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[](http://crossmark.crossref.org/dialog/?doi=10.1016/j.eij.2020.08.002&domain=pdf)Bi-parametric distance and similarity measures of picture fuzzy sets and their applications in medical diagnosis

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The concept of picture fuzzy sets (PFS) is a generalization of ordinary fuzzy sets and intuitionistic fuzzy sets, which is characterized by positive membership, neutral membership, and negative membership functions. Keeping in mind the importance of similarity measures and applications in data mining, med- ical diagnosis, decision making, and pattern recognition, several studies have been proposed in the liter- ature. Some of those, however, cannot satisfy the axioms of similarity and provide counter-intuitive cases. In this paper, we propose new similarity measures for *PFSs* based on two parameters *t* and *p*, where *t* identifies the level of uncertainty and *p* is the *Lp* norm. The properties of the bi-parametric similarity and distance measures are discussed. We provide some counterexamples for existing similarity measures in the literature and show how our proposed similarity measure is important and applicable to the pattern recognition problems. In the end, we provide an application of a proposed similarity measure for medical diagnosis.

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1. Introduction

Many classical theories such as fuzzy set theory [[1]](#_bookmark31), probability theory, vague set theory [[2]](#_bookmark31), rough set theory [[3]](#_bookmark31), intuitionistic fuzzy set (IFS) [[4]](#_bookmark31), and the interval mathematics [[5]](#_bookmark31) are well known and effectively model uncertainties. These approaches show their inherent difficulties because of intensive quantity and type of

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uncertainties as pointed out by Molodtsov [[6]](#_bookmark31). In [[6]](#_bookmark31), Molodtsov defined the soft set which is an absolutely new logical instrument for dealing uncertainties. Nowadays, many authors work to hybri- dize the different models with the soft set and achieved results in many applicable theories. The fuzzy soft set and intuitionistic fuzzy soft set (IFSS) were defined by Maji et al. [[7,8]](#_bookmark31). Then the further extensions of soft sets like the generalized fuzzy soft set [[9]](#_bookmark31), the interval-valued fuzzy soft set [[10]](#_bookmark32), the soft rough set [[11]](#_bookmark33), the vague soft set [[12]](#_bookmark34), the trapezoidal fuzzy soft set [[13]](#_bookmark35), the neutro- sophic soft set [[14]](#_bookmark36), the intuitionistic neutrosophic soft set [[15]](#_bookmark42), the multi-fuzzy soft set [[16]](#_bookmark43), and the hesitant fuzzy soft set [[17]](#_bookmark44) were introduced. Agarwal defined the generalized intuitionistic fuzzy soft set (GIFSS) as the cross product of IFSS and IFS [[18]](#_bookmark45). Later, Feng el al. [[19]](#_bookmark50) pointed out that this cross product was not well defined. Feng el al. clarified and reformulated the GIFSS as the combination of an IFSS over the universe of discourse and the IFS in the parameter set. The soft decernibility matrix approach for GIFSS was used to solve decision making problems in reference [[20]](#_bookmark52).

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In [[21]](#_bookmark54), Coung introduced the notion of PFS by including an extra membership; the ‘‘neutral membership degree”. The PFS is the gener- alization of fuzzy sets and IFSs [[22]](#_bookmark55). Coung and Kreinovich discussed PFS for computational intelligence problems [[22]](#_bookmark55). The PFS model uncertainties more effectively and directly applied to solve the daily life problems. Coung and Kreinovich explained t-norm operators for PFSs [[23]](#_bookmark57). Fuzzy logic operators for PFSs were determined by Cuong and Hai [[24]](#_bookmark59). Correlation coefficients of PFS and their applications in clustering analysis were introduced by Sing [[25]](#_bookmark62). With the help of novel fuzzy calculations based on the PFS domain time arrange- ment gauging and climate estimating was given by Son and Thong [[26]](#_bookmark63). Son [[27]](#_bookmark64), defined picture fuzzy separation, generalized distance and association measures, and connected them to tackle group inves- tigation under the PFS condition. Wei [[28]](#_bookmark65), exhibited picture fuzzy aggregation operators method and applied it to multi attribute deci- sion making (MADM) for ranking enterprise resource planning (ERP) structures. Wei [[29]](#_bookmark67), researched a basic leadership technique in light of the picture fuzzy weighted cross-entropy and used this to rank the choices. Garg [[30]](#_bookmark69), contemplated aggregation operations on PFSs and applied it to multi criteria decision-making (MCDM) problems. Pic- ture fuzzy hamarcher, geometric and Dombi aggregation operators and their applications to the MADM were demonstrated by Wei, Wang and Jana, respectively [[31–33]](#_bookmark71). For more about aggregation operators of PFSs, we refer to [[34–37]](#_bookmark37).

Yang et al. [[38]](#_bookmark37) defined the hybrid model of soft set and PFS and called it picture fuzzy soft set (PFSS). The generalized picture fuzzy soft set (GPFSS) and their applications in decision making were dis- cussed by Khan et al. [[39]](#_bookmark37). The soft decernibility matrix approach for PFSSs and GPFSSs approach was used to solve decision making problems [[38,40]](#_bookmark37). The decernibility matrix approach is important when you have to start the process with predefined conditions on membership functions, i.e., threshold values. The applications of GPFSS in concept selection were discussed in [[41]](#_bookmark37).

In this paragraph, we discuss the distance and similarity mea- sures of PFSs and the methods derived based on them. The distance and similarity measures have application in data mining, medical diagnosis, decision making and pattern recognition. In [[42]](#_bookmark37), Wei defined the picture fuzzy cross entropy and use it for MADM prob- lems. Similarity measures for PFSs based on cosine and cotangent functions and applied them for strategic decision making [[43]](#_bookmark38). Peng [[44]](#_bookmark39), determined an algorithm for PFS that was implemented to decision making problems. In [[45]](#_bookmark40), Wei interpreted cosine, weighted cosine, set theoretic, weighted set theoretic, grey and weighted grey similarity measure for picture fuzzy sets and applied them to the pattern recognition problems. Wang et al.

[[46]](#_bookmark41) used projection based VIKOR method for risk evaluation of construction project by using PFSs. Meksavang et al. [[47]](#_bookmark46) used extended VIKOR method for sustainable supplier selection using PFSs and discussed its application in beef industry. An applications of PFSs in cleaner production of gold mines by using integrated EDAS-ELECTRE was discussed by Liang [[48]](#_bookmark47). Picture fuzzy TOPSIS method based on the linear programming and distance measure was discussed by Sindhu et al. [[49]](#_bookmark48). The picture fuzzy entropy based similarity measures and its applications in MADM were practiced by Thao [[50]](#_bookmark49). Nhung et al. [[51]](#_bookmark51) adopted the dissimilarity measures for PFS and discussed its applications in MADM. A pro- jection model was exercised by Wei et al. [[52]](#_bookmark53), in order to measure the similarity. The picture fuzzy generalized dice similarity mea- sures were presented by Wei and Gao [[53]](#_bookmark56). The generalized picture distance measures and picture association measures, and their application in clustering analysis were discussed by Son [[54]](#_bookmark58). Ganei et al. [[55]](#_bookmark60) defined some new correlation coefficients of PFSs and discussed their applications in pattern recognition, medical diagnosis and clustering analysis.

In this paragraph, we discuss the applications of IFSs and type-2 fuzzy sets in medical diagnosis. Szmidt and Kacprzyk discuss the

medical diagnosis problems using IFSs based on the similarity and distance measures [[56–58]](#_bookmark61). De et al. [[59]](#_bookmark66) extended the San- chez’s approach for medical diagnosis using max–min-max com- position based on IFSs. The pattern recognition and medical diagnosis problems were discussed by Vlachos et al. [[60]](#_bookmark68) using cross-entropy in IF settings. Wei et al. [[61]](#_bookmark70) introduced the entropy measure for interval valued IFSs and construct similarity measures based on the proposed entropy. Then similarity measures were used to solve pattern recognition and medical diagnosis problems. The distance measures for interval values IFSs were used for med- ical diagnosis of headache in [[63]](#_bookmark72). Boran and Akay introduced the bi-parametric distance and similarity measures for IFSs and dis- cussed their applications in pattern recognition and medical diag- nosis problems in [[62]](#_bookmark73). In reference [[64]](#_bookmark74) Own studied advantages of type-2 fuzzy and switching relation between type-2 fuzzy sets and IFSs. Then the switching results are applied in medical diagno- sis. For advance study about medical diagnosis by using type-2 fuzzy sets, we refer you to [[65–70]](#_bookmark75).

We adopt the generalize model of fuzzy set (PFS) for medical diagnosis. The PFSs are comparatively a new extension of fuzzy sets which describe the human opinions that has more answers like acceptance, rejection, neutral and desist, which cannot be cor- rectly presented in fuzzy sets and IFSs. The PFSs are categorized by three objects, the degree of belonging (membership), the degree of neutral belonging and the degree of non-belonging (non- membership) such that the total of these three degrees must not be more than one. Basically, PFSs play an important role in situa- tions involving more types of answers like no, yes, refusal and abstain. To model such situations, a PFS is an appropriate choice.

The main contributions of our work are:

* We extend the idea of bi-parametric distance and similarity measures for PFSs. The properties of the proposed similarity measures are discussed.
* The interpretation of the novel distance measure and the func- tionality of level of uncertainty *t* are discussed.
* Applications in pattern recognition for the proposed bi- parametric distance and similarity measures are discussed, as well as, the counter intuitive cases for existing distance and similarity measures are discussed.
* The medical diagnosis problem is discussed based on the bi- parametric distance and similarity measures.

Some of the proposed similarity measure for *PFSs* has some problems which are pointed out in Section [5](#_bookmark14). To improve the idea of the similarity measure, we proposed the new similarity mea- sure based on two parameters *p* and *t*, where *p* is the *Lp* norm and *t* identifies the level of uncertainty. The remaining paper is organized as follows: The preliminaries are presented in Sections

1. In Section [3](#_bookmark6), we proposed the new distance and similarity mea- sures for *PFSs*. In Section [4](#_bookmark11), we give the explanation of both param- eters *p* and *t*, and interpretation of novel distance measure. In Section [5](#_bookmark14), we provide some counterexamples for already proposed similarity measures. To support the proposed similarity measure a numerical example of medical diagnosis is presented in Section [6](#_bookmark21). Comparison analysis and conclusion are presented in Sections [7 and 8](#_bookmark28).
2. Preliminaries

In this section, we provide some basic definitions of fuzzy set, intuitionistic fuzzy set, picture fuzzy set and some similarity measures.

A fuzzy set is defined by Zadeh [[1]](#_bookmark31), which handles uncertainty based on the view of gradualness effectively.

