦ ⊥C⊗D* = *h ◦* (*⊥C ⊗* 1*D*) *◦ λD ◦* mix(*f ◦ f†*)

` ˛¸ x

Tr*g* (*f◦f†*)

` ˛¸ x

*g*

so we also have commutation of the more general diagram

**C**pos(*A, A*) m)ix

Σ

**C**purif(I*, A*)

(6)

Tr*g*

v

*g ◦−*

v

**C**pos(*B, B*) ) **C**purif(I*, B*)

Σ mix

where *g* now only has to be purifiable. Diagram ([5](#_bookmark7)) now also easily follows.

From diagram ([6](#_bookmark9)) in the above proof it follows that axiom ([2](#_bookmark4)) in Definition [3.1](#_bookmark3) can in fact be extended from pure operations to all purifiable operations.

**Corollary 4.3** *In a* (*⊗, †, ⊥*)*-category for all f, g ∈* **C**purif *we have*

(7) Tr*f* = Tr*g ⇐⇒ f ◦ ⊥*dom(*f* ) = *g ◦ ⊥*dom(*g*) *.*

# Recovering Selinger’s CPM-construction

Denote by **C**pos the graph with the same objects as **C** but morphisms restricted to the positive ones. [3](#_bookmark10) We will now present Selinger’s CPM-construction of [[11](#_bookmark23)], slightly modified such that it fits better the needs of this paper. Given a (*⊗, †,* 1’)-

Σ

3 Note that above we implicitly made the convention **C**pos := (**C**Σ)pos.

category **C** define a new category **CPM**(**C**) which has the same objects as **C**, but which has as morphisms

**CPM**(**C**)(*A, B*) := **C**pos(*A∗ ⊗ B, A∗ ⊗ B*)

with *♦* as composition and hence which has 1*A*’ *◦* ( 1*A*’)*†* as identities. Selinger went on showing that **CPM**(**C**) is again a (*⊗, †,* 1’)-category and in particular that **CPM**(**FdHilb**) is the category which has completely positive maps as morphisms and (not necessarily normalized) density matrices as its *elements* i.e. morphisms with as type C *→ H*. Note here that if *f ∈* **CPM**(**C**)(*A, B*) = **C**pos(*A∗ ⊗B, A∗ ⊗B*) then by positivity *f* = *g† ◦ g*, and each choice for such a *g†* : *C → A∗ ⊗ B* yields in fact a purification for the operation *f* in the sense of Section [3](#_bookmark2).

**Theorem 5.1** *If* **C** *carries a ⊥-structure then* **CPM**(**C**Σ)**C**purif*.*

**Proof:** By Proposition [4.2](#_bookmark8) we have

**C**purif(*A, B*)**C**purif(I*, A∗ ⊗ B*)**C**pos(*A∗ ⊗ B, A∗ ⊗ B*) D=ef*.* **CPM**(**C**Σ)(*A, B*) and diagrams ([1](#_bookmark1)) and ([5](#_bookmark7)) guarantee that also composition carries over.

Σ

Selinger also introduced the canonical identity-on-objects mapping

*F***CPM** : **C** *→* **CPM**(**C**) :: *f '→* *f* ’ *◦* ( *f* ’)*†* which due to the variances (cf. composition in **CPM**(**C**) is *♦*)

(8)

**C**(*A, B*) *F***CP**)**M**

**C**(*A∗ ⊗ B†, A∗ ⊗ B*) =: **CPM**(**C**)(*A, B*)

provides a functorial passage from **C** to **CPM**(**C**), and the intended interpretation of the range of this functor are pure operations/states. In general *F***CPM** is not faithful and this is due to the fact that in general **C** does not satisfy preparation- state agreement. [4](#_bookmark12)

**Lemma 5.2** *For a* (*⊗, †,* 1’)*-category* **C** *the following are equivalent:*

1. **C**pos *satisﬁes the preparation-state agreement axiom* ;
2. **C**pos *F***CPM**[**C**pos];
3. **CPM**(**C**) *satisﬁes the preparation-state agreement axiom* ;
4. **CPM**(**C**)**CPM**(*F***CPM**[**C**]) ;
5. **CPM**(**C**)**CPM**(**C***'*) *for some* **C***' which satisﬁes preparation-state agree- ment* ;

*where all isomorphisms are assumed to be canonical ones.*

**Proof:** Equivalences **1** *⇔* **2** and **3** *⇔* **4** follow by the fact that the preparation-state agreement axiom can be stated as *f* = *g ⇔ F***CPM**(*f* ) = *F***CPM**(*g*), and **1** *⇔* **3** fol- lows straightforwardly by the definition of **CPM**(**C**). **3**,**4** *⇒* **5**: if **CPM**(**C**) satisfies the preparation-state agreement axiom then so does *F***CPM**[**C**], hence **5** follows by

4 In [[3](#_bookmark15)] the preparation-state agreement axiom was derived as a fixed point with respect to *F***CPM**, which was introduced as a construction which ‘eliminates global phases’, independent of the Selinger’s **CPM**- construction.