Definition 2.1. [[1]](#_bookmark31) A membership function nΑ^ : Y^ → [0; 1] defines

Definition 2.7. [[43]](#_bookmark38) For two *PFSs* Α^ and S^ in Y^ , a cosine similarity

the fuzzy set Α^ over the Y^ , where nΑ^ (*y*) particularized the

measure by using degree of refusal membership between Α^ and S^

membership of an element *y* ∈ Y^ in fuzzy set Α^ .

is defined as follows:

### ^ ^ 1 X np

*m*

CS5(Α; S) =

*cos*

[|n^ (*yj*)— n^ (*yj*)| V |g^ (*yj*)

In [[21]](#_bookmark54), Cuong defines the PFS, which is an extension of fuzzy set

*m*

2

Α

S

Α

and applicable in many real life problems. By adding an extra

membership function, namely, the degree of the neutral member-

*j*=1

—g (*y* )| V |m (*y* )— m (*y* )| V |p (*y* )— p (*y* )|]};

S^

*j*

Α^

*j*

S^

*j*

Α^

*j*

S^

*j*

ship function, the picture fuzzy set is obtained. Basically, the model of the picture fuzzy set may be adequate in situations when we face human opinions involving more answers of the type: yes, abstain, no, refusal. Voting can be a good example of picture fuzzy set because it involves the situation of more answers of the type: yes, abstain, no, refusal.

where V is the maximum operation.

Definition 2.8. [[43]](#_bookmark38) For two *PFSs* Α^ and S^ in Y^ , a cotangent similar- ity measure by using degree of refusal membership between Α^ and S^ is defined as follows:

Definition 2.2. [[21]](#_bookmark54) A PFS ^ ^

Α over the universe Y is defined as

## ^ ^ 1 X

np p

4

4

Α

S

Α

^ ^ —gS^ (*yj*)| V |mΑ^ (*yj*)— mS^ (*yj*)| V |pΑ^ (*yj*)— pS^ (*yj*)|]};

*m*

CS6(Α; S) =

*cot*

+

[|n^ (*yj*)— n^ (*yj*)| V |g^ (*yj*)

*m*

*j*=1

Α= {(*y*; nΑ^ ; gΑ^ ; tΑ^ )|*y* ∈ Y};

where nΑ^ : Y^ → [0; 1]; gΑ^ : Y^ → [0; 1] and #Α^ : Y^ → [0; 1] are the degree of positive membership, neutral membership and degree of negative membership, respectively. Furthermore, it is required

that 0 6 n^ + g^ + t^ 6 1. Then for *y* ∈ Y^ ; p^ (*y*)=

where V is the maximum operation.

Definition 2.9. [[43]](#_bookmark38) For two t7£*s* Α^ and S^ in Y^ , the similarity measure based on cosine function by using degree of refusal mem-

Α Α Α

1

Α bership are defined as follows:

— (nΑ^ (*y*)+ gΑ^ (*y*)+ tΑ^ (*y*)) is called the degree of refusal member-

*m*

C (Α; S) =

*cos*

[|n^ (*yj*)— n^ (*yj*)| + |g^ (*yj*)— g^ (*yj*)|

ship of *y* in Α^ . For PFS (n (*y*); g (*y*); t (*y*)) are said to picture fuzzy

Α^

Α^

Α^

*j*=1

value (*PFV* ) or picture fuzzy number (*PFN*) and each *PFV* can be

### 7 ^ ^ 1 X np

denoted by *q* = (n*q*; g*q* ; t*q* ), where n*q*; g*q* and t*q* ∈ [0; 1], with condi-

*m*

2

Α

S

Α

S

tion that 0 6 n*q* + g*q*

Α

Α

+ t*q* 6 1.

+|m^ (*yj*)— m^ (*yj*)| + |p^ (*yj*)— p^ (*yj*)|]}

In [[43]](#_bookmark38), Wei defines some similarity measures for *PFSs* based on cosine and cotangent functions.

S

S

Definition 2.3. [[43]](#_bookmark38) For two *PFSs* Α^ and S^ in Y^ , a cosine similarity measure between Α^ and S^ is defined as follows:

where V and p are the maximum operation and refusal membership

degree, respectively.

Definition 2.10. [[45]](#_bookmark40) For two *PFSs* Α^ and S^ in Y^ , a set theoretic similarity measure between Α^ and S^ is defined as follows:

1 *m* ^ *y* )n^ (*y* )+ g^ (*y* )g^ (*y* )+ m^ (*y* )m^ (*y* )

STS(Α^ ; S^ ) =

*m*

^ *y*

^ *y* )+

^ *y* )g^ (*y* )+ m^ (*y* )m^ (*y* )

X nΑ( *j* S *j* Α *j* S *j* Α *j* S *j* .

*CS*1(Α^ ; S^) = X

1

nΑ( *j* )nS( *j*

gΑ( *j* S *j* Α *j* S *j*

qﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃﬃ .

*j*=1

*m*

*max*

2

2

2

2

2

2

{nΑ^ (*yj* )+ gΑ^ (*yj* )+ mΑ^ (*yj* ); nS^ (*yj* )+ gS^ (*yj* )+ mS^ (*yj* )}

*m j*=1 qnﬃﬃﬃ2ﬃﬃﬃ(ﬃﬃ*y*ﬃﬃ*j*ﬃ)ﬃﬃﬃ+ﬃﬃﬃﬃﬃgﬃﬃﬃ2ﬃﬃﬃ(ﬃﬃ*y*ﬃﬃ*j*ﬃ)ﬃﬃﬃ+ﬃﬃﬃﬃﬃmﬃﬃ2ﬃﬃﬃ(ﬃﬃ*y*ﬃﬃﬃ*j*ﬃ)ﬃﬃ

n2 (*yj* )+ g2 (*yj* )+ m2 (*yj* )

Α^ Α^

Α^ S^

S^ S^

Definition 2.11. [[53]](#_bookmark56) For two t7£*s* Α^ and S^ in Y^ , Dice similarity

Definition 2.4. [[43]](#_bookmark38) For two *PFSs* Α^ and S^ in Y^ , a cosine similarity measure between Α^ and S^ is defined as follows:

measures between Α^ and S^ is defined as follows:

1 *m* 2(n^ (*y* )n^ (*y* )+ g^ (*y* )g^ (*y* )+ m^ (*y* )m^ (*y* ))

*m*

2 *y*

2

2

2

2

2

£1(Α^ ; S^ ) =

*d*

X Α *j* S *j* Α *j* S *j* Α *j* S *j*

CS2(Α^ ; S^) = 1 X*cos*np [|n (*y* )— n (*y* )| V |g (*y* )— g (*y* )| V |m (*y* )— m (*y* )|]o;

*m*

*j*=1 nΑ^ ( *j* )+ gΑ^ (*yj*)+ mΑ^ (*yj*)+ nS^ (*yj*)+ gS^ (*yj*)+ mS^ (*yj*)

*m j*=1

2 Α^ *j*

S^ *j*

Α^ *j*

S^ *j*

Α^ *j*

S^ *j*

1 *m* 2(n^ (*y* )n^ (*y* )+ g^ (*y* )g^ (*y* )+ m^ (*y* )m^ (*y* )) + p^ (*y* )p^ (*y* )

X

where V

is the maximum operation.

£2(Α^ ; S^) =

*m*

*d*

Α *j* S *j* Α *j* S *j* Α *j* S *j* Α *j* S *j*

[n2 (*y* )+ g2 (*y* )+ m2 (*y* )+ p2 (*y* )] + [n2 (*y* )+ g2 (*y* )+ m2 (*y* )+ p2 (*y* )]

*j*=1

Α^ *j*

Α^ *j*

Α^ *j*

Α^ *j*

S^ *j*

S^ *j*

S^ *j*

S^ *j*

Definition 2.5. [[43]](#_bookmark38) For two *PFSs* Α^ and S^ in Y^ , a cosine similarity

P*m* 12(n^ (*yj*)n^ (*yj*)+ g^ (*yj*)g^ (*yj*)+ m^ (*yj*)m^ (*yj*))

*d* P*m* (n2 (*y* )+ g2 (*y* )+ m2 (*y* )) + P*m* (n2 (*y* )+ g2 (*y* )+ m2 (*y* ))

£3 (Α^ ; S^ ) = *j*= Α S Α S Α S

measure between Α^ and S^ is defined as follows:

*j*=1

Α^

*j*

Α^

*j*

Α^

*j*

*j*=1

S^

*j*

S^

*j*

S^

*j*

CS3(Α^ ; S^) =

1 X*cos*np [|n (*y* )— n (*y* )| + |g (*y* )— g (*y* )| + |m (*y* )— m (*y* )|]o.

^ *j* ^ *j* ^ *j* ^ *j* ^ *j* ^ *j*

*m*

Α

S

1. Biparametric distance and similarity measures for *P/Ss*

*m j*=1 4

Α S Α S

In this section, we define new distance and similarity measure for *PFSs* based on two parameters and give their proof.

Definition 2.6. [[43]](#_bookmark38) For two *PFSs* Α^ and S^ in Y^ , a cotangent similar-

Definition 3.1. A distance measure between *PFSs* Α^

and S^ is a

ity measure between Α^ and S^ is defined as follows:

mapping

Ð^ : *PFS* × *PFS* → [0; 1], which satisfies the following

CS4(Α^ ; S^) =

1 X

*cot*np + p [|n^ (*yj* )— n^ (*yj* )| V |g^ (*yj* )— g^ (*yj* )| V |m^ (*yj* )— m^ (*yj* )|]o;

properties:

^ ^ ^

*m j*=1

*m*

4 4 Α S

Α S Α S

(D1) 0 6 Ð(Α; S) 6 1

(D2) Ð^ (Α^ ; S^) = 0 →⇒ Α^ = S^

where V is the maximum operation.

(D3) Ð^ (Α^ ; S^) = Ð^ (S^; Α^ )

(D4) If Α^ C S^ C C^ then Ð^ (Α^ , C^) P Ð^ (Α^ , S^) and Ð^ (Α^ , C^) P Ð^ (S^, C^).

Definition 3.2. A similarity measure between *PFSs* Α^ and S^ is a mapping £^ : *PFS* × *PFS* → [0, 1], which satisfies the following

— (1 + *t*) 6 (*t*gΑ^ (*yj*)+ mΑ^ (*yj*)— nΑ^ (*yj*))— (*t*gS^ (*yj*)+ mS^ (*yj*)

— nS^ (*yj*)) 6 (1 + *t*).

It means that

0 6 |(*t*g^ *yj* m^ *yj* n^ *yj t*g^ *yj* m^ *yj* n^ *yj p*

Α( )+ Α( )— Α( ))— ( S ( )+ S ( )— S ( ))|

properties:

(S1) 0 6 £^(Α^ , S^) 6 1

(S2) £^(Α^ , S^) = 1 →⇒ Α^ = S^

(S3) £^(Α^ , S^) = £^(S^, Α^ )

(S4) If Α^ C S^ C C^ then £^(Α^ , C^) 6 £^(Α^ , S^) and £^(Α^ , C^) 6 £^(S^, C^).

Definition 3.3. A bi-parametric distance measure between two

*PFSs* Α^ and S^ is a mapping Ð^ : Α^ × S^ → [0, 1] which is defined as

## " 1 X

*m*

Ð^ (Α^ , S^) =

|*t*(nΑ^ (*yj*)— nS^ (*yj*))— (gΑ^ (*yj*)— gS^ (*yj*))

3*m*(*t*

+

1)*p*

*j*=1

### 6 (1 + *t*)*p*. (3)

Also for remaining equation we have

—1 6 (*t*mΑ^ (*yj*)+ gΑ^ (*yj*)— nΑ^ (*yj*)) 6 *t*

—*t* 6 —(*t*mS^ (*yj*)+ gS^ (*yj*)— nS^ (*yj*)) 6 1

then we have

; —(1 + *t*) 6 (*t*mΑ^ (*yj*)+ gΑ^ (*yj*)— nΑ^ (*yj*))— (*t*mS^ (*yj*)+ gS^ (*yj*)

— nS^ (*yj*)) 6 (1 + *t*).

It means that

0 6 |(*t*m^ (*yj*)+ g^ (*yj*)— n^ (*yj*))— (*t*m^ (*yj*)+ g^ (*yj*)— n^ (*yj*))|*p*

Α

Α

Α

S

S

S

6 (1 + *t*)*p*. (4)

—(m^ (*yj*)— m^ (*yj*))|*p* + |*t*(g^ (*yj*)— g^ (*yj*))— (n^ (*yj*)— n^ (*yj*))

Finally we have the following inequality:

Α S Α S Α S

+ (m^ (*yj*)— m^ (*yj*))|*p* + |*t*(m^ (*yj*)— m^ (*yj*))— (n^ (*yj*)— n^ (*yj*))

Α

S

Α

S

Α

S

" 1 X

g *y* g *y*

+(

Α^ ( *j*)—

Α S Α S Α S Α S

*p* 1

1 3*m*(*t* + 1) Α S Α S

*m*

0 6

*p*

|*t*(n^ (*yj*)— n^ (*yj*))— (g^ (*yj*)— g^ (*yj*))

*j*=1

—(m (*y* )— m (*y* ))|*p* + |*t*(g (*y* )— g (*y* ))— (n (*y* )— n (*y* ))

Α^

*j*

S^

*j*

Α^

*j*

S^

*j*

Α^

*j*

S^

*j*

where *t* = 3, 4, ... and *p* = 1, 2, 3, ... are identifies the level of uncer-

S^ ( *j*))|

*p* , ( )

tainty and *Lp* norm, respectively.

+ (m^ (*yj*)— m^ (*yj*))|*p* + |*t*(m^ (*yj*)— m^ (*yj*))— (n ^ (*yj*)— n^ (*yj*))+ (g^ (*yj*)— g^ (*yj*))|*p*

Theorem 3.4. Ð^ (Α^ , S^) is the distance measure between two *PFSs* Α^

## 0 6 Ð^ (Α^ , S^) 6 1.

(D2) Since for two *PFSs* Α^

and

S^, if

Α^ = S^, then

and S^ in Y^ .

Proof. (D1) Let Α^ and S^ be two *PFSs*.

We can write the following equations:

|*t*(nΑ^ (*yj*)— nS^ (*yj*))— (gΑ^ (*yj*)— gS^ (*yj*))— (mΑ^ (*yj*)— mS^ (*yj*))|

= |(*t*nΑ^ (*yj*)— gΑ^ (*yj*)— mΑ^ (*yj*))— (*t*nS^ (*yj*)— gS^ (*yj*)— mS^ (*yj*))|,

|*t*(g^ (*yj*)— g^ (*yj*))— (n^ (*yj*)— n^ (*yj*))+ (m^ (*yj*)— m^ (*yj*))|

nΑ^ (*yj*) = nS^ (*yj*), gΑ^ (*yj*) = gS^ (*yj*) and mΑ^ (*yj*)= mS^ (*yj*), therefore, nΑ^ (*yj*)— nS^ (*yj*)= 0, gΑ^ (*yj*)— gS^ (*yj*) = 0 and mΑ^ (*yj*)— mS^ (*yj*) = 0. Therefore, the distance measure is equal to zero.

(D3) For two *PFSs* Α^ and S^, we can write the following equations:

## Α( )— S ( ))— ( Α( )— S ( ))— ( Α( )— S ( ))|

|*t*(n^ *yj* n^ *yj* g^ *yj* g^ *yj* m^ *yj* m^ *yj p*

= |(—1){*t*(nS^ (*yj*)— nΑ^ (*yj*))— (gS^ (*yj*)— gΑ^ (*yj*)) — (mS^ (*yj*)

— m^ (*y* ))}|*p* ,

Α S Α S Α S Α *j*

= |(*t*gΑ^ (*yj*)+ mΑ^ (*yj*)— nΑ^ (*yj*))— (*t*gS^ (*yj*)) + mS^ (*yj*)— nS^ (*yj*))|,

|*t*(mΑ^ (*yj*)— mS^ (*yj*))— (nΑ^ (*yj*)— nS^ (*yj*))+ (gΑ^ (*yj*)— gS^ (*yj*))|

= |(*t*m^ (*yj*)+ g^ (*yj*)— n^ (*yj*))— (*t*m^ (*yj*)) + g^ (*yj*)— n^ (*yj*))|.

|*t*(g^ (*yj*)— g^ (*yj*))— (n^ (*yj*)— n^ (*yj*))+ (m^ (*yj*)— m^ (*yj*))|*p*

= |(—1){*t*(gS^ (*yj*)— gΑ^ (*yj*))— (nS^ (*yj*)— nΑ^ (*yj*)) + (mS^ (*yj*)

Α S Α S Α S

— m^ (*y* ))}|*p* ,

Α Α Α

S S S Α *j*

Since from the definition of PFS, we have 0 6 nΑ^ (*yj*) 6

|*t*(m (*y* )— m (*y* ))— (n (*y* )— n (*y* ))+ (g (*y* )— g (*y* ))|*p*

1, 0 6 g (*y* ) 6 1, 0 6 m (*y* ) 6 1, 0 6 n (*y* ) 6 1, 0 6 g (*y* ) 6 1 and

Α^ *j*

S^ *j*

Α^ *j*

S^ *j*

Α^ *j*

S^ *j*

Α^ *j*

Α^ *j*

S^ *j*

S^ *j*

0 6 mS^ (*yj*) 6 1, and therefore we have the following inequalities:

—1 6 (*t*nΑ^ (*yj*)— gΑ^ (*yj*)— mΑ^ (*yj*)) 6 *t*

—*t* 6 —(*t*nS^ (*yj*)— gS^ (*yj*)— mS^ (*yj*)) 6 1

then we have

= |(—1){*t*(mS^ (*yj*)— mΑ^ (*yj*))— (nS^ (*yj*)— nΑ^ (*yj*))+ (gS^ (*yj*)

— g^ (*yj*))}|*p*.

Α

Based on the definition of absolute value, we have

|*t*(n^ (*yj*)— n^ (*yj*))— (g^ (*yj*)— g^ (*yj*))— (m^ (*yj*)— m^ (*yj*))|*p*

Α S Α S Α S

— (1 + *t*) 6 (*t*nΑ^ (*yj*)— gΑ^ (*yj*)— mΑ^ (*yj*))— (*t*nS^ (*yj*)— gS^ (*yj*)

— m (*y* )) 6 (1 + *t*).

= |{*t*(nS^ (*yj*)— nΑ^ (*yj*))— (gS^ (*yj*)— gΑ^ (*yj*))— (mS^ (*yj*)

— m^ (*yj*))}|*p* , (5)

S^ *j* Α

It means that

0 6 |(*t*n^ (*yj*)— g^ (*yj*)— m^ (*yj*))— (*t*n^ (*yj*)— g^ (*yj*)— m^ (*yj*))|*p*

|*t*(g^ (*yj*)— g^ (*yj*))— (n^ (*yj*)— n^ (*yj*))+ (m^ (*yj*)— m^ (*yj*))|*p*

Α Α Α

Α S Α S Α S

S S S

= |{*t*(gS^ (*yj*)— gΑ^ (*yj*))— (nS^ (*yj*)— nΑ^ (*yj*))+ (mS^ (*yj*)

### 6 (1 + *t*)*p*. (2)

Α *j*

Similarly, we have the following inequalities:

— m^ (*y* ))}|*p* , (6)

|*t*(m^ (*yj*)— m^ (*yj*))— (n^ (*yj*)— n^ (*yj*))+ (g^ (*yj*)— g^ (*yj*))|*p*

—1 6 (*t*g^ (*yj*)+ m^ (*yj*)— n^ (*yj*)) 6 *t*

Α S Α S Α S

Α Α Α

—*t* 6 —(*t*gS^ (*yj*)+ mS^ (*yj*)— nS^ (*yj*)) 6 1

= |{*t*(mS^ (*yj*)— mΑ^ (*yj*))— (nS^ (*yj*)— nΑ^ (*yj*))+ (gS^ (*yj*)

— g (*y* ))}|*p*. (7)

then we have

Α^ *j*

Thus, Ð^ (Α^ , S^) = Ð^ (S^, Α^ ).

|(*t*m^ (*yj*)+ g^ (*yj*)— n^ (*yj*))— (*t*m^(*yj*)) + g^(*yj*)— n^(*yj*))|*p*

(D4) Let Α^ , S^ and C^ are three *PFSs*. The distance between Α^ and

Α

Α

Α

C

C

C

S^, and Α^ and C^ are the following:

P |(*t*mΑ^ (*yj*)+ gΑ^ (*yj*)— nΑ^ (*yj*))— (*t*mS^ (*yj*)) + gS^ (*yj*)

— n^ (*yj*))|*p*, (10)

S

Ð^ (Α^ , S^) =

1

3*m*(*t* +

"

1)*p*

X*j*=1

|*t*(nΑ^ (*yj*)— nS^ (*yj*)) — (gΑ^ (*yj*)— gS^ (*yj*))

and this completes the fourth property for distance measure, i.e.,

Ð^ (Α^ , C^) P Ð^ (Α^ , S^) and Ð^ (Α^ , C^) P Ð^ (S^, C^). h

—(m^ (*yj*)— m^ (*yj*))|*p* + |*t*(g^ (*yj*)— g^ (*yj*)) — (n^ (*yj*)— n^ (*yj*))