**4** for **C***'* := *F***CPM**[**C**]. **5** *⇒* **3**: if **C***'* satisfies the preparation-state agreement axiom then so does **CPM**(**C***'*) and hence so does **CPM**(**C**).

The equivalent conditions **1–5** in Lemma [5.2](#_bookmark11) do not require **C** itself to sat- isfy the preparation-state agreement axiom i.e., equivalently, **C** *F***CPM**[**C**]. A counter example is **FdHilb**. But they are slightly stronger than only requiring that *F***CPM**[**C**] satisfies the preparation-state agreement axiom i.e., equivalently, *F***CPM**[**C**] *F***CPM**[*F***CPM**[**C**]].

**Theorem 5.3** *If* **C** *is a* (*⊗, †,* 1’)*-category then*

*⊥A* := *F***CPM**(1*A*) and **CPM**(**C**)Σ := *F***CPM**[**C**]

*deﬁne a ⊥-structure on* **CPM**(**C**) *iff the equivalent conditions* **1–5** *in Lemma* [*5.2*](#_bookmark11) *hold.*

**Proof:** Since **CPM**(**C**)Σ(*A, B*) = *F***CPM**[**C**(*A, B*)] and the fact that positivity is a compositionally defined property with *F***CPM** being functorial we have

**CPM**(**C**)pos(*A†, A*) = *F***CPM**[**C**pos(*A†, A*)] *.*

Σ

Hence, since we also have that

**CPM**(**C**)purif(I*, A*) D=ef*.* **C**pos(I*∗ ⊗ A†,* I*∗ ⊗ A*)**C**pos(*A†, A*)

condition **2** in Lemma [5.2](#_bookmark11) (i.e. the restriction of *F***CPM** to **C**pos is faithful) suffices in the light of Proposition [4.2](#_bookmark8) to establish a *⊥*-structure on **CPM**(**C**).

Hence we can indeed conclude that:

*⊥*-structure *≡* CPM-construction **+** preparation-state agreement

That is, more precisely, carrying a *⊥*-structure coincides with the subcategory of purifiable operations being isomorphic to a category **CPM**(**C**) which is the re- sult of applying Selinger’s CPM-construction to a category **C** which satisfies the preparation-state agreement axiom (cf. **5** in Lemma [5.2](#_bookmark11)), and this satisfaction of the preparation-state agreement axiom of that underlying category in turns coin- cides with the subcategory of purifiable operations, or equivalently, **CPM**(**C**) itself satisfying the preparation-state agreement axiom (cf. **3** in Lemma [5.2](#_bookmark11)).

# Introducing black triangle, and outlook

Graphically (cf. [[4](#_bookmark18)]), Selinger’s CPM-construction, of which we now consider the covariant version of [[11](#_bookmark23)] (and not the version considered above), boils down to ‘restricting’ to operations of the shape:

*B*

*f*

*C*

*C\**

*B\**

*f\**

*A*

*A\**

where *f* : *A ⊗ C → B* is a (co)purification of the operation of type *A → B* under consideration. This pictures carries some sort of redundancy in that they both involve *f* and a copy of it subjected to (*−*)*∗*. We can reduce this notation by introducing a new primitive notion, referred to above as maximally mixed states, and depicted as a black triangle:

*B*

*f*

*C*

*A*

which is subject to the graphical counterpart to axiom ([2](#_bookmark4)). In this representation quantitative notions such as Reimpell and Werner’s *channel ﬁdelity*, Schumacher’s *entanglement ﬁdelity* and Devetak’s *entanglement generating capacity* (see [[9](#_bookmark20)] and references therein) emerge naturally as:



*f*

*C*



*f*

*C*



*g\**

*f*

*g*

*C*

We intend to systematically analyse these important quantitative notions of quantum information theory in this qualitative manner, and cast them within a uniform theory. We expect that new canonical and unifying notions will emerge. This work is still in progress, and hence is not fully represented here, but we do expect a compositional theory on quantum informatic resources to emerge, which substantially extends the recent proposals by Devetak, Harrow and Winter in [[6](#_bookmark19)].

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