*m*

Α S Α S Α S

Definition 3.5. The weighted bi-parametric distance measure

+(m^ (*yj*)— m^ (*yj*))|*p* + |*t*(m^ (*yj*)— m^ (*yj*)) — (n^ (*yj*)

Α S Α S Α

—n^ (*yj*)) + (g^ (*yj*)— g^ (*yj*))|*p*

between two *PFSs* Α^ and S^ is a mapping Ð^ : Α^ × S^ → [0, 1] which

is defined as

S Α S

"

## ^ ^ " 1 X

Ð^ (Α^ , C^) = 1 X |*t*(n (*y* )— n (*y* )) — (g (*y* )— g (*y* ))

*m*

3*m*(*t* + 1)*p*

Α^

*j*

C^

*j*

Α^

*j*

C^

*j*

*j*=1

### Ð^ x(Α, S) =

3*m*(*t* + 1)*p*

*m*

*j*=1

—g^ (*yj*))— (m^ (*yj*)— m^ (*yj*))|*p*

+ |*t*(gΑ^ (*yj*)— gS^ (*yj*))

x^ *j* |*t*(nΑ^ (*yj* )— nS^ (*yj* ))— (gΑ^ (*yj* )

—(m^ (*yj*)— m^(*yj*))|*p* + |*t*(g^ (*yj*)— g^(*yj*)) — (n^ (*yj*) *p*

S Α S

Α C Α C Α

*y p*

—(nΑ^ (*yj*)— nS^ (*yj*))+ (mΑ^ (*yj*)— mS^ (*yj*))| + |*t*(mΑ^ (*yj*)

—nC^( *j*)) + (mΑ^ (*yj*)— mC^ (*yj*))| + |*t*(mΑ^ (*yj*)— mC^ (*yj*)) — (nΑ^ (*yj*)

—m (*y* ))— (n (*y* )— n (*y* ))+ (g (*y* )— g (*y* ))|*p*

—n (*y* )) + (g (*y* )— g (*y* ))|*p*

C^

*j*

Α^

*j*

C^

*j*

S^ *j*

Α^ *j*

S^ *j*

Α^ *j*

S^ *j*

We can write the following equations:

|*t*(n^ (*y* )— n^ (*y* ))— (g^ (*y* )— g^ (*y* ))— (m^ (*y* )— m^ (*y* ))|

Α

*j*

S

*j*

Α

*j*

S

*j*

Α

*j*

S

*j*

= |(*t*nΑ^ (*yj*)— gΑ^ (*yj*)— mΑ^ (*yj*))— (*t*nS^ (*yj*)— gS^ (*yj*)— mS^ (*yj*))|,

|*t*(gΑ^ (*yj*)— gS^ (*yj*))— (nΑ^ (*yj*)— nS^ (*yj*))+ (mΑ^ (*yj*)— mS^ (*yj*))|

= |(*t*gΑ^ (*yj*)+ mΑ^ (*yj*)— nΑ^ (*yj*))— (*t*gS^ (*yj*)) + mS^ (*yj*)— nS^ (*yj*))|,

|*t*(mΑ^ (*yj*)— mS^ (*yj*))— (nΑ^ (*yj*)— nS^ (*yj*))+ (gΑ^ (*yj*)— gS^ (*yj*))|

= |(*t*mΑ^ (*yj*)+ gΑ^ (*yj*)— nΑ^ (*yj*))— (*t*mS^ (*yj*)) + gS^ (*yj*)— nS^ (*yj*))|,

where *t* = 3, 4, ... , *p* = 1, 2, 3, .. .are the level of uncertainty and *Lp*

norm and x^ = {x^ 1, x^ 2, .. . , x^ *m*} is the weight vector with each

x^ *j* ∈ [0, 1] and P*m* x^ *j* = 1.

*j*=1

Theorem 3.6. Ð^ x(Α^ , S^) is the distance measure between two *PFSs*

Α^ and S^ in Y^ .

Proof. (D1) If we multiply x^ *j* with inequalities [(2)–(4)](#_bookmark9), then we have

0 6 x^ |(*t*n (*y* )— g (*y* )— m (*y* ))— (*t*n (*y* )— g (*y* )— m (*y* ))|*p*

*j* Α^ *j*

Α^ *j*

Α^ *j*

S^ *j*

S^ *j*

S^ *j*

|*t*(nΑ^ (*yj*)— nC^(*yj*))— (gΑ^ (*yj*)— gC^(*yj*))— (mΑ^ (*yj*)— mC^(*yj*))|

= |(*t*nΑ^ (*yj*)— gΑ^ (*yj*)— mΑ^ (*yj*))— (*t*nC^(*yj*)— gC^(*yj*)— mC^(*yj*))|,

|*t*(gΑ^ (*yj*)— gC^(*yj*))— (nΑ^ (*yj*)— nC^(*yj*))+ (mΑ^ (*yj*)— mC^(*yj*))|

= |(*t*gΑ^ (*yj*)+ mΑ^ (*yj*)— nΑ^ (*yj*))— (*t*gC^(*yj*)) + mC^(*yj*)— nC^(*yj*))|,

|*t*(mΑ^ (*yj*)— mC^(*yj*))— (nΑ^ (*yj*)— nC^(*yj*))+ (gΑ^ (*yj*)— gC^(*yj*))|

= |(*t*mΑ^ (*yj*)+ gΑ^ (*yj*)— nΑ^ (*yj*))— (*t*mC^(*yj*)) + gC^(*yj*)— nC^(*yj*))|.

^

6 x^ *j* (1 + *t*)*p*

0 6 x^ *j* |(*t*g^ (*yj*)+ m^ (*yj*)— n^ (*yj*))— (*t*g^ (*yj*)+ m^ (*yj*)— n^ (*yj*))|*p*

Α Α Α S S S

6 x^ *j*(1 + *t*)*p*

0 6 x^ *j* |(*t*g^ (*yj*)+ m^ (*yj*)— n^ (*yj*))— (*t*g^ (*yj*)+ m^ (*yj*)— n^ (*yj*))|*p*

Α Α Α S S S

6 x^ *j* (1 + *t*)*p*.

For all *yj* ∈ Y^ *j* = 1, ... , *m*, we have

*m*

*j*

Α^

*j*

Α^

*j*

Α^

*j*

S^

*j*

S^

*j*

If Α C S^ C C^, then 0 6 nΑ^ (*yj*) 6 nS^ (*yj*) 6 nC^(*yj*) 6 1, 1 P

Α^

*j*

S^

*j*

C^

*j*

Α^

*j*

S^

*j*

g (*y* ) P g (*y* ) P g (*y* ) P 0 and 1 P m (*y* ) P m (*y* ) P

0 6 Xx^ |(*t*n (*y* )— g (*y* )— m (*y* ))— (*t*n (*y* )— g (*y* )

mC^ (*yj*) P 0. Therefore we have the following inequalities: *m m*

*j*=1

— m^ (*yj*))|*p* 6 Xx^ *j*(1 + *t*)*p*0 6 Xx^ *j* |(*t*g^ (*yj*)+ m^ (*yj*)— n^ (*yj*))

C

C

C

S

S

S

*j*=1

*j*=1

(*t*n^(*yj*)— g^(*yj*)— m^(*yj*)) P (*t*n^ (*yj*)— g^ (*yj*)— m^ (*yj*))

S

Α

Α

Α

P (*t*n^ (*y* )— g^ (*y* )— m^ (*y* )) *m*

Α *j* Α *j* Α *j*

— (*t*g (*y* )+ m (*y* )— n (*y* ))|*p* 6 Xx^ (1 + *t*)*p*0

(*t*gΑ^ (*yj*)+ mΑ^ (*yj*)— nΑ^ (*yj*)) P (*t*gS^ (*yj*)) + mS^ (*yj*)— nS^ (*yj*))

X

S^ *j*

*m*

S^ *j*

S^ *j*

*j*

*j*=1

P (*t*gC^(*yj*)) + mC^(*yj*)— nC^(*yj*))

6 x^ *j*|(*t*gΑ^ (*yj*)+ mΑ^ (*yj*)— nΑ^ (*yj*))— (*t*gS^ (*yj*)+ mS^ (*yj*)

*j*=1

*m*

S^

*j*

*j*

(*t*mΑ^ (*yj*)+ gΑ^ (*yj*)— nΑ^ (*yj*)) P (*t*mS^ (*yj*)) + gS^ (*yj*)— nS^ (*yj*))

P (*t*mC^(*yj*)) + gC^(*yj*)— nC^(*yj*)).

— n (*y* ))|*p* 6 Xx^ (1 + *t*)*p*.

*j*=1

So it is easy to see that:

|(*t*n (*y* )— g (*y* )— m (*y* ))— (*t*n (*y* )— g (*y* )— m (*y* ))|*p*

Since *m* x^ *j* = 1, therefore *m* x^ *j*(1 + *t*)*p* is equal to the

(1 + *t*)*p* and after addition of the above equations, we get

*m*

*j*=1

*j*=1

Α^ *j*

Α^ *j*

Α^ *j*

C^ *j*

C^ *j*

C^ *j*

P |(*t*n^ (*yj*)— g^ (*yj*)— m^ (*yj*))— (*t*n^ (*yj*)— g^ (*yj*)— m^ (*yj*))|*p*, (8)

Α

Α

Α

S

S

S

P P

|(*t*g^ (*yj*)+ m^ (*yj*)— n^ (*yj*))— (*t*g^ (*yj*)) + m^(*yj*)— n^(*yj*))|*p*

## " 1 X

3*m*(*t*

+

1)*p*

x^ *j* |*t*(nΑ^ (*yj* )— nS^ (*yj*))— (gΑ^ (*yj*)— gS^ (*yj* ))

*p*

0 6

*j*=1

Α Α Α

C C C

—(mΑ^ (*yj*)— mS^ (*yj*))|

+ |*t*(gΑ^ (*yj*)— gS^ (*yj*))— (nΑ^ (*yj*)— nS^ (*yj*))

P |(*t*gΑ^ (*yj*)+ mΑ^ (*yj*)— nΑ^ (*yj*))— (*t*gS^ (*yj*)) + mS^ (*yj*)

+(m^ (*yj*)— m^ (*yj*))|*p* + |*t*(m^ (*yj*)— m^ (*yj*))— (n ^ (*yj*)— n^ (*yj*))

S

Α

*y p* 9

Α S Α S

### — nS^ ( *j*))| , ( )

+(g (*y* )— g (*y* ))|*p* 1 6 1

Α^ *j* S^ *j*

*p*

### 0 6 Ð^ x(Α^ , S^) 6 1.

(D2) Since for two *PFSs* Α^ and S^, if Α^ = S^, then

where *t* = 3, 4, ... , *p* = 1, 2, 3, .. . are called the level of uncertainty and *Lp* norm, respectively and x^ = {x^ 1, x^ 2, .. . , x^ *m*} is the weight

n^ (*yj*)= n^ (*yj*), g^ (*yj*)= g^ (*yj*) and m^ (*yj*) = m^ (*yj*), therefore, nΑ^ (*yj*)— nS^ (*yj*) = 0, gΑ^ (*yj*)— gS^ (*yj*) = 0 and mΑ^ (*yj*)— mS^ (*yj*) = 0. Therefore, the weighted distance measure, Ð^ x(Α^ , S^), is equal to zero.

Α

S

Α

S

Α

S

(D3) If we multiply x^ *j* with Eqs. [(5)–(7)](#_bookmark8), then we have

vector with each x^ *j* ∈ [0, 1] and P*m* 1x^ *j* = 1.

Theorem 3.9. £^(Α^ , S^) and £^x(Α^ , S^) are similarity measure between two *PFSs* Α^ and S^ in Y^ .

*j*=

x^ *j*|*t*(n^ (*yj*)— n^ (*yj*))— (g^ (*yj*)— g^ (*yj*))— (m^ (*yj*)— m^ (*yj*))|*p*

Α S Α S Α S

1. Interpretation of novel distance measure for *P/Ss*

= x^ *j* |{*t*(n^ (*yj*)— n^ (*yj*))— (g^ (*yj*)— g^ (*yj*))— (m^ (*yj*)— m^ (*yj*))}|*p*,

S Α S Α S Α

x^ *j*|*t*(g^ (*yj*)— g^ (*yj*))— (n ^ (*yj*)— n^ (*yj*))+ (m^ (*yj*)— m^ (*yj*))|*p*

Α S Α S Α S

The interpretation of novel distance measure and functionality of level of uncertainty *t* are explained in this section.

Let (n , g , m ) and (n , g , m ) be two *PFNs*. n takes the value

= x^ |{*t*(g (*y* )— g (*y* ))— (n (*y* )— n (*y* ))+ (m (*y* )— m (*y* ))}|*p*,

Α^ Α^ Α^

S^ S^ S^ Α^

*j* S^ *j*

Α^ *j*

S^ *j*

Α^ *j*

S^ *j*

Α^ *j*

from [n^ , p^ ], g^ takes the value from [g^ , p^ ] and m^

takes the

Α Α Α Α Α Α

x^ *j*|*t*(m^ (*y* )— m^ (*y* ))— (n^ (*y* )— n^ (*y* ))+ (g^ (*y* )— g^ (*y* ))|*p*

value from [mΑ^ , pΑ^ ]. Shaded area of [Fig. 1](#_bookmark13) represents the all possible values for (n^ , g^ , m^ ).

Α *j* S *j*

Α *j* S *j*

Α *j* S *j*

Α Α Α

= x^ *j* |{*t*(m^ (*yj*)— m^ (*yj*))— (n^ (*yj*)— n^ (*yj*))+ (g^ (*yj*)— g^ (*yj*))}|*p*.

S Α S Α S Α

Undoubtedly, the center of gravity (centroid) *G* of tetrahedron

*ABCD* in [Fig. 1](#_bookmark13) is the most informative point among all the possible

points. The center of gravity (n' , g' , m' ) can be easily found as

Thus, Ð^ x(Α^ , S^) = Ð^ x(S^, Α^ ).

Α^ Α^ Α^

^ (n^ + pΑ^ , g^ + pΑ^ , m^ + pΑ^ ). Since p^ = 1 — (n ^ + g^ + m^ ), therefore,

(D4) If we multiply x*j* with inequalities [(8)–(10)](#_bookmark10), then we have

Α 4 Α 4 Α 4

Α Α Α Α

(n' , g' , m' ) takes the form

x^ |(*t*n (*y* )— g (*y* )— m (*y* ))— (*t*n (*y* )— g (*y* )— m (*y* ))|*p*

Α^ Α^ Α^

*j* Α^ *j*

Α^ *j*

Α^ *j*

C^ *j*

C^ *j*

C^ *j*

P x^ |(*t*n (*y* )— g (*y* )— m (*y* ))— (*t*n (*y* )— g (*y* )— m (*y* ))|*p*,

*j*

Α^

*j*

Α^

*j*

Α^

*j*

S^

*j*

S^

*j*

S^

*j*

n + pΑ^ , g

+ pΑ^ , m

+ pΑ^

x^ |(*t*g (*y* )+ m (*y* )— n (*y* ))— (*t*g (*y* )) + m (*y* )— n (*y* ))|*p*

Α^

4

Α^

4

Α^

4

*j*

Α^

*j*

Α^

*j*

Α^

*j*

C^

*j*

C^

*j*

C^

*j*

P x^ *j*|(*t*g^ (*y* )+ m^ (*y* )— n^ (*y* ))— (*t*g^ (*y* )) + m^ (*y* )— n^ (*y* ))|*p*,

4

4

4

= 1 + 3nΑ^ — gΑ^ — mΑ^ , 1 + 3gΑ^ — nΑ^ — mΑ^ , 1 + 3mΑ^ — gΑ^ — nΑ^ .

Α *j* Α *j* Α *j*

S *j* S *j* S *j*

Similarly, we also have the pair (n' , g' , m' ) for *PFN* (n^ , g^ , m^ ) as

x^ *j*|(*t*m^ (*yj*)+ g^ (*yj*)— n^ (*yj*))— (*t*m^(*yj*)) + g^(*yj*)— n^(*yj*))|*p*

follows:

S^ S^ S^

S S S

Α Α Α

P x^ |(*t*m (*y* )+ g (*y* )— n (*y* ))— (*t*m (*y* )) + g (*y* )— n (*y* ))|*p*,

Α^

Α^

Α^

S^

S^

S^

*j*

*j*

*j*

C C C

*j*

*j*

*j*

*j*

# p p p

since

n

+

S^ , g

+

S^ , m

+

S^

S^

4

S^

4

S^

4

weighted distance measure i.e.,

x^ *j* P 0, therefore, we have the fourth property for the

Ð^ x(Α^ , C^) P Ð^ x(Α^ , S^) and

= 1 + 3nS^ — gS^ — mS^ , 1 + 3gS^ — nS^ — mS^ , 1 + 3mS^ — gS^ — nS^ .

Ð^ x(Α^ , C^) P Ð^ x(S^, C^). Hence from (D1)— (D4), Ð^ x(Α^ , S^) is a distance The absolute difference between (n' , g' , m' ) and (n' , g' , m' ) is

4

4

4

measure. h

defined in Eqs. [(12)–(14)](#_bookmark12):

Α^ Α^ Α^

S^ S^ S^

Definition 3.7. A bi-parametric similarity measure between two

*PFSs* Α^ and S^ is a mapping £^ : Α^ × S^ → [0, 1] which is defined as

n'

' 1 + 3nΑ^ — gΑ^ — mΑ^ 1 + 3nS^ — gS^ — mS^

S^

" *m*  4

— n = —

=

Α^

3(nΑ^

4

— nS^ )— (g

Α^

— gS^ )— (m

Α^

### (12)

£^(Α^ , S^) = 1 — 1 X |*t*(n (*y* )— n (*y* ))— (g (*y* )

*j*=1

3*m*(*t* + 1)*p*

Α^

*j*

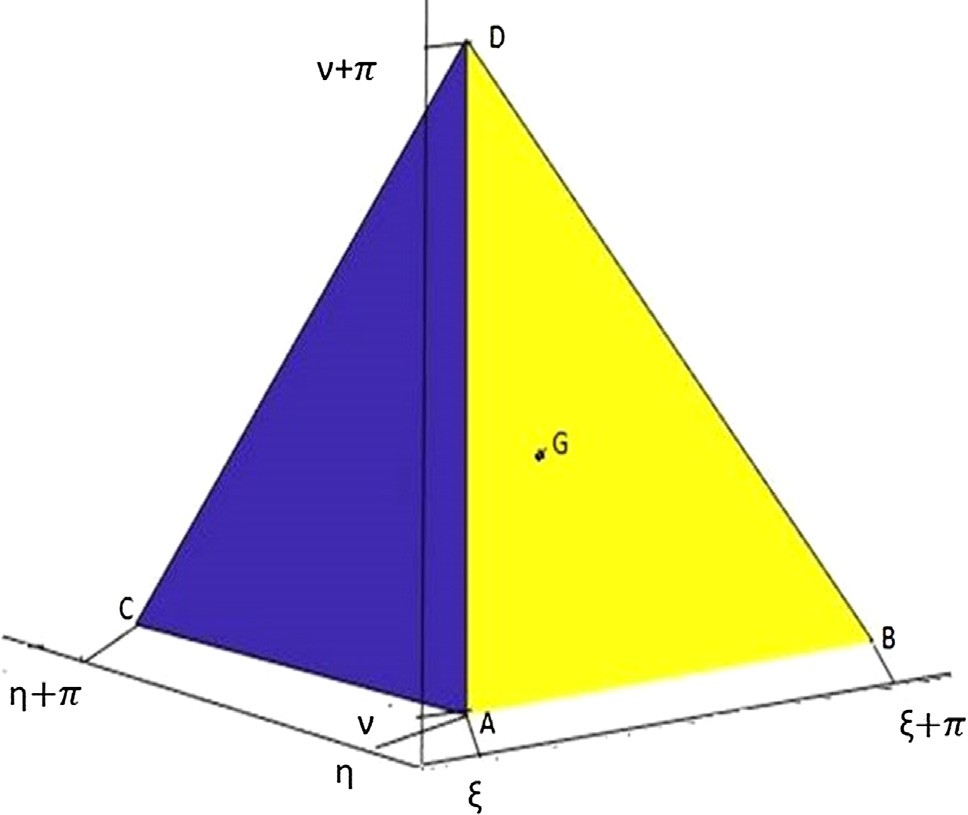
S^

*j*

Α^

*j*

—g^ (*yj*))— (m^ (*yj*)— m^ (*yj*))|*p* + |*t*(g^ (*yj*)— g^ (*yj*))— (n ^ (*yj*)

S Α S Α S Α

4

— mS^ )

—n^ (*yj*))+ (m^ (*yj*)— m^ (*yj*))|*p* + |*t*(m^ (*yj*)— m^ (*yj*))— (n^ (*yj*)

S Α S Α S Α

—n (*y* ))+ (g (*y* )— g (*y* ))|*p* 1 , (11)

*p*

S^ *j*

Α^ *j*

S^ *j*

where *t* = 3, 4, ... and *p* = 1, 2, 3, .. . are called the level of uncer- tainty and *Lp* norm, respectively.

Definition 3.8. The weighted bi-parametric similarity measure between two *PFSs* Α^ and S^ is a mapping £^ : Α^ × S^ → [0, 1] which is defined as

^ ^ ^

## " 1 X

3*m*(*t* + 1)*p*

*j*=1

—g^ (*yj*))— (m^ (*yj*)— m^ (*yj*))|*p* + |*t*(g^ (*yj*)— g^ (*yj*))— (n ^ (*yj*))

*m*

£x(Α, S) = 1 —

x^ *j* |*t*(nΑ^ (*yj* )— nS^ (*yj* ))— (gΑ^ (*yj* )

S Α S Α S Α

—n^ (*yj*))+ (m^ (*yj*)— m^ (*yj*))|*p* + |*t*(m^ (*yj*)— m^ (*yj*))— (n^ (*yj*)

S Α S Α S Α

—n (*y* ))+ (g (*y* )— g (*y* ))|*p* 1 ,

*p*

S^ *j*

Α^ *j*

S^ *j*

Fig. 1. The Possible Values for (n, g, m).

1 + 3g^ — n^ — m^ 1 + 3g^ — n^ — m^

' Α Α Α S S S

g' — g^ = 4 — 4

Α^ S

and negative membership function. Hence the Eq. [(17)](#_bookmark19) takes the form:

3(g^ — g^ )— (n^ — n^ )— (m^ — m^ )

" #1

Α S Α S Α S

=

### 4

(13)

1 X |n (*y* )' — n (*y* )'|*p* + |g (*y* )' — g (*y* )'|*p* + |m (*y* )' — m (*y* )'|*p*}

### 1 + 3m

m'

^

— m' =

Α^

Α^

Α^ —

S^

S^

— g — n

1 + 3m — g

### — n

3*m j*=1

" 1 X

*m p*

^ *j* ^ *j* ^ *j* ^ *j* ^ *j* ^ *j*

*m*

S^

=

3*m* × 4*p*

|3(nΑ^ (*yj*)— nS^ (*yj*))— (gΑ^ (*yj*)— gS^ (*yj*))— (mΑ^ (*yj*)— mS^ (*yj*))|

Α

S

Α S Α S

*p*

Α S 4 4

^

3(mΑ^ — mS^ )— (nΑ^ — nS^ )— (gΑ^ — gS^ )

*j*=1

+|3(g^ (*yj*)— g^ (*yj*)) — (n^ (*yj*)— n^ (*yj*))+ (m^ (*yj*)— m^ (*yj*))|*p*

Α

S

Α

S

Α

S

### (14)

=

*p*

*p*

### 4

After tanking power *p* to the |n'

— n' |, |g'

— g' | and |m'

— m' |, we

i1

+ 3(mΑ^ (*yj*)— mS^ (*yj*))— (nΑ^ (*yj*)— nS^ (*yj*))+ (gΑ^ (*yj*)— gS^ (*yj*))

(18)

get

Α^ S^ Α^ S^

Α^ S^

It is easy to see that Eq. [(18)](#_bookmark16) is the special case of Eq. [(1)](#_bookmark7) when

*t* = 3. We shall have Eq. [(18)](#_bookmark16) as the distance measure between

(n^ , g^ , m^ ) and (n^ , g^ , m^ ) when we parametrize (n' , g' , m' ) =

*p* 1 *p*

Α Α Α

( pΑ^

pΑ^

S S S

pΑ^

' ' '

Α^

pS^

Α^ Α^

pS^

n' — n' = *p* 3(n^ — n^ )— (g^ — g^ )— (m^ — m^ )

nΑ^ + 4 , gΑ^ + 4 , mΑ^ + 4 ) and (nS^ , gS^ , mS^ )= (nS^ + 4 , gS^ + 4 , mS^ +

Α^ S^ 4 Α S

*p*

*p*

Α S Α S

pS^ ). In Eq. [(1)](#_bookmark7), the effect of hesitation margin in the computation

is adjusted by the parameter *t*. When we need to neglect the effect of hesitation margin, the value of *t* should be high and the effect of

4

g'

* g'

1

### = 3(g

*p*

* g )— (n
* n )— (m
* m )

hesitation margin is taken into account when the value of *t* is very

Α^ S^ 4*p*

Α^ S^

Α^ S^

Α^ S^

low in the computation.

m'

*p*

* m'

1

### = 3(m

* m )— (n

### n )— (g

* g )

— g' |*p* and |m'

— m' |*p* is obtained in

1. Applications in pattern recognition and counter examples

## Α^

S^ 4*p*

Α^ S^

Α^ S^

Α^ S^

Α^ S^

The mean of |n'

— n' |*p*, |g'

Eq. [(15)](#_bookmark15) as follows:

that our proposed similarity measure is applicable to pattern

Α^ S^

Α^ S^

similarity measures. We have seen that the already proposed mea- sures cannot classify the unknown pattern while the bi-parametric

### 1 n|n' — n' |*p* + |g'

In this section, we provide several counterexamples for existing

3

Α^

S^

Α^

* g' |*p* + |m' — m' |*p*o

similarity measure classifies the unknown pattern. This guarantee

### = 1 h 3(n

S^

Α^

S^

3 × 4*p*

Α^

S^

Α^

S^

* n )— (g — g )— (m

*p*

* m )

Α^

S^

*p*i

recognition problems.

+ 3(gΑ^

### + 3(m

Α^

* gS^ )— (nΑ^
* m )— (n

S^

Α^

* nS^ )— (mΑ^

### n )— (g

S^

Α^

* mS^ )
* g ) (15)

*p*

S^

Example 5.1. In this example, we have seen that the second condition of [Definition 3.2](#_bookmark7) (*S*2) is not satisfied for cosine similarity

measure *CS*1 (Definition 2.3), i.e., if

Α^ = {(*a*, *a*, *a*)/*y* |*y* ∈

The *p* root of Eq. [(15)](#_bookmark15) is found as:

*j*

*j*

Y^ , 1 6 *j* 6 *m*} and S^ = {(*b*, *b*, *b*)/*yj* |*yj* ∈ Y^ , 1 6 *j* 6 *m*} are two *PFSs*

in Y^ with 0 6 *a*, *b* ≤ 1, 0 6 *a* + *a* + *a* ≤ 1, 0 6 *b* + *b* + *b* ≤ 1 and

1 ^ ^

1 ^ ^

1 n ' ' *p* ' ' *p* ' ' *p* o *p*

3 |nΑ^ — nS^ |

+ |gΑ^ — gS^ |

+ |mΑ^ — mS^ |

### 1 *p*

=

3 × 4*p* 3(nΑ^ — nS^ )— (gΑ^ — gS^ )— (mΑ^ — mS^ )

*a* – *b*, then Α – S. But, we have seen that *CS* (Α, S) = 1.

For example, let Y^ = {*y*1, *y*2, *y*3} and *PFSs* in Y^ are

Α = {(0.20, 0.20, 0.20)/*y*1, (0.30, 0.30, 0.30)/*y*2, (0.25, 0.25, 0.25)/*y*3}

### + 3(g

*p*

*p*

Α^

S^

Α^

S^

Α^

S^

S^

Α^

S^ )

)]*p*

1

— g )— (n

— n )— (m — m ) ^

### + 3(m

Α^

— m )— (n

— n )— (g — g

### (16)

S^ = {(0.31, 0.31, 0.31)/*y* , (0.27, 0.27, 0.27)/*y* , (0.33, 0.33, 0.33)/*y* }.

For more than one feature such as (*yj* ), 1 6 *j* 6 *m*, Eq. [(16)](#_bookmark18) can be defined in Eq. [(17)](#_bookmark19) as follows:

S^

Α^

1

2

3

Clearly, Α^ – S^ but *CS*1(Α^ , S^) = 1. Hence *CS*1 is not effective for these cases and not reliable to find the similarity measure between *PFSs*. But when we find the similarity measure from proposed bi- parametric similarity measure with *t* = 3 and *p* = 2, we get

Sp(Α^ , S^) = 0.949406.

### " 1 X t

*m*

|n^ (*yj*)' — n^ (*yj*)'|*p* + |g^ (*yj*)' — g^ (*yj*)'|*p*

3*m*

Α

S

Α

S

Example 5.2. Let *Q*

and *Q*

be two known patterns with class

*j*=1

1

2

+|m (*y* )' — m (*y* )'|*p*} 1

*p*

labels *Z*1 and *Z*2, respectively, are given. The *PFSs* are used to repre-

Α^ *j*

S^ *j*

sents the patterns in Y^ = {*y* , *y* , *y* } as follows:

## " 1 X

*m*

=

|3(nΑ^ (*yj*)— nS^ (*yj*))— (gΑ^ (*yj*)— gS^ (*yj*))

3*m* × 4*p*

*j*=1

1 2 3

—(m^ (*yj*)— m^ (*yj*))|*p* + |3(g^ (*yj*)— g^ (*yj*))— (n^ (*yj*)— n^ (*yj*))

*Q* 1 = {(0.20, 0.20, 0.10)/*y*1, (0.15, 0.15, 0.15)/*y*2, (0.25, 0.15, 0.10)/*y*3}

*Q* 2 = {(0.30, 0.10, 0.10)/*y*1 , (0.20, 0.20, 0.05)/*y*2 , (0.20, 0.30, 0.00)/*y*3 }.

*P* = {(0.40, 0.40, 0.20)/*y* , (0.30, 0.30, 0.30)/*y* , (0.50, 0.30, 0.20)/*y* }.

Α S Α S Α S

—(m (*y* )— m (*y* ))| + 3(m (*y* )— m (*y* ))— (n (*y* )— n (*y* ))

*p*

Α^

*j*

S^

*j*

Α^

*j*

S^

*j*

Α^

*j*

S^

*j*

*P* is the unknown pattern which is given as follows:

—(g (*y* )— g (*y* )) *p* i*p*

Α^ *j* S^ *j*

1

### (17)

1 2 3

To obtain distance measure and the reliability of the proposed method, we replace the negative sign from positive sign in second and third term in Eq. [(17)](#_bookmark19) for neutral membership function

Our aimed to find out the class of unknown pattern *P* belongs to. When we use set theoretic similarity measure *STM* (Definition 2.10), we get the same similarity measure i.e., *STM*(*P*, *Q* 1) = *STM*(*P*, *Q* 2)= 0.5. Hence in this case we can’t decide the class of unknown pattern *P*. But when we find the similarity

measure from proposed bi-parametric similarity measure with

*t* = 3 and *p* = 2, we get Sp(*P*, *Q* 1) = 0.907298 and

t

Sp(*P*, *Q* 2) = 0.870163. Since Sp(*P*, *Q* 1) > Sp(*P*, *Q* 2), therefore, the

Example 5.5. Let *Q* 1 and *Q* 2 be two known patterns with class labels *Z*1 and *Z*2, respectively, are given. The *PFSs* are used to repre-

sents the patterns in Y^ = {*y* , *y* , *y* } as follows:

t t t

1 2 3

unknown pattern *P* belongs to class *Z*1.

### *Q* 1 = {(0.4, 0.1, 0.3)/*y*1, (0.5, 0.1, 0.2)/*y*2 , (0.6, 0.1, 0.3)/*y*3 }

Example 5.3. Let *Q* 1

and *Q* 2

be two known patterns with class

### *Q* 2 = {(0.6, 0.1, 0.3)/*y*1, (0.1, 0.2, 0.5)/*y*2 , (0.5, 0.3, 0.0)/*y*3}.

labels *Z*1 and *Z*2, respectively, are given. The *PFSs* are used to repre- sents the patterns in Y^ = {*y*1, *y*2 , *y*3} as follows:

### *Q* 1 = {(0.3, 0.2, 0.1)/*y*1, (0.5, 0.1, 0.2)/*y*2 , (0.6, 0.1, 0.3)/*y*3 }

*Q* 2 = {(0.6, 0.1, 0.3)/*y*1, (0.1, 0.2, 0.5)/*y*2 , (0.6, 0.3, 0.1)/*y*3 }.

*P* is the unknown pattern which is given as follows:

### *P* = {(0.5, 0.3, 0.2)/*y*1, (0.3, 0.4, 0.2)/*y*2, (0.4, 0.3, 0.2)/*y*3}.

Our aimed to find out the class of unknown pattern *P* belongs to. When we use cosine similarity measure CS2 (Definition 2.4), we get the same similarity measure i.e., CS2(*P*, *Q* 1)= CS2(*P*, *Q* 2)= 0.93104. Hence in this case we can’t decide the class of unknown pattern *P*. Also when we use the cotangent similarity measure CS4, we get the same similarity measures i.e., CS4(*P*, *Q* 1)= CS4(*P*, *Q* 2)= 0.688629. Hence in this case we can’t decide the class of unknown pattern *P* by using cotangent similarity

measure CS4. The bi-parametric similarity measure successfully applies for *t* = 3 and *p* = 2. We get Sp(*P*, *Q* 1)= 0.844098 and Sp(*P*, *Q* 2)= 0.864087. Since Sp(*P*, *Q* 2) > Sp(*P*, *Q* 1), therefore, the

t

t

t

t

*P* is the unknown pattern which is given as follows:

### *P* = {(0.5, 0.3, 0.2)/*y*1, (0.3, 0.4, 0.2)/*y*2, (0.4, 0.3, 0.2)/*y*3}.

Our aimed to find out the class of unknown pattern *P* belongs to. But when we use cosine similarity measure CS5 with degree of refusal membership (Definition 2.7), we get the same similarity measure i.e., CS5(*P*, *Q* 1)= CS5(*P*, *Q* 2)= 0.93104. Also when we use the cotan- gent similarity measure CS6 with degree of refusal membership (Definition 2.8), we get the same similarity measures i.e., CS6(*P*, *Q* 1)= CS6(*P*, *Q* 2)= 0.688629. Hence in this case we can’t decide the class of unknown pattern *P* by using cotangent similarity measure CS6 with degree of refusal membership function. Hence in this case we can’t decide the class of unknown pattern *P*. But when

we find the similarity measure from proposed bi-parametric simi- larity measure with *t* = 3 and *p* = 2, we get Sp(*P*, *Q* 1)= 0.844098 and Sp(*P*, *Q* 2)= 0.864087. Since Sp(*P*, *Q* 2) > Sp(*P*, *Q* 1), therefore, the

t

t t t

unknown pattern *P* belongs to class *Z*2.

1. Application in medical diagnoses

unknown pattern *P* belongs to class *Z*2.

Let *U*^ = {*Joy*, *Pob*, *Aya*, *Ted*} represents the set of four patients, *V*^

Example 5.4. Let *Q* 1 and *Q* 2 be two known patterns with class labels *Z*1 and *Z*2, respectively, are given. The *PFSs* are used to repre- sents the patterns in Y^ = {*y*1, *y*2 , *y*3} as follows:

### *Q* 1 = {(0.3, 0.2, 0.1)/*y*1, (0.5, 0.1, 0.2)/*y*2, (0.4, 0.3, 0.3)/*y*3}

*Q* 2 = {(0.6, 0.1, 0.3)/*y*1, (0.1, 0.3, 0.4)/*y*2, (0.3, 0.1, 0.2)/*y*3}.

*P* is the unknown pattern which is given as follows:

### *P* = {(0.5, 0.3, 0.2)/*y*1, (0.3, 0.4, 0.2)/*y*2, (0.6, 0.2, 0.2)/*y*3}.

Our aimed to find out the class of unknown pattern *P* belongs to. But when we use cosine similarity measure CS3 (Definition 2.5), we get the same similarity measure i.e., CS3(*P*, *Q* 1)= CS3(*P*, *Q* 2) =0.941998. Hence in this case we can’t decide the class of unknown pattern *P*. But when we find the similarity measure from proposed bi- parametric similarity measure with *t* = 3 and *p* = 2, we get

Sp(*P*, *Q* 1)= 0.836064 and Sp(*P*, *Q* 2)= 0.876678. Since Sp(*P*, *Q* 2) >

represent their symptoms, where *V*^ = {*Chest Pain*, *Cough*, *Stomach Pain*, *Headche*, *Temprature*} and the set *W*^ represents the diagnosis, where *W*^ ={*Stomach Problem*, *Malaria*, *Viral Fever*, *Heart Problem*, *Typhoid*}. *PFNs* are given as relation *U*^ → *V*^ → *W*^

in [Tables 1 and 2](#_bookmark25).

For a proper diagnosis, we calculated similarity measure between patients and diagnosis in the context of symptoms. For all diagnosis this process is done. If similarity measure is high then the patient suffer from that particular disease. The similarity mea- sure between patients and diagnosis is presented in [Table 3](#_bookmark26). According to the similarity measure in [Table 3](#_bookmark26), Joy suffers from Stomach problem, Pob suffers from Heart problem, Aya suffers from Viral fever and Ted suffers from Stomach problem.

When we compare our result with already proposed methods in [Table 4](#_bookmark26), we have seen that the results are same but only using CS2

and CS4, we can’t discriminate that Aya suffers from stomach prob-

t t t

Sp(*P*, *Q* 1), therefore, the unknown pattern *P* belongs to class *Z*2. lem or headache.

t

Table 1

Symptoms Characteristics for the Patients.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Chest pain | Cough | Stomach pain | Headache | Temperature |
| Joy | (0.8,0.1,0.1) | (0.6,0.2,0.1) | (0.1,0.1,0.8) | (0.6,0.1,0.2) | (0.2,0.1,0.6) |
| Pob | (0.0,0.2,0.8) | (0.4,0.2,0.4) | (0.6,0.1,0.2) | (0.1,0.1,0.7) | (0.1,0.1,0.8) |
| Aya | (0.8,0.1,0.1) | (0.8,0.1,0.1) | (0.0,0.2,0.6) | (0.2,0.1,0.7) | (0.0,0.2,0.5) |
| Ted | (0.6,0.1,0.2) | (0.5,0.1,0.4) | (0.3,0.1,0.4) | (0.7,0.1,0.2) | (0.3,0.2,0.4) |

Table 2

Symptoms Characteristics for the Diagnosis.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Chest pain | Cough | Stomach pain | Headache | Temperature |
| Stomach problem | (0.6,0.0,0.0) | (0.3,0.2,0.5) | (0.1,0.1,0.7) | (0.4,0.1,0.3) | (0.1,0.1,0.7) |
| Malaria | (0.7,0.0,0.1) | (0.2,0.2,0.6) | (0.0,0.1,0.9) | (0.7,0.0,0.0) | (0.1,0.1,0.8) |
| Viral fever | (0.3,0.3,0.3) | (0.6,0.1,0.1) | (0.2,0.1,0.7) | (0.2,0.1,0.6) | (0.1,0.0,0.9) |
| Heart problem | (0.1,0.1,0.7) | (0.2,0.1,0.4) | (0.8,0.0,0.1) | (0.2,0.1,0.7) | (0.2,0.1,0.7) |
| Typhoid | (0.1,0.1,0.8) | (0.0,0.1,0.8) | (0.2,0.0,0.8) | (0.2,0.0,0.8) | (0.8,0.1,0.1) |

Table 3

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Symptoms Characteristics | for the Diagnosis. |  | | | |
|  | Stomach problem | Malaria | Viral fever | Heart problem | Typhoid |
| Joy | 0.842282 | 0.796489 | 0.755988 | 0.562583 | 0.542561 |
| Pob | 0.612809 | 0.517688 | 0.744559 | 0.891028 | 0.642171 |
| Aya | 0.770236 | 0.654433 | 0.806674 | 0.578693 | 0.52645 |
| Ted | 0.821348 | 0.766458 | 0.737798 | 0.663783 | 0.609434 |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Table 4  All the Considered | Results. |  | | | | |
|  |  | Stomach problem | Malaria | Viral fever | Heart problem | Typhoid |
|  | Joy | 0.920408 | 0.883993 | 0.852514 | 0.541238 | 0.481780 |
|  | Pob | 0.629256 | 0.501889 | 0.812292 | 0.975644 | 0.692161 |
| *CS*1 | Aya | 0.868131 | 0.729510 | 0.915494 | 0.578503 | 0.493098 |
|  | Ted | 0.906828 | 0.881773 | 0.799051 | 0.699248 | 0.631653 |
|  | Joy | 0.937301 | 0.916919 | 0.876501 | 0.682359 | 0.614248 |
|  | Pob | 0.760103 | 0.662606 | 0.856119 | 0.973036 | 0.765234 |
| CS2 | Aya | 0.881185 | 0.762295 | 0.881185 | 0.660370 | 0.602154 |
|  | Ted | 0.915027 | 0.869175 | 0.817447 | 0.780667 | 0.67976 |
|  | Joy | 0.952134 | 0.931842 | 0.895582 | 0.694304 | 0.638913 |
|  | Pob | 0.770674 | 0.647313 | 0.877439 | 0.976080 | 0.764781 |
| CS3 | Aya | 0.894519 | 0.794582 | 0.909742 | 0.717893 | 0.647573 |
|  | Ted | 0.937874 | 0.899481 | 0.862786 | 0.814174 | 0.72981 |
|  | Joy | 0.734154 | 0.715092 | 0.64894 | 0.451595 | 0.396816 |
|  | Pob | 0.558057 | 0.489356 | 0.629878 | 0.803065 | 0.556537 |
| CS4 | Aya | 0.646181 | 0.528936 | 0.646181 | 0.489985 | 0.427494 |
|  | Ted | 0.658297 | 0.623433 | 0.533366 | 0.493648 | 0.397558 |
|  | Joy | 0.908893 | 0.909593 | 0.876501 | 0.682359 | 0.614248 |
|  | Pob | 0.760103 | 0.662606 | 0.856119 | 0.961026 | 0.765234 |
| CS5 | Aya | 0.852777 | 0.754969 | 0.881185 | 0.64836 | 0.602154 |
|  | Ted | 0.903017 | 0.857165 | 0.817447 | 0.780667 | 0.67976 |
|  | Joy | 0.690751 | 0.689584 | 0.64894 | 0.451595 | 0.396816 |
|  | Pob | 0.558057 | 0.489356 | 0.629878 | 0.780317 | 0.556537 |
| CS6 | Aya | 0.602778 | 0.503429 | 0.646181 | 0.467237 | 0.427494 |
|  | Ted | 0.635549 | 0.600684 | 0.533366 | 0.493648 | 0.397558 |
|  | Joy | 0.780610 | 0.776750 | 0.718477 | 0.458849 | 0.428452 |
|  | Pob | 0.549545 | 0.460958 | 0.700649 | 0.840259 | 0.603189 |
| STS | Aya | 0.668152 | 0.561164 | 0.689217 | 0.467620 | 0.417647 |
|  | Ted | 0.736332 | 0.723483 | 0.623126 | 0.537819 | 0.462145 |

To observe the consistency and effect of the parameter *p*, we have prepared [Table 5](#_bookmark27) and [Fig. 2](#_bookmark29). For different values of *p*, we have seen that the values of similarity measures decreases when we increase the value of *p* but the order of the diagnosis remains same. From [Fig. 2](#_bookmark29), we have seen that the graph for *p* = 2 remains above and for *p* = 8 remains lower. No graph intersect or cut each other

and hence consistent and reliable for all value of *p*. This shows that if we increase or decrease the value for *p*, we get the same diagno- sis (order). For different values of *t*, we get the same diagnosis but their value depends on the data because *t* identifies the level of uncertainty and purely depends on the data set that we have provided.

Table 5

Bi-parametric Similarity Measure for *p* = 2 to *p* = 8.

Joy Pob Aya Ted

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| *p* = 2 |  | *Stomach problem*  0.842282 |  |  | *Heart problem*  0.891028 |  |  | *Viral fever*  0.806674 |  |  | *Stomach problem*  0.821348 |  |
| *p* = 3 |  | *Stomach problem*  0.809326 |  |  | *Heart problem*  0.881166 |  |  | *Viral fever*  0.764421 |  |  | *Stomach problem*  0.811558 |  |
| *p* = 4 |  | *Stomach problem*  0.782716 |  |  | *Heart problem*  0.873604 |  |  | *Viral fever*  0.732354 |  |  | *Stomach problem*  0.80347 |  |
| *p* = 5 |  | *Stomach problem*  0.762076 |  |  | *Heart problem*  0.867324 |  |  | *Viral fever*  0.70739 |  |  | *Stomach problem*  0.796708 |  |
| *p* = 6 |  | *Stomach problem*  0.746042 |  |  | *Heart problem*  0.86186 |  |  | *Viral fever*  0.687400 |  |  | *Stomach problem*  0.790998 |  |
| *p* = 7 |  | *Stomach problem*  0.733378 |  |  | *Heart problem*  0.857004 |  |  | *Viral fever*  0.671029 |  |  | *Stomach problem*  0.786128 |  |
| *p* = 8 |  | *Stomach problem*  0.733378 |  |  | *Heart problem*  0.857004 |  |  | *Viral fever*  0.671029 |  |  | *Stomach problem*  0.786128 |  |

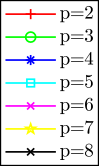


Fig. 2. Bi-parametric Similarity Measure for *p* = 2 to *p* = 8.

1. Comparison analysis

A comparison between bi-parametric similarity measure and already proposed similarity measure is conducted to illustrate the superiority of the bi-parametric similarity measure.

We have seen from [Example 5.1](#_bookmark17) that the second condition of [Definition 3.2](#_bookmark7) (*S*2) is not satisfied for cosine similarity measure *CS*1, i.e., £^(Α^ , S^) = 1 even Α^ – S^. Also we provide a general criteria when second condition of [Definition 3.2](#_bookmark7) (*S*2) is not satisfied for

cosine similarity measure *CS*1. In [Example 5.2](#_bookmark20), we have seen that the *STM* can not classify the unknown pattern and i.e., *STM*(*P*, *Q* 1) = *STM*(*P*, *Q* 2) even *Q* 1 – *Q* 2. In [Example 5.3](#_bookmark22), we have seen that the CS2 can not classify the unknown pattern i.e., CS2(*P*, *Q* 1)= CS2(*P*, *Q* 2) even *Q* 1 – *Q* 2. Also in [Example 5.3](#_bookmark22), we have seen that the similarity measure CS4 fails to classify unknown pat-

tern i.e., CS4(*P*, *Q* 1) = CS4(*P*, *Q* 2) even *Q* 1 – *Q* 2. In [Example 5.4](#_bookmark24), we

have seen that the CS3 can not classify the unknown pattern and i.e., CS3(*P*, *Q* 1) = CS3(*P*, *Q* 2) even *Q* 1 – *Q* 2. In [Example 5.5](#_bookmark23), we have seen that the CS5 can not classify the unknown pattern i.e., CS5(*P*, *Q* 1)= CS5(*P*, *Q* 2) even *Q* 1 – *Q* 2. Also in [Example 5.5](#_bookmark23), we have seen that the similarity measure CS6 fails to classify unknown pat- tern i.e., CS6(*P*, *Q* 1)= CS6(*P*, *Q* 2) even *Q* 1 – *Q* 2. But in all [Examples](#_bookmark17)

[5.1–5.5](#_bookmark17), the bi-parametric similarity measure classify the unknown pattern and hence successfully applicable to the pattern recogni-

tion problems. Also in medical diagnosis problems CS2 and CS4

can not diagnosis Aya (see [Table 4](#_bookmark26)).

A comprehensive comparison with the existing similarity mea- sures is discussed by considering four cases. In first case, suppose *A* = (0.4, 0.4, 0.2), *B* = (0.2, 0.2, 0.1) and *C* = (0.3, 0.1, 0.1) be three

distinct PFVs. We have calculated the similarity measure between *A* and *B*, as well as, *A* and *C*. The second condition for similarity measure is not satisfied for CS1, i.e., *A* – *B* but CS1(*A*, *B*) = 1. The similarity measures CS3, CS5, CS6, CS7 and STS have no capabilities to distinguish between the PFVs *B* and *C* from the PFV *A* perspec- tive. In second case, we suppose *D* = (0.5, 0.3, 0.2), *E* = (0.3, 0.2, 0.1)) and *F* = (0.6, 0.1, 0.3) be three distinct PFVs. The similarity measures CS2, CS3 and CS4 have no capabilities to distin- guish between the PFVs *E* and *F* from the PFV *D* perspective. Simi- larly, in third case, the PFVs *G* = (0.5, 0.3, 0.2), *H* = (0.3, 0.2, 0.1)) and *I* = (0.6, 0.1, 0.3) are considered. All the previously proposed similarity measures for PFSs have no capabilities to distinguish

between the PFVs *H* and *I* from the PFV *G* perspective. The fourth case consist of two distinct PFVs *K* = (0.2, 0.2, 0.2), *L* = (0.3, 0.3, 0.3) but CS1(*K*, *L*)= 1. The bi-parametric similarity mea- sure remains consistent for all cases and gave us a satisfactory results. The summary of the four cases presented in [Table 6](#_bookmark30).

1. Conclusion

In this paper, we have proposed new similarity measures for *PFSs* called bi-parametric similarity measures based on two param- eters *t* and *p*, where *t* is the level of uncertainty and *p* is the *Lp* norm. We have defined bi-parametric distance measures, weighted bi-parametric distance measures, bi-parametric similarity mea- sures and weighted bi-parametric similarity measures and dis- cussed their properties. Also, we discussed the comprehensive interpretation of the proposed similarity measure. We discuss some special cases ([(5.1)–(5.5)](#_bookmark17)), where already proposed similarity measure fails to classify the unknown pattern while the proposed similarity measure successfully applied to the pattern recognition problems. A numerical example is proposed to solve medical diag- nosis problems and we have seen that the results are consistent and does not change with the values of the parameters (See [Table 5](#_bookmark27) and [Fig. 2](#_bookmark29)).

In the future direction, we will apply the bi-parametric similar- ity measure to data mining, medical diagnosis, decision making, complex group decision making, linguistic summarization risk analysis, pattern recognition, clustering analysis, image processing and image restoration problems.

*d d d*

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Table 6  Comparison Table. |  | | | | | | |
|  | S(A, B) | S(A, C) | S(D, E) | S(D, F) | S(G, H) | S(G, I) | S(K, L) |
|  | (0.4,0.4,0.2) | (0.4,0.4,0.2) | (0.5,0.3,0.2) | (0.5,0.3,0.2) | (0.3,0.3,0.3) | (0.3,0.3,0.3) | (0.2,0.2,0.2) |
|  | (0.2,0.2,0.1) | (0.3,0.1,0.1) | (0.3,0.2,0.1) | (0.6,0.1,0.3) | (0.2,0.2,0.1) | (0.1,0.2,0.2) | (0.3,0.3,0.3) |
| CS1 | 1. | 0.9045 | 0.9972 | 0.9328 | 0.9623 | 0.9623 | 1. |
| CS2 | 0.9511 | 0.891 | 0.9511 | 0.9511 | 0.9511 | 0.9511 | 0.9877 |
| CS3 | 0.9239 | 0.9239 | 0.9511 | 0.9511 | 0.9511 | 0.9511 | 0.9724 |
| CS4 | 0.7265 | 0.6128 | 0.7265 | 0.7265 | 0.7265 | 0.7265 | 0.8541 |
| CS5 | 0.7071 | 0.7071 | 0.809 | 0.9511 | 0.809 | 0.809 | 0.891 |
| CS6 | 0.4142 | 0.4142 | 0.5095 | 0.7265 | 0.5095 | 0.5095 | 0.6128 |
| CS7 | 0.7071 | 0.7071 | 0.809 | 0.9511 | 0.809 | 0.809 | 0.891 |
| STS | 0.5 | 0.5 | 0.6053 | 0.7222 | 0.5556 | 0.5556 | 0.6667 |
| £1 | 0.8 | 0.766 | 0.8846 | 0.9286 | 0.8333 | 0.8333 | 0.9231 |
| £2 | 0.5143 | 0.5 | 0.6765 | 0.9286 | 0.6452 | 0.6452 | 0.7857 |
| £3 | 0.8 | 0.766 | 0.8846 | 0.9286 | 0.8333 | 0.8333 | 0.9231 |
| Sp 0.0947 | | 0.1493 | 0.0707 | 0.1041 | 0.1041 | 0.0707 | 0.0629 |

t

